Finiteness of the Number of Compatibly Split Subvarieties

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We prove that there are only finitely many compatibly split closed subschemes of a Frobenius split scheme.

1 Introduction

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and let \( X \) be a scheme over \( k \) (always assumed to be separated of finite type over \( k \)). The following is the main theorem of this note and we give here its complete and self-contained proof.

\textbf{Theorem 1.1.} Assume that \( X \) is Frobenius split by a splitting \( \sigma \in \text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \), where \( F \) is the absolute Frobenius morphism (cf. [1, Section 1.1]). Then, there are only finitely many closed subschemes of \( X \) that are compatibly split (under \( \sigma \)).

2 Proof of Theorem 1.1

We first prove the following proposition that is of independent interest.

\textbf{Proposition 2.1.} Let \( X \) be a nonsingular irreducible scheme that is Frobenius split by \( \sigma \in \text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong H^0(X, F_*(\omega_X^{1-p})) \), where \( \omega_X \) is the dualizing sheaf of \( X \) (cf. [1,
Proposition 1.3.7), and let \( Y \subsetneq X \) be a compatibly split closed subscheme of \( X \). Then,

\[
Y \subset Z(\bar{\sigma}),
\]

where \( Z(\bar{\sigma}) \) denotes the set of zeroes of \( \bar{\sigma} \) and \( \bar{\sigma} \) is the section of \( F_*(\omega_X^{1-p}) \) obtained from \( \sigma \) via the above identification.

It may be remarked that \( Z(\bar{\sigma}) \) is not compatibly split in general unless \( \bar{\sigma} \) is a \((p-1)\)th power (cf. \cite[Proposition 1.3.11]{[1]}).

**Proof.** Since any irreducible component of a compatibly split closed subscheme is compatibly split (cf. \cite[Proposition 1.2.1]{[1]}), we can assume that \( Y \) is irreducible. Assume, for contradiction, that \( Y \cap (X \setminus Z(\bar{\sigma})) \neq \emptyset \). Then, \( Y^{\text{reg}} \cap (X \setminus Z(\bar{\sigma})) \neq \emptyset \), where \( Y^{\text{reg}} \) is the nonsingular locus of \( Y \).

Take \( y \in Y^{\text{reg}} \cap (X \setminus Z(\bar{\sigma})) \). Choose a system of local parameters \( \{t_1, \ldots, t_m, t_{m+1}, \ldots, t_n\} \) at \( y \in X \) such that \( \{t_1, \ldots, t_m\} \) restrict to a system of local parameters at \( y \in Y \) and \( \langle t_{m+1}, \ldots, t_n \rangle \) is the completion of the defining ideal of \( Y \) in \( X \) at \( y \). (This is possible since both \( X \) and \( Y \) are nonsingular at \( y \).) By assumption, \( \bar{\sigma} \) is a unit in the local ring \( O_{X,Y} \). Moreover, \( \sigma \) induces a splitting \( \hat{\sigma} \) of the power series ring \( k[[t_1, \ldots, t_n]] \) compatibly splitting the ideal \( \langle t_{m+1}, \ldots, t_n \rangle \). Now, since \( \hat{\sigma} \) does not vanish at \( y \), \( \hat{\sigma}((t_1 \cdots t_n)^{p-1}) \) is a unit in the ring \( k[[t_1, \ldots, t_n]] \) (cf. \cite[Proposition 1.3.7]{[1]}). In particular, \( \hat{\sigma} \) does not keep the ideal \( \langle t_{m+1}, \ldots, t_n \rangle \) stable. This is a contradiction to the assumption. Hence, \( Y \subset Z(\bar{\sigma}) \), proving the proposition. \( \blacksquare \)

**Proof of Theorem 1.1.** We prove Theorem 1.1 by induction on the dimension of \( X \). If \( \dim X = 0 \), then the theorem is clear. So assume that \( \dim X = n \) and the theorem is true for schemes of dimension \(< n \). By \cite[Proposition 1.2.1]{[1]}, we can assume without loss of generality that \( X \) is irreducible. Let \( Y \subset X \) be a compatibly split irreducible closed subscheme. Then, either \( Y \subset X^{\text{sing}} \) (where \( X^{\text{sing}} \) is the singular locus of \( X \)) or \( Y \cap X^{\text{reg}} \neq \emptyset \). In the latter case, by Proposition 2.1,

\[
Y \cap X^{\text{reg}} \subset Z(\bar{\sigma}^o),
\]

where \( Z(\bar{\sigma}^o) \) denotes the set of zeroes of the splitting \( \bar{\sigma}^o \) of the open subset \( X^{\text{reg}} \) of \( X \) viewed as a section of \( F_*(\omega_X^{1-p}) \). Thus, in this case,

\[
Y \subset \overline{Z(\bar{\sigma}^o)},
\]

\( \overline{Z(\sigma^o)} \) being the closure of \( Z(\sigma^o) \) in \( X \). Hence, in either case,

\[
Y \subset \overline{Z(\sigma^o)} \cup X^{\text{sing}}.
\]
Considering the irreducible components, the same inclusion (1) holds for any compatibly split closed subscheme $Y \subset X$ such that $Y \neq X$.

Let $\{Y_i\}_{i \in I}$ be the collection of all the distinct compatibly split closed subschemes $Y_i \subset X$ and let $Y := \bigcup_{i \in I} Y_i$. Observe that $X$ being a scheme of finite type over $k$ and $Y$ a subscheme of $X$, $Y$ has only finitely many irreducible components. Since the ideal sheaf $\mathcal{I}_Y = \bigcap_{i \in I} \mathcal{I}_{Y_i}$ and each $\mathcal{I}_{Y_i}$ is stable under the splitting $\sigma$ of $X$, the closed subscheme $Y$ is compatibly split. In particular, by (1), for each $i \in I$,

$$Y_i \subset Z(\overline{\sigma}) \cup X^\text{sing},$$

and hence $Y \subset Z(\overline{\sigma}) \cup X^\text{sing}$. Since $\dim(Z(\overline{\sigma}) \cup X^\text{sing}) < \dim X$; in particular, one has $\dim Y < \dim X$. Thus, by the induction hypothesis (applying the theorem with $X$ replaced by $Y$), $I$ is a finite set. This completes the proof of the theorem.

**Remark 2.2.** Karl Schwede has also obtained the above theorem in a recent preprint [4]. His argument (obtained independently of our work) is similar to ours but he uses the theory of tight closure and test ideals to obtain a replacement for our Proposition 2.1. As pointed out by Schwede, when $X$ is projective, the theorem also follows from [2, Corollary 3.2]. (See also [5] for another proof via the study of Frobenius actions on Artinian modules.)

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**References**


