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Finiteness of the Number of Compatibly Split Subvarieties

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We prove that there are only finitely many compatibly split closed subschemes of a Frobenius split scheme.

1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$ and let X be a scheme over k (always assumed to be separated of finite type over k). The following is the main theorem of this note and we give here its complete and self-contained proof.

Theorem 1.1. Assume that X is Frobenius split by a splitting $\sigma \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, where F is the absolute Frobenius morphism (cf. [1, Section 1.1]). Then, there are only finitely many closed subschemes of X that are compatibly split (under σ).

2 Proof of Theorem 1.1

We first prove the following proposition that is of independent interest.

Proposition 2.1. Let X be a nonsingular irreducible scheme that is Frobenius split by $\sigma \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \simeq H^0(X, F_*(\omega_X^{1-p}))$, where ω_X is the dualizing sheaf of X (cf. [1,

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Proposition 1.3.7]), and let $Y \underset{\neq}{\subset} X$ be a compatibly split closed subscheme of X. Then,

$$Y \subset Z(\bar{\sigma}),$$

where $Z(\bar{\sigma})$ denotes the set of zeroes of $\bar{\sigma}$ and $\bar{\sigma}$ is the section of $F_*(\omega_X^{1-p})$ obtained from σ via the above identification.

It may be remarked that $Z(\bar{\sigma})$ is not compatibly split in general unless $\bar{\sigma}$ is a $(p-1)$th power (cf. [1, Proposition 1.3.11]).

Proof. Since any irreducible component of a compatibly split closed subscheme is compatibly split (cf. [1, Proposition 1.2.1]), we can assume that Y is irreducible. Assume, for contradiction, that $Y \cap (X \backslash Z(\bar{\sigma})) \neq \emptyset$. Then, $Y^{\text{reg}} \cap (X \backslash Z(\bar{\sigma})) \neq \emptyset$, where Y^{reg} is the nonsingular locus of Y.

Take $y \in Y^{\text{reg}} \cap (X \backslash Z(\bar{\sigma}))$. Choose a system of local parameters $\{t_1, \dots, t_m, t_{m+1}, \dots, t_n\}$ at $y \in X$ such that $\{t_1, \dots, t_m\}$ restrict to a system of local parameters at $y \in Y$ and $\langle t_{m+1}, \dots, t_n \rangle$ is the completion of the defining ideal of Y in X at y. (This is possible since both X and Y are nonsingular at y.) By assumption, $\bar{\sigma}$ is a unit in the local ring $\mathcal{O}_{X,y}$. Moreover, σ induces a splitting $\hat{\sigma}$ of the power series ring $k[[t_1, \dots, t_n]]$ compatibly splitting the ideal $\langle t_{m+1}, \dots, t_n \rangle$. Now, since $\bar{\sigma}$ does not vanish at y, $\hat{\sigma}((t_1 \cdots t_n)^{p-1})$ is a unit in the ring $k[[t_1, \dots, t_n]]$ (cf. [1, Proposition 1.3.7]). In particular, $\hat{\sigma}$ does not keep the ideal $\langle t_{m+1}, \dots, t_n \rangle$ stable. This is a contradiction to the assumption. Hence, $Y \subset Z(\bar{\sigma})$, proving the proposition. ∎

Proof of Theorem 1.1. We prove Theorem 1.1 by induction on the dimension of X. If $\dim X = 0$, then the theorem is clear. So assume that $\dim X = n$ and the theorem is true for schemes of dimension $< n$. By [1, Proposition 1.2.1], we can assume without loss of generality that X is irreducible. Let $Y \underset{\neq}{\subset} X$ be a compatibly split irreducible closed subscheme. Then, either $Y \subset X^{\text{sing}}$ (where X^{sing} is the singular locus of X) or $Y \cap X^{\text{reg}} \neq \emptyset$. In the latter case, by Proposition 2.1,

$$Y \cap X^{\text{reg}} \subset Z(\bar{\sigma^o}),$$

where $Z(\bar{\sigma^o})$ denotes the set of zeroes of the splitting $\bar{\sigma^o}$ of the open subset X^{reg} of X viewed as a section of $F_*(\omega_{X^{\text{reg}}}^{1-p})$. Thus, in this case,

$$Y \subset \overline{Z(\bar{\sigma^o})},$$

$\overline{Z(\bar{\sigma^o})}$ being the closure of $Z(\bar{\sigma^o})$ in X. Hence, in either case,

$$Y \subset \overline{Z(\bar{\sigma^o})} \cup X^{\text{sing}}. \tag{1}$$

Considering the irreducible components, the same inclusion (1) holds for any compatibly split closed subscheme $Y \subset X$ such that $Y \neq X$.

Let $\{Y_i\}_{i \in I}$ be the collection of all the distinct compatibly split closed subschemes $Y_i \subsetneq X$ and let $Y := \overline{\bigcup_{i \in I} Y_i}$. Observe that X being a scheme of finite type over k and Y a subscheme of X, Y has only finitely many irreducible components. Since the ideal sheaf $\mathcal{I}_Y = \bigcap_{i \in I} \mathcal{I}_{Y_i}$ and each $\mathcal{I}_{Y_i}$ is stable under the splitting σ of X, the closed subscheme Y is compatibly split. In particular, by (1), for each $i \in I$,

$$Y_i \subset \overline{Z(\bar{\sigma}^o)} \cup X^{\text{sing}},$$

and hence $Y \subset \overline{Z(\bar{\sigma}^o)} \cup X^{\text{sing}}$. Since $\dim(\overline{Z(\bar{\sigma}^o)} \cup X^{\text{sing}}) < \dim X$; in particular, one has $\dim Y < \dim X$. Thus, by the induction hypothesis (applying the theorem with X replaced by Y), I is a finite set. This completes the proof of the theorem. ■

Remark 2.2. Karl Schwede has also obtained the above theorem in a recent preprint [4]. His argument (obtained independently of our work) is similar to ours but he uses the theory of tight closure and test ideals to obtain a replacement for our Proposition 2.1. As pointed out by Schwede, when X is projective, the theorem also follows from [2, Corollary 3.2]. (See also [5] for another proof via the study of Frobenius actions on Artinian modules.)

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