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On Positivity in T-Equivariant K-Theory of Flag Varieties

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We prove some general results on the T-equivariant K-theory $K_T(G/P)$ of the flag variety G/P, where G is a complex semisimple algebraic group, P is a parabolic subgroup, and T is a maximal torus contained in P. In particular, we make a conjecture about a positivity phenomenon in $K_T(G/P)$ for the product of two basis elements written in terms of the basis of $K_T(G/P)$ given by the dual of the structure sheaf (of Schubert varieties) basis. (For the full flag variety G/B, this dual basis is closely related to the basis given by Kostant–Kumar.) This conjecture is parallel to (but different from) the conjecture of Griffeth–Ram for the structure constants of the product in the structure sheaf basis. We give explicit expressions for the product in the T-equivariant K-theory of projective spaces in terms of these bases. In particular, we establish our conjecture and the conjecture of Griffeth–Ram in this case.

1 Introduction

Let X denote the partial flag variety G/P, where G is a complex semisimple simply connected algebraic group and P is a parabolic subgroup of G containing a fixed Borel subgroup B. The group B acts with finitely many orbits on X, and the closures of these orbits (called the Schubert varieties) are indexed by W^P, the set of minimal length coset representatives of W/W_P (where W is the Weyl group of G and W_P is the Weyl group

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of P); the Schubert variety corresponding to $w \in W^P$ is denoted by X_w^P. The Poincaré duals of the fundamental classes $[X_w^P]$ (called the Schubert classes) form a basis for the cohomology $H^*(X)$. The structure constants of the multiplication in $H^*(X)$ with respect to this basis have long been known to be non-negative.

This positivity result has been generalized in different directions. If T is a maximal torus of B, then the equivariant cohomology $H_T^*(X)$ is a free module over $H_T^*(\mathrm{pt})$, the equivariant cohomology ring of a point, again with a basis consisting of Schubert classes. As proved by Graham [7] using methods of Kumar and Nori [12], the structure constants in this basis again have a positivity property generalizing the nonequivariant positivity. Similarly, the Grothendieck group $K(X)$ has a basis consisting of classes of structure sheaves $[\mathcal{O}_{X_w^P}]$ of Schubert varieties. As proved by Brion [3], the structure constants of the multiplication in $K(X)$ have a predictable alternating sign behavior, which we will refer to as a *positivity property*.

The positivity in $H_T^*(X)$ and the positivity in $K(X)$ each imply the positivity in $H^*(X)$. Our aim in this paper is to discuss a positivity property for the multiplication in the T-equivariant Grothendieck group $K_T(X)$ encompassing the positivity both in $H_T(X)$ and $K(X)$. One subtlety in the T-equivariant K-theory is that $K_T(X)$ has two natural quite different bases: the basis consisting of classes of structure sheaves of Schubert varieties (called the *structure sheaf* basis), and the dual basis with respect to the natural pairing on $K_T(X)$ (called the *dual structure sheaf* basis).

Surprisingly, both the structure sheaf basis and the dual structure sheaf basis of $K_T(X)$ seem to exhibit the positivity phenomenon. Let $R(T)$ denote the representation ring of T, which is a free abelian group with basis consisting of the characters e^λ. Let Δ denote the set of roots of Lie G with respect to Lie T, Δ^+ the set of positive roots (chosen so that these are the roots of Lie B), and $\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset \Delta^+$ the set of simple roots. Let $\{\xi_P^w\}$ denote the dual basis to the $R(T)$-basis $\{[\mathcal{O}_{X_w^P}]\}$ of $K_T(X)$. Write

$$[\mathcal{O}_{X_u^P}][\mathcal{O}_{X_v^P}] = \sum_{w \in W^P} b_{u,v}^w(P)[\mathcal{O}_{X_w^P}],$$

and

$$\xi_P^u \xi_P^v = \sum_{w \in W^P} p_{u,v}^w(P) \xi_P^w,$$

for (unique) elements $b_{u,v}^w(P)$ and $p_{u,v}^w(P)$ of $R(T)$. Griffeth and Ram conjectured a positivity property for the coefficients $b_{u,v}^w(P)$ (see [9, Conjecture 5.9]). Specifically, their conjecture asserts that

$$(-1)^{\dim(X)+\ell(u)+\ell(v)+\ell(w)} b_{u,v}^{w}(P) \in \mathbb{Z}_{+}[\mathrm{e}^{\alpha_i}-1]_{\alpha_i \in \Pi} \tag{1}$$

(see Conjecture 3.10 and Remarks 3.11 and 3.12, where we discuss their conventions). The validity of this conjecture for $P = B$ implies its validity for any P (cf. Lemma 3.13).

In this paper, we conjecture that the coefficients $p_{u,v}^{w}(P)$ also exhibit the following positivity:

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\ p_{u,v}^{w}(P) \in \mathbb{Z}_{+}[\mathrm{e}^{-\alpha_i}-1]_{\alpha_i \in \Pi} \tag{2}$$

(see Conjecture 3.1). It is not clear if the validity of the conjecture for $P = B$ implies that for an arbitrary P. On the other hand, this conjecture is compatible with the inclusion of flag varieties associated to Levi subgroups (see Proposition 3.3). Although the coefficients $b_{u,v}^{w}(P)$ and $p_{u,v}^{w}(P)$ are related (see Propositions 4.1 and 4.3), it is not clear if one conjecture implies the other.

The nonequivariant analogs of both these conjectures hold. More precisely, if $F : R(T) \to \mathbb{Z}$ is the forgetful map (sending each e^{λ} to 1), then

$$(-1)^{\dim(X)+\ell(u)+\ell(v)+\ell(w)} F\left(b_{u,v}^{w}(P)\right) \geq 0$$

and

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\ F\left(p_{u,v}^{w}(P)\right) \geq 0.$$

The first inequality is proved in [3]. The second inequality can be easily deduced from [3, Theorem 1] (see Remark 3.7). In fact, we conjecture that an equivariant generalization of [3, Theorem 1] holds: Let T' be a subtorus of T. If $Y \subset X$ is a T'-stable irreducible subvariety with rational singularities, and we write in $K_{T'}(X)$

$$[\mathcal{O}_Y] = \sum_{w \in W^P} a_w^Y [\mathcal{O}_{X_w^P}],$$

then

$$(-1)^{\mathrm{codim}\ Y + \mathrm{codim}\ X_w^P}\, a_w^Y \in \mathbb{Z}_+\big[\mathrm{e}^{-\alpha_i}{}_{|T'} - 1\big]_{\alpha_i \in \Pi}$$

(see Conjecture 7.1). By Proposition 3.6, this would imply Conjecture 3.1.

The main purpose of this paper is to prove some results giving evidence for Conjectures 3.1 and 7.1. The most substantial results are explicit formulas for the coefficients $p_{u,v}^w(P)$ and $b_{u,v}^w(P)$ in case $X = \mathbb{P}^n$ (see Theorems 6.5 and 6.14). We also deduce recurrence relations (Theorems 6.6 and 6.15) for the coefficients $p_{u,v}^w(P)$ and $b_{u,v}^w(P)$ that imply Conjectures 3.1 and 3.10 for $X = \mathbb{P}^n$. The proofs of these results presented here were suggested to us by one of the referees; they are somewhat shorter than our original proofs. We also prove that in the case $P = B$, Conjecture 3.1 holds for the coefficients $p_{u,1}^w$ and $p_{u,s}^w$, where 1 and s are (respectively) the identity element of W and a simple reflection (see Proposition 3.8 and Remark 3.9). We verify Conjecture 7.1 in the case Y is any opposite Schubert variety X_P^w (cf. Proposition 7.6 and Remark 7.7(a)).

A secondary purpose of this paper is to collect various results relating different bases of $K_T(X)$, and relations among the structure constants in these bases. Among the natural bases of $K_T(X)$ are the structure sheaf basis, the dual structure sheaf basis, and the basis of the dualizing sheaves of Schubert varieties. Also, one can take opposite Schubert varieties in place of Schubert varieties. The positivity conjectures have different formulations in terms of these different bases. We describe some of the relations between these bases and structure constants, in the hope that this paper will serve as a useful reference for other workers in this area.

The contents of the paper are as follows. Section 1 lays down the basic notation. Section 2 contains some preliminary results on $K_T(G/P)$. In particular, it identifies the dual structure sheaf basis and also the basis of the dualizing sheaves of Schubert varieties (cf. Propositions 2.1 and 2.2). Section 3 contains the statement of our positivity conjecture (Conjecture 3.1). We prove the conjecture for the coefficients $p_{u,1}^w$ and $p_{u,s}^w$ (for any simple reflection s) in the case $P = B$ (cf. Proposition 3.8 and Remark 3.9). We have verified the conjecture by an explicit calculation for any rank-2 group in the case $P = B$ (see the appendix for G of type A_2 and B_2). By a result of Brion, the nonequivariant analog of this conjecture holds (Remark 3.7). This section also contains the positivity conjecture of Griffeth and Ram (cf. Conjecture 3.10) and its equivalent reformulation in terms of the dualizing sheaves (cf. Proposition 3.14). It is shown that the validity of the Griffeth–Ram conjecture for $P = B$ implies its validity for any P (cf. Lemma 3.13). Section 4 contains some relations between the structure constants with respect to the structure sheaf basis and the dual structure sheaf basis (cf. Propositions 4.1 and 4.3). Section 5 proves that the structure constants with respect to either basis in the case $P = B$ lie in the subring $\mathbb{Z}[e^{-\alpha_i} - 1]_{\alpha_i \in \Pi}$ of $R(T)$ (cf. Theorem 5.1 and Corollary 5.2). Section 6 contains the explicit formula for the structure constants in the case $X = \mathbb{P}^n$ in the dual structure sheaf basis, and the recurrence relation implying Conjecture 3.1 in this case (cf. Theorems 6.5, 6.6, and 6.12). Similar results are also obtained in the structure sheaf basis (cf. Theorems 6.14, 6.15, and 6.19). Section 7 contains our more general conjecture asserting the positivity of the coefficients of the class of the structure sheaf of a T'-stable subvariety Y of G/P with rational singularities written in terms of the structure sheaf basis (cf. Conjecture 7.1), for any subtorus $T' \subset T$. We prove this conjecture in the special case where Y is any opposite Schubert variety in any G/P (cf. Proposition 7.6 and Remark 7.7(a)).

We include explicit formulas for the product in the dual basis $\{\xi^w\}_{w\in W}$ of $K_T(G/B)$ for G of type A_2 and B_2 in the appendix.

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1.1 Definitions and notation

We work with noetherian schemes over the ground field of complex numbers. By a variety we mean quasi-projectivè (not necessarily irreducible) variety.

Let X be a smooth algebraic variety with an action of a torus T. Let $K_T(X)$ denote the Grothendieck group of T-equivariant coherent sheaves on X; because X is smooth, $K_T(X)$ may be identified with the Grothendieck group of T-equivariant vector bundles on X (cf. [5, Proposition 5.1.28]). Thus, $K_T(X)$ is a ring; we will sometimes write the multiplication in $K_T(X)$ using the notation of tensor product. The class in $K_T(X)$ of a T-equivariant coherent sheaf $\mathcal{F}$ will be denoted by $[\mathcal{F}]$. In particular, if $Y \subset X$ is a T-stable closed subscheme, then the structure sheaf of Y defines a class $[\mathcal{O}_Y]$ in $K_T(X)$; if Y is Cohen–Macaulay and equidimensional, then its dualizing sheaf ω_Y defines a class $[\omega_Y]$ in $K_T(X)$. Let $* : K_T(X) \to K_T(X)$ denote the standard involution taking a vector bundle to its dual and e^{λ} to $e^{-\lambda}$. If $Y \supset Z$ are closed T-stable subschemes of X, then $\mathcal{O}_Y(-Z)$ is the ideal sheaf of Z in Y. Thus, viewed as an element of $K_T(X)$, $[\mathcal{O}_Y(-Z)] = [\mathcal{O}_Y] - [\mathcal{O}_Z]$.

Recall that $R(T)$ is a free abelian group (freely) generated by the characters e^{λ} of T. If V is any representation of T, we write $\mathrm{ch}\, V$ for the corresponding element of $R(T)$

(a linear combination of e^λ). The group $K_T(X)$ is an $R(T)$-module. If X is proper, and $\mathcal{F}$ is a T-equivariant coherent sheaf on X, write $h^i(X, \mathcal{F}) = \text{ch } H^i(X, \mathcal{F})$ and

$$\chi(X, \mathcal{F}) := \sum_{p \geq 0} (-1)^p h^p(X, \mathcal{F}) \in R(T).$$

We extend this definition to define $\chi(X, \gamma)$ for any $\gamma \in K_T(X)$. For X proper, there is a pairing

$$\langle \cdot, \cdot \rangle : K_T(X) \otimes_{R(T)} K_T(X) \to R(T)$$

given by

$$\langle v_1, v_2 \rangle = \chi(X, v_1 \cdot v_2).$$

If $\mathcal{F}$ is supported on a T-stable closed subscheme Y, then, viewing $\mathcal{F}$ as a sheaf on Y, we have $\chi(X, \mathcal{F}) = \chi(Y, \mathcal{F})$.

Let G be a semisimple connected simply connected complex algebraic group. For the rest of the paper, T will denote a maximal torus of G. Let B be a Borel subgroup of G containing T and, as above, let Δ denote the set of roots and Δ^+ the set of positive roots, chosen so that the roots of Lie B are positive. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Delta^+$ denote the simple roots, and let s_i denote the simple reflection corresponding to α_i. Let $Q^+ := \sum_i \mathbb{Z}_+ \alpha_i$. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Let B^- denote the Borel subgroup of G such that $B \cap B^- = T$.

Let $P \supset B$ be a (standard) parabolic subgroup of G and let W_P be its Weyl group. Let W^P be the set of the minimal length coset representatives in W/W_P. For $w \in W^P$, let X_w^P (respectively, X_P^w) be the Schubert variety (respectively, the opposite Schubert variety) defined by

$$X_w^P = \overline{BwP/P} \subset G/P$$

and

$$X_P^w = \overline{B^- wP/P} \subset G/P.$$

(Here and elsewhere, we use the same notation for elements of W and lifts of those elements to G.) Set closed and reduced subschemes:

$$\partial X_w^P = \bigsqcup_{\substack{v \in W^P \\ v < w}} BvP/P,$$

and

$$\partial X_P^w = \bigsqcup_{\substack{v \in W^P \\ v > w}} B^- vP/P.$$

Except in Section 6, we will abbreviate X_w^B and X_B^w by X_w and X^w, respectively. If λ is a character of P and $\mathbb{C}_\lambda$ is the corresponding 1-dimensional representation of P, then let $\mathcal{L}(\lambda)$ denote the line bundle $G \times_P \mathbb{C}_{\lambda^{-1}}$ on G/P.

Given an element $\gamma \in K_T(G/B)$, we write $\gamma(w)$ for the pullback of γ to $K_T(\{wB\}) = R(T)$.

2 Preliminary Results on $K_T(G/P)$

Recall that X_w^P and X_P^w are Cohen–Macaulay (cf. [1, Cor. 3.4.4]), and hence their dualizing sheaves $\omega_{X_w^P}$ and $\omega_{X_P^w}$ make sense.

It is well known that $\{[\mathcal{O}_{X_w^P}]\}_{w \in W^P}$ is a $R(T)$-basis of $K_T(G/P)$, and so is $\{[\mathcal{O}_{X_P^w}]\}_{w \in W^P}$. For any $w \in W^P$, set $\xi_P^w = [\mathcal{O}_{X_P^w}(-\partial X_P^w)] \in K_T(G/P)$.

The next proposition is known and has been observed, for example, by Knutson (see [4, Section 8]).

Proposition 2.1. For any $v, w \in W^P$,

$$\langle [\mathcal{O}_{X_w^P}], \xi_P^v \rangle = \delta_{v,w},$$

i.e., $\{[\mathcal{O}_{X_w^P}]\}_{w \in W^P}$ and $\{\xi_P^w\}_{w \in W^P}$ are dual bases under the above pairing. □

Proof. Since the intersections $X_w^P \cap X_P^v$ and $X_w^P \cap \partial X_P^v$ are proper (∂X_P^v is also Cohen–Macaulay since it is of pure codimension 1 in the Cohen–Macaulay variety X_P^v), we get (by [3, Lemma 1])

$$\langle [\mathcal{O}_{X_w^P}], \xi_P^v \rangle = \chi\big(G/P, \mathcal{O}_{X_w^P \cap X_P^v}\big(-X_w^P \cap \partial X_P^v\big)\big).$$

By [2, Proposition 1],

$$\chi\big(G/P, \mathcal{O}_{X_w^P \cap X_P^v}\big) = 1 \text{ (or 0)}$$

according as

$$X_w^P \cap X_P^v \neq \emptyset \ \big(\text{or } X_w^P \cap X_P^v = \emptyset\big).$$

Similarly,

$$\chi\big(G/P, \mathcal{O}_{X_w^P \cap \partial X_P^v}\big) = 1 \text{ (or 0)}$$

according as

$$X_w^P \cap \partial X_P^v \neq \emptyset \ \big(\text{or } X_w^P \cap \partial X_P^v = \emptyset\big).$$

Now,

$$\begin{aligned} X_w^P \cap X_P^v \neq \emptyset &\Leftrightarrow w \geq v \quad \text{and} \\ X_w^P \cap \partial X_P^v \neq \emptyset &\Leftrightarrow \text{there exists a } \theta \in W^P \text{ such that } w \geq \theta > v \\ &\Leftrightarrow w > v. \end{aligned}$$

Combining the above, we get the proposition. ■

Let $\{\tau^w\}_{w \in W}$ be the Kostant–Kumar $R(T)$-basis of $K_T(G/B)$ (cf. [11, Remark 3.14]). We abbreviate ξ_B^w by ξ^w (as noted above, X_w^B and X_B^w are abbreviated as X_w and X^w, respectively).

The next proposition gives some of the relations between various T-equivariant sheaves on G/B and between elements of $K_T(G/B)$.

Proposition 2.2. For any $w \in W$

(a) $\omega_{X_w} \simeq \mathrm{e}^{-\rho}\mathcal{L}(-\rho) \otimes \mathcal{O}_{X_w}(-\partial X_w)$ as T-equivariant sheaves.
(b) $\omega_{X^w} \simeq \mathrm{e}^{\rho}\mathcal{L}(-\rho) \otimes \mathcal{O}_{X^w}(-\partial X^w)$ as T-equivariant sheaves.
(c) $*\tau^w = \xi^{w^{-1}} = \mathrm{e}^{-\rho}[\mathcal{L}(\rho)][\omega_{X_{w^{-1}}}]$ as elements of $K_T(G/B)$.
(d) $\mathrm{e}^{\rho}[\mathcal{L}(\rho)](*\tau^w) = (-1)^{\ell(w)} * [\mathcal{O}_{X_{w^{-1}}}]$.
(e) $*\xi^w = (-1)^{\ell(w)} e^{\rho}[\mathcal{L}(\rho)][\mathcal{O}_{X^w}]$.

□

Proof. By [16, Theorem 4.2], as nonequivariant sheaves,

$$\omega_{X_w} \simeq \mathcal{L}(-\rho) \otimes \mathcal{O}_{X_w}(-\partial X_w).$$

We now determine ω_{X_w} as a T-equivariant sheaf. Since BwB/B is a smooth open subset of X_w, $\omega_{X_w}|_{(BwB/B)}$ is the canonical line bundle. The fiber of ω_{X_w} at the T-fixed point $wB \in BwB/B$ as a T-module is given by the character $|\Delta^- \cap w\Delta^+| = \sum_{\alpha\in\Delta^-\cap w\Delta^+} \alpha = w\rho - \rho$. The fiber of $\mathcal{L}(-\rho)$ at wB has weight $w\rho$ and clearly the fiber of $\mathcal{O}_{X_w}(-\partial X_w)$ at wB has weight 0. Combining the above, we get (a). (Here, we have used the fact that on a reflexive sheaf $\mathcal{S}$ of rank 1 on an irreducible projective T-variety X, there exists at most one T-equivariant structure such that the induced T-module structure on the stalk of $\mathcal{S}$ at a T-fixed point $x_0 \in X$ is trivial.)

The proof of (b) is similar.

By [11, Proposition 3.39], for any $v, w \in W$,

$$\chi\left(X_{v^{-1}}, *\tau^w\right) = \left\langle \mathcal{O}_{X_{v^{-1}}}, *\tau^w\right\rangle = \delta_{v,w}. \tag{3}$$

By the preceding proposition, this implies that $*\tau^w = \xi^{w^{-1}}$, proving the first equality of (c). The second equality of (c) follows from (b) and the definition of ξ^w. By [3, §2] (which holds equivariantly by the same proof), for any closed T-stable Cohen–Macaulay equidimensional subvariety $Y \subset G/B$, we have

$$*[\mathcal{O}_Y] = (-1)^{\operatorname{codim} Y}[\omega_Y] \cdot *[\omega_{G/B}].$$

Part (d) follows by combining (c) with this equation for $Y = X^{w^{-1}}$, using the fact that $\omega_{G/B} \cong \mathcal{L}(-2\rho)$. Part (e) follows by combining (c) and (d). ■

3 A Positivity Conjecture for $K_T(G/P)$

3.1 Positivity in the dual Schubert basis

We make the following conjecture concerning the multiplication in $K_T(G/P)$ in terms of the basis $\{\xi_P^w\}$.

Conjecture 3.1. For any (standard) parabolic subgroup P and any $u, v \in W^P$, express

$$\xi_P^u\, \xi_P^v = \sum_{w\in W^P} p_{u,v}^w(P)\, \xi_P^w, \tag{4}$$

for some (unique) $p_{u,v}^w(P) \in R(T)$. Then,

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\, p_{u,v}^w(P) \in \mathbb{Z}_+[\mathrm{e}^{-\alpha_i} - 1]_{\alpha_i\in\Pi},$$

where the notation $\mathbb{Z}_+[e^{-\alpha_i}-1]_{\alpha_i\in\Pi}$ means polynomials in $\{e^{-\alpha_i}-1\}_{\alpha_i\in\Pi}$ with coefficients in $\mathbb{Z}_+$. □

We will write simply $p^w_{u,v}$ for $p^w_{u,v}(B)$.

Remark 3.2.

(a) By an explicit case by case calculation, we have verified the validity of the above conjecture for $P=B$ and any rank-2 group G. The expression for the product $\xi^u\xi^v$ for G of types A_2 and B_2 is included in the Appendix.

(b) We show the validity of our conjecture when $G=SL_{n+1}$ and P is the standard maximal parabolic subgroup corresponding to the first node (so that $G/P=\mathbb{P}^n$) in Section 6. □

The next proposition gives a relation between structure constants under the inclusion of flag varieties associated to Levi subgroups. In the special case where the flag varieties are projective spaces, this result also follows from our explicit calculation of the structure constants (see Corollary 6.11).

Proposition 3.3. Let G, P, T be as in the above conjecture and let L be the Levi subgroup of P containing T. Let Q be a standard parabolic subgroup of G contained in P and let $Q_L := L\cap Q$ be the corresponding parabolic subgroup of L. Then, for any $u, v, w \in (W_P)^{Q_L}$,

$$p^w_{u,v}(Q_L) = p^w_{u,v}(Q),$$

where $p^w_{u,v}(Q_L)$ are the structure constants for the flag variety L/Q_L. (Observe that $(W_P)_{Q_L}$ can canonically be identified with W_Q and $(W_P)^{Q_L}$ is canonically embedded in W^Q.) □

Proof. Observe that the canonical inclusion $i : L/Q_L \hookrightarrow G/Q$ takes the Schubert variety $X^{Q_L}_w \subset L/Q_L$ isomorphically onto the Schubert variety $X^Q_w \subset G/Q$, for any $w\in (W_P)^{Q_L}$. For $w\in W^Q$, we claim that $i^*(\xi^w_Q)$ equals $\xi^w_{Q_L}$ if $w\in (W_P)^{Q_L}$, and is 0 otherwise. Indeed, for $u\in (W_P)^{Q_L}$,

$$\chi\left(X^{Q_L}_u, i^*(\xi^w_Q)\right) = \chi\left(X^Q_u, \xi^w_Q\right) = \delta_{u,w},$$

proving the claim.

For $u, v \in (W_P)^{Q_L}$, we have

$$\xi^u_{Q_L}\,\xi^v_{Q_L} = \sum_{w\in (W_P)^{Q_L}} p^w_{u,v}(Q_L)\,\xi^w_{Q_L}.$$

On the other hand, since i^* is a ring homomorphism, we have

$$\xi^u_{Q_L}\,\xi^v_{Q_L} = i^*(\xi^u_Q\,\xi^v_Q) = i^*\left(\sum_{w\in W^Q} p^w_{u,v}(Q)\,\xi^w_Q\right) = \sum_{w\in (W_P)^{Q_L}} p^w_{u,v}(Q)\,\xi^w_{Q_L}.$$

Comparing these two expressions, we get the proposition. ■

Lemma 3.4. Let $\pi : G/B \to G/P$ denote the projection. Then,

$$\pi^*(\xi^v_P) = \sum_{u\in vW_P} \xi^u, \qquad \text{for any } v\in W^P.$$

□

Proof. We have $\langle \pi^*(\xi^v_P), [\mathcal{O}_{X_u}]\rangle = \langle \xi^v_P, \pi_*[\mathcal{O}_{X_u}]\rangle$. Further, $\pi_*[\mathcal{O}_{X_u}] = [\mathcal{O}_{\pi(X_u)}]$ by [1, Theorem 3.3.4(a)]. Thus, by Proposition 2.1, $\langle \xi^v_P, \pi_*[\mathcal{O}_{X_u}]\rangle$ is 0 unless $\pi(X_u) = X^P_v$, and this holds if and only if $u\in vW_P$. The lemma follows from this, together with Proposition 2.1 applied to the case of G/B. ■

Unlike Conjecture 3.10 below due to Griffeth–Ram, the validity of Conjecture 3.1 for $P = B$ does not seem to give the validity of the conjecture for an arbitrary (standard) parabolic P. In fact, we have the following proposition relating the structure constants for P and B.

Proposition 3.5. For any $u, v, w \in W^P$,

$$p^w_{u,v}(P) = \sum_{\substack{u'\in uW_P\\ v'\in vW_P}} p^w_{u',v'}(B).$$

□

Proof. Since

$$\xi^u_P\xi^v_P = \sum_{w\in W^P} p^w_{u,v}(P)\xi^w_P,$$

taking π^* and using the above lemma, we get

$$\sum_{\substack{u'\in uW_P\\ v'\in vW_P}} \xi^{u'}\xi^{v'} = \sum_{w\in W^P}\left(p^w_{u,v}(P)\sum_{w'\in wW_P}\xi^{w'}\right),$$

i.e.,

$$\sum_{\theta \in W} \sum_{\substack{u' \in uW_P \\ v' \in vW_P}} p^{\theta}_{u',v'}(B)\xi^{\theta} = \sum_{w \in W^P} \sum_{w' \in wW_P} p^{w}_{u,v}(P)\xi^{w'}.$$

Equating the coefficients from the two sides, we get the proposition. ■

Let D be the diagonal map $G/P \to G/P \times G/P$. This, of course, induces the push-forward map

$$D_* : K_T(G/P) \longrightarrow K_T(G/P) \otimes_{R(T)} K_T(G/P),$$
$$D_*[\mathcal{F}] = \sum_{p \geq 0} (-1)^p [R^p D_* \mathcal{F}],$$

and also the pullback (product) map

$$D^* : K_T(G/P) \otimes_{R(T)} K_T(G/P) \longrightarrow K_T(G/P).$$

Here, we have identified $K_T(G/P \times G/P)$ with $K_T(G/P) \otimes_{R(T)} K_T(G/P)$ (cf. [5, Theorem 5.6.1]). The next proposition gives another description of the coefficients $p^w_{u,v}(P)$.

Proposition 3.6. For any $u, v, w \in W^P$,

$$D_*\big[\mathcal{O}_{X^P_w}\big] = \sum_{u,v \in W^P} p^w_{u,v}(P)\big[\mathcal{O}_{X^P_u}\big] \boxtimes \big[\mathcal{O}_{X^P_v}\big],$$

where $[\mathcal{O}_{X^P_u}] \boxtimes [\mathcal{O}_{X^P_v}] \in K_T(G/P) \otimes_{R(T)} K_T(G/P)$ denotes the external product defined as $p_1^*([\mathcal{O}_{X^P_u}]) \cdot p_2^*([\mathcal{O}_{X^P_v}])$ (p_1, p_2 being the projections $G/P \times G/P \to G/P$ to the first and the second factors, respectively). □

Proof. This follows from functorial properties of K-theory. To see this, for any space Y, write π_Y for the projection from Y to a point. Write $X = G/P$. By definition, $\chi(X, \mathcal{F}) = \pi_{X*}(\mathcal{F})$. By definition of the coefficients $p^w_{u,v}(P)$ and Proposition 2.1,

$$\begin{aligned} p^w_{u,v}(P) &= \pi_{X*}\big(\xi^u_P \xi^v_P \cdot \big[\mathcal{O}_{X^P_w}\big]\big) \\ &= \pi_{X*}\big(D^*\big(\xi^u_P \boxtimes \xi^v_P\big) \cdot \big[\mathcal{O}_{X^P_w}\big]\big) \\ &= (\pi_{X\times X})_*\, D_*\big(D^*\big(\xi^u_P \boxtimes \xi^v_P\big) \cdot \big[\mathcal{O}_{X^P_w}\big]\big) \\ &= (\pi_{X\times X})_*\big(\big(\xi^u_P \boxtimes \xi^v_P\big) \cdot D_*\big[\mathcal{O}_{X^P_w}\big]\big). \end{aligned}$$

Since $\{\xi^u_P \boxtimes \xi^v_P\}$ and $\{[\mathcal{O}_{X^P_u}] \boxtimes [\mathcal{O}_{X^P_v}]\}$ are dual bases of $K_T(X \times X)$, the lemma follows. ■

Remark 3.7. The nonequivariant analog of the preceding proposition holds with the same proof. Combining this with [3, Theorem 1], we see that the structure constants $F(p^w_{u,v}(P))$ for the nonequivariant multiplication in the basis $\{\xi^u_P\}_u$ (cf. equation (4) of Conjecture 3.1; here $F : R(T) \to \mathbb{Z}$ is the forgetful map) satisfy

$$(-1)^{\ell(w)+\ell(u)+\ell(v)} F\left(p^w_{u,v}(P)\right) \in \mathbb{Z}_+.$$

□

For any subset $S \subset \{1,\ldots,\ell\}$ (including $S = \emptyset$), let W_S be the subgroup of W generated by the simple reflections $\{s_i, i \in S\}$. Recall that $Q^+ := \sum_i \mathbb{Z}_+\alpha_i$.

Proposition 3.8. For any $u, w \in W$, and any $S \subset \{1,\ldots,\ell\}$, we have

$$(-1)^{\ell(w)+\ell(u)} \sum_{v \in W_S} p^w_{u,v} \in \sum_{\beta \in Q^+} \mathbb{Z}_+ \mathrm{e}^{-\beta}.$$

In particular,

$$(-1)^{\ell(w)+\ell(u)}\ p^w_{u,1} \in \mathbb{Z}_+[\mathrm{e}^{-\alpha_i} - 1]_{\alpha_i \in \Pi}.$$

□

Proof. As in Proposition 3.6, write

$$D_*[\mathcal{O}_{X_w}] = \sum_{u,v \in W} p^w_{u,v} [\mathcal{O}_{X_u}] \boxtimes [\mathcal{O}_{X_v}].$$

Pairing this with $\xi^u \boxtimes \mathcal{L}(-\rho_S)$, we get

$$\chi\left(X_w, \xi^u \cdot \mathcal{L}(-\rho_S)\right) = \sum_{v \in W} p^w_{u,v}\ \chi(X_v, \mathcal{L}(-\rho_S)), \tag{5}$$

where ρ_i is the ith fundamental weight and $\rho_S := \sum_{i \notin S} \rho_i$.

We claim that, for any $v \notin W_S$,

$$H^i(X_v, \mathcal{L}(-\rho_S)) = 0 \qquad \text{for all } i \geq 0. \tag{6}$$

Let $P = P_S$ be the parabolic subgroup corresponding to the subset S, i.e., the Levi subgroup of P_S containing T has for its simple roots $\{\alpha_i\}_{i \in S}$. Then, the line bundle $\mathcal{L}(-\rho_S)$

is the pullback of a line bundle on G/P. Moreover, by [1, Theorem 3.3.4],

$$H^i(X_v, \mathcal{L}(-\rho_S)) \cong H^i(X_{v'}, \mathcal{L}(-\rho_S)), \tag{7}$$

where v' is the coset representative of minimal length in the coset vW_S. Take s_j, $j \notin S$, such that $v's_j < v'$. Then, the standard projection $\pi : X_{v'} \to X_{v'}^{P_j}$ is a $\mathbb{P}^1$-fibration and $\mathcal{L}(-\rho_S)$ has degree -1 along the fibers of π, where $P_j = P_{\{j\}}$. Hence, $R^i\pi_*\mathcal{L}(-\rho_S) = 0$ for all i; (6) follows from this and the Leray spectral sequence together with (7).

For $v \in W_S$, by (7),

$$H^i(X_v, \mathcal{L}(-\rho_S)) \cong H^i(X_1, \mathcal{L}(-\rho_S)).$$

Thus, for $v \in W_S$,

$$\begin{aligned} H^i(X_v, \mathcal{L}(-\rho_S)) &= 0 \quad \text{if } i > 0 \text{ and} \\ \text{ch } H^0(X_v, \mathcal{L}(-\rho_S)) &= \mathrm{e}^{\rho_S}. \end{aligned}$$

Thus, by (5),

$$\chi(X_w, \xi^u \cdot \mathcal{L}(-\rho_S)) = \left(\sum_{v \in W_S} p_{u,v}^w\right) \mathrm{e}^{\rho_S},$$

i.e.,

$$\sum_{v \in W_S} p_{u,v}^w = \mathrm{e}^{-\rho_S}\, \chi(X_w, \xi^u \cdot \mathcal{L}(-\rho_S)). \tag{8}$$

Now,

$$\chi(X_w, \xi^u \cdot \mathcal{L}(-\rho_S)) = \chi(X_w \cap X^u, \mathcal{L}(-\rho_S)(-X_w \cap \partial X^u)).$$

By [3, Theorem 4], this equals

$$(-1)^{\ell(u)+\ell(w)} w_o\big(* \operatorname{ch}\big(H^0\big(X_{w_o u} \cap X^{w_o w}, \mathcal{L}(\rho_S)\big(- X_{w_o u} \cap \partial X^{w_o w}\big)\big)\big)\big), \tag{9}$$

where w_o is the longest element of W. (Brion's result is stated nonequivariantly, but if we change his duality formula to the following:

$$c_v^w(\lambda) = (-1)^{\ell(v)+\ell(w)} w_o \cdot \big(* c_{w_o w}^{w_o v}(-\lambda)\big),$$

then it remains true T-equivariantly by a similar proof.) By [2, Proposition 1], the restriction map

$$H^0(G/B,\mathcal{L}(\rho_S)) \longrightarrow H^0\big(X_{w_o u}\cap X^{w_o w},\mathcal{L}(\rho_S)\big)$$

is surjective. Also,

$$H^0\big(X_{w_o u}\cap X^{w_o w},\mathcal{L}(\rho_S)\big(-X_{w_o u}\cap \partial X^{w_o w}\big)\big)\subset H^0\big(X_{w_o u}\cap X^{w_o w},\mathcal{L}(\rho_S)\big).$$

Since $H^0(G/B,\mathcal{L}(\rho_S))$ is the irreducible G-module with the highest weight $-w_o\rho_S$, we see that

$$\mathrm{e}^{-\rho_S}\, w_o\big(*\,\mathrm{ch}\big(H^0\big(X_{w_o u}\cap X^{w_o w},\mathcal{L}(\rho_S)\big(-X_{w_o u}\cap \partial X^{w_o w}\big)\big)\big)\big)\in \sum_{\beta\in Q^+}\mathbb{Z}_+\,\mathrm{e}^{-\beta}. \tag{10}$$

Combining (8)–(10), we get the proposition. ■

Remark 3.9. Comparing the expression obtained in the above proof for $p^w_{u,s_i}+p^w_{u,1}$ and $p^w_{u,1}$ (for any simple reflection s_i), it can be shown that

$$(-1)^{\ell(u)+\ell(w)+1}\, p^w_{u,s_i}\in \mathbb{Z}_+\,[\mathrm{e}^{-\alpha_i}-1]_{\alpha_i\in\Pi}.$$

□

3.2 Positivity in the structure sheaf basis

We recall below the conjecture of Griffeth and Ram on the nonnegativity of the product in the structure sheaf basis [9, Conjecture 5.9]. They verified their conjecture for rank-2 groups by an explicit case by case calculation. In this section, we prove that the validity of their conjecture for $P=B$ implies its validity for every (standard) parabolic subgroup P, and give an equivalent formulation of their conjecture in terms of dualizing sheaves.

Conjecture 3.10. For any standard parabolic subgroup P and $u,v\in W^P$, express

$$\big[\mathcal{O}_{X^u_P}\big]\big[\mathcal{O}_{X^v_P}\big]=\sum_{w\in W^P} c^w_{u,v}(P)\big[\mathcal{O}_{X^w_P}\big]\in K_T(G/P), \tag{11}$$

for some (unique) $c^w_{u,v}(P)\in R(T)$.

Then,

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}c^w_{u,v}(P)\in \mathbb{Z}_+[\mathrm{e}^{-\alpha_i}-1]_{\alpha_i\in\Pi}.$$

□

Remark 3.11. Express

$$[\mathcal{O}_{X_u^P}][\mathcal{O}_{X_v^P}] = \sum_{w \in W^P} b_{u,v}^w(P)[\mathcal{O}_{X_w^P}].$$

Then, an equivalent formulation of the above conjecture asserts that

$$(-1)^{\dim(G/P)+\ell(u)+\ell(v)+\ell(w)} b_{u,v}^w(P) \in \mathbb{Z}_+[\mathrm{e}^{\alpha_i} - 1]_{\alpha_i \in \Pi}.$$

In fact,

$$c_{u,v}^w(P) = w_o\left(b^{w_o w w_o^P}_{w_o u w_o^P, w_o v w_o^P}(P)\right),$$

where w_o (respectively w_o^P) is the longest element of W (respectively W_P). This can be seen by the proof of Lemma 7.5. Moreover, an argument similar to the proof of Proposition 3.6 shows that the structure constants $b_{u,v}^w(P)$ are described by the equation:

$$D_*(\xi_P^w) = \sum_{u,v \in W^P} b_{u,v}^w(P)\xi_P^u \boxtimes \xi_P^v.$$

□

Remark 3.12. We note that Griffeth and Ram state their conjecture in terms of the structure constants with respect to the basis $[\mathcal{O}_{X_w}]$. As stated, their conjecture involves $\mathrm{e}^{-\alpha_i} - 1$ rather than $\mathrm{e}^{\alpha_i} - 1$. However, their conventions are different from ours. Their formulas for the products $[\mathcal{O}_{X_1}]^2$ in rank-2 indicate that for them X_1 is the point in X where the tangent space to X is spanned by positive root vectors, which is opposite to our conventions. □

Lemma 3.13. We have

$$c_{u,v}^w(P) = c_{u,v}^w(B), \quad \text{for any } u, v, w \in W^P. \tag{12}$$

Hence, the validity of the above Conjecture 3.10 for $P = B$ implies its validity for every (standard) parabolic subgroup P. □

Proof. Let $\pi : G/B \to G/P$ be the standard projection. Since π is a T-equivariant smooth morphism, for any T-stable closed subvariety $Z \subseteq G/P$,

$$\pi^*[\mathcal{O}_Z] = [\mathcal{O}_{\pi^{-1}(Z)}].$$

Thus,

$$\pi^*\big[\mathcal{O}_{X_P^w}\big] = \big[\mathcal{O}_{\pi^{-1}(X_P^w)}\big] \\ = [\mathcal{O}_{X^w}], \tag{13}$$

since $B^- wP/P = w_o B w_o w P/P$ and $w_o w$ is the longest element in the W_P-orbit $w_o w W_P$.

Since π^* is a ring homomorphism, (12) follows from (13). ■

As a consequence of this proposition, we will simply write $c_{u,v}^w$ for $c_{u,v}^w(P)$.

The following proposition provides an equivalent formulation of Conjecture 3.10 in terms of dualizing sheaves.

Proposition 3.14. For any standard parabolic subgroup P and any $u, v \in W^P$, express

$$\big[\omega_{X_P^u}\big] \cdot \big[\omega_{X_P^v}\big] = \sum_{w \in W^P} d_{u,v}^w(P)\big[\omega_{X_P^w}\big][\omega_{G/P}], \tag{14}$$

for some (unique) $d_{u,v}^w(P) \in R(T)$. Then,

$$d_{u,v}^w(P) = (-1)^{\ell(u)+\ell(v)+\ell(w)} * \big(c_{u,v}^w\big). \tag{15}$$

In particular, Conjecture 3.10 is equivalent to the conjecture that $d_{u,v}^w(P) \in \mathbb{Z}_+[\mathrm{e}^{\alpha_i} - 1]_{\alpha_i \in \Pi}$.

Moreover, by Lemma 3.13,

$$d_{u,v}^w(P) = d_{u,v}^w(B), \quad \text{for any } u, v, w \in W^P.$$

□

Proof. By [3, §2],

$$* \big[\mathcal{O}_{X_P^v}\big] = (-1)^{\ell(v)}\big[\omega_{X_P^v}\big] \cdot *[\omega_{G/P}]. \tag{16}$$

(Even though Brion proves this result nonequivariantly, the same proof works T-equivariantly.) Multiply equation (14) by $*[\omega_{G/P}]^2$ to get

$$\big[\omega_{X_P^u}\big](*[\omega_{G/P}])\big[\omega_{X_P^v}\big] * [\omega_{G/P}] = \sum_{w \in W^P} d_{u,v}^w(P)\big[\omega_{X_P^w}\big] * [\omega_{G/P}].$$

By (16), the above equation reduces to

$$\left(* \left[\mathcal{O}_{X_P^u}\right]\right) \cdot *\left[\mathcal{O}_{X_P^v}\right] = \sum_{w \in W^P} (-1)^{\ell(u)+\ell(v)+\ell(w)} d_{u,v}^w(P) * \left[\mathcal{O}_{X_P^w}\right].$$

Comparing this with the identity (11) of Conjecture 3.10, we get

$$d_{u,v}^w(P) = (-1)^{\ell(u)+\ell(v)+\ell(w)} * \left(c_{u,v}^w\right).$$

This proves (15). ■

4 Relations Between Structure Constants

In this section, we restrict to the case $P = B$, and prove two relations (Propositions 4.1 and 4.3) between the structure constants in the structure sheaf basis and the structure constants in the dual basis. However, we do not know how to use these relations to relate the two positivity Conjectures 3.1 and 3.10.

Write

$$[\mathcal{L}(-\rho)][\mathcal{O}_{X^\theta}] = \sum_{w \in W} d_w^\theta [\mathcal{O}_{X^w}],$$

for some (unique) $d_w^\theta \in R(T)$. For some explicit formulas for d_w^θ, see [3] and [13]–[15].

The following proposition gives a relation between the structure constants $c_{u,v}^w$ and $p_{u,v}^w$.

Proposition 4.1. For any $u, v, w \in W$,

$$c_{u,v}^w = (-1)^{\ell(u)+\ell(v)} \sum_{\theta \in W} (-1)^{\ell(\theta)} \, e^{-\rho} \, d_w^\theta * p_{u,v}^\theta.$$

□

Proof. By Proposition 2.2(e),

$$*\xi^w = (-1)^{\ell(w)} \, e^{\rho} \, [\mathcal{L}(\rho)][\mathcal{O}_{X^w}].$$

We have

$$(*\xi^u)(*\xi^v) = \sum_{\theta\in W} \left(*\, p^{\theta}_{u,v}\right)(*\xi^{\theta}),$$

and hence

$$\begin{aligned}[\mathcal{O}_{X^u}][\mathcal{O}_{X^v}] &= (-1)^{\ell(u)+\ell(v)} \sum_{\theta\in W} (-1)^{\ell(\theta)} \left(*\, p^{\theta}_{u,v}\right) \mathrm{e}^{-\rho}\, [\mathcal{L}(-\rho)][\mathcal{O}_{X^{\theta}}] \\ &= (-1)^{\ell(u)+\ell(v)} \sum_{\theta,w\in W} (-1)^{\ell(\theta)} \left(*\, p^{\theta}_{u,v}\right) \mathrm{e}^{-\rho} d^{\theta}_{w} [\mathcal{O}_{X^w}].\end{aligned}$$

From this the proposition follows. ■

Before stating the second relation between structure constants, we compare the bases $\{\xi^v\}_{v\in W}$ and $\{[\mathcal{O}_{X^v}]\}_{v\in W}$ of $K_T(G/B)$. Let

$$\mu(v,w) = \begin{cases} (-1)^{\ell(v)+\ell(w)}, & \text{if } v \le w \\ 0, & \text{otherwise} \end{cases}$$

denote the Möbius function of the Weyl group W.

Lemma 4.2. For any $v \in W$, write

$$[\mathcal{O}_{X^v}] = \sum_{w\in W} e_{v,w}\, \xi^w.$$

Then,

$$\begin{aligned} e_{v,w} &= 1, \quad \text{if } v \le w \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Thus,

$$\xi^v = \sum_{w} \mu(v,w)[\mathcal{O}_{X^w}].$$

□

Proof. By Proposition 2.1,

$$e_{v,w} = \left\langle [\mathcal{O}_{X_w}], [\mathcal{O}_{X^v}] \right\rangle = \chi\left(G/B, [\mathcal{O}_{X_w}] \cdot [\mathcal{O}_{X^v}]\right).$$

Since $X_w \cap X^v$ is a proper intersection, by [3, Lemma 1], $[\mathcal{O}_{X_w}] \cdot [\mathcal{O}_{X^v}] = [\mathcal{O}_{X_w \cap X^v}]$. Thus, $e_{v,w} = \chi(X_w \cap X^v, \mathcal{O}_{X_w \cap X^v})$. By [2, Proposition 1] (cf. proof of Proposition 2.1),

$$\chi\left(X_w \cap X^v, \mathcal{O}_{X_w \cap X^v}\right) = 1 \text{ (or 0)}$$

according as $X_w \cap X^v$ is nonempty (or empty), i.e.,

$$\begin{aligned} \chi\left(X_w \cap X^v, \mathcal{O}_{X_w \cap X^v}\right) &= 1, \quad \text{if } v \leq w \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

This proves the first part of the lemma.

Define the matrix

$$E = (e_{v,w})_{v,w \in W}.$$

Then, by [6, §3], E^{-1} is the Möbius function, i.e.,

$$(E^{-1})_{v,w} = \mu(v, w).$$

From this, the second part of the lemma follows. ■

We can now state the second relation between the structure constants.

Proposition 4.3. For any $u, v, w \in W$,

$$c^w_{u,v} = (-1)^{\ell(w)} \sum_{\substack{u \leq y \\ v \leq z \\ \theta \leq w}} (-1)^{\ell(\theta)} p^\theta_{y,z}. \tag{17}$$

Similarly,

$$p^w_{u,v} = (-1)^{\ell(u)+\ell(v)} \sum_{\substack{u \leq y \\ v \leq z \\ \theta \leq w}} (-1)^{\ell(y)+\ell(z)} c^\theta_{y,z}. \tag{18}$$

□

Proof. By Lemma 4.2,

$$\begin{aligned}[\mathcal{O}_{X^u}]\,[\mathcal{O}_{X^v}] &= \left(\sum_y e_{u,y}\,\xi^y\right)\cdot\left(\sum_z e_{v,z}\,\xi^z\right)\\ &= \sum_{y,z} e_{u,y}e_{v,z}\,\xi^y\xi^z\\ &= \sum_{y,z,\theta} e_{u,y}e_{v,z}\,p^{\theta}_{y,z}\xi^{\theta}\\ &= \sum_{y,z,\theta,w} e_{u,y}e_{v,z}\,p^{\theta}_{y,z}\,\mu(\theta,w)[\mathcal{O}_{X^w}]\\ &= \sum_{\substack{u\le y\\ v\le z\\ \theta\le w}} p^{\theta}_{y,z}\,(-1)^{\ell(\theta)+\ell(w)}[\mathcal{O}_{X^w}].\end{aligned}$$

Thus, equating the coefficients in the $\{[\mathcal{O}_{X^w}]\}_w$ basis, we get

$$c^w_{u,v} = (-1)^{\ell(w)}\sum_{\substack{u\le y\\ v\le z\\ \theta\le w}} (-1)^{\ell(\theta)}\,p^{\theta}_{y,z}.$$

The proof of (18) is similar. ■

5 Multiplicative Structure Constants Lie in $\mathbb{Z}[\mathrm{e}^{-\alpha_i}-1]$

Let Z denote the center of G, and let $T' = T/Z$. The map $T \to T'$ induces an injection $R(T') \hookrightarrow R(T)$ whose image is the subring $\mathbb{Z}[\mathrm{e}^{\beta}]_{\beta\in\Delta}$ of $R(T)$. Of course, $\mathbb{Z}[\mathrm{e}^{\beta}]_{\beta\in\Delta} = \mathbb{Z}[\mathrm{e}^{\beta}-1]_{\beta\in\Delta}$; writing the ring in this way emphasizes the relationship with the positivity conjectures.

The main result of this section is the following theorem concerning the structure constants in the dual structure sheaf basis, in the case $P = B$.

Theorem 5.1. With the notation as in Conjecture 3.1, for any $u, v, w \in W$, we have

$$p^w_{u,v} \in \mathbb{Z}[\mathrm{e}^{-\alpha_i}-1]_{\alpha_i\in\Pi}.$$

Moreover, $p^w_{u,v} = 0$ unless $u, v \le w$. □

Proof. We first show that $p^w_{u,v} \in R(T')$. Because Z acts trivially on $X = G/B$, the action of T on G/B factors through the action of $T' = T/Z$. Therefore, there is a canonical map

$K_{T'}(X) \to K_T(X)$ compatible with the map $R(T') \to R(T)$. By the cellular fibration lemma [5, Lemma 5.5.1], $K_{T'}(X)$ is free over $R(T')$ and the classes of $\mathcal{O}_{X_w}$ in $K_{T'}(X)$ form a basis. Since the class of $\mathcal{O}_{X_w}$ in $K_{T'}(X)$ maps to the class of the same sheaf in $K_T(X)$, and the map $K_{T'}(X) \to K_T(X)$ is a ring homomorphism, the structure constants $b^w_{u,v}$ of the multiplication in $K_T(X)$ with respect to this basis must be the images of the corresponding structure constants in $K_{T'}(X)$, and hence must lie in $R(T')$. Thus, by (18) and Remark 3.11, $p^w_{u,v}$ must lie in $R(T')$ as well.

By Proposition 2.2(c),

$$\xi^w = *\tau^{w^{-1}}, \qquad \text{for any } w \in W. \tag{19}$$

By [11, Proposition 2.22(h)], $p^w_{u,v} = 0$ unless $u, v \leq w$. We will prove the proposition by induction on $\ell(w)$. The proposition is true for $w = 1$ since $p^1_{u,v} = 0$ unless $u = v = 1$. Moreover, $p^1_{1,1} = 1$, since by [11, Proposition 2.22(b)],

$$\tau^1\tau^1 = \tau^1 + \sum_{w \neq 1} d^w\tau^w,$$

for some $d_w \in R(T)$.

Fix $u, v \in W$ and assume by induction that $p^\theta_{u,v} \in \mathbb{Z}[e^{-\alpha_i} - 1]$ for all $\theta < w$. Write

$$\tau^{u^{-1}} \cdot \tau^{v^{-1}} = \sum_{\theta < w} a^\theta_{u,v}\tau^{\theta^{-1}} + a^w_{u,v}\tau^{w^{-1}} + \sum_{\delta \not\leq w} a^\delta_{u,v}\tau^{\delta^{-1}}.$$

By (19),

$$a^w_{u,v} = *p^w_{u,v}, \qquad \text{for any } u, v, w \in W. \tag{20}$$

Now,

$$\tau^{u^{-1}}(w^{-1})\tau^{v^{-1}}(w^{-1}) = \sum_{\theta < w} a^\theta_{u,v}\tau^{\theta^{-1}}(w^{-1}) + a^w_{u,v}\tau^{w^{-1}}(w^{-1}), \tag{21}$$

as $\tau^{\delta^{-1}}(w^{-1}) = 0$ for $\delta \not\leq w$ by [11, Proposition 2.22(b)]. By [8] or [18],

$$\tau^{\theta^{-1}}(w^{-1}) \in \mathbb{Z}[e^{\alpha_i} - 1]_{\alpha_i \in \Pi}. \tag{22}$$

Moreover, by [11, Proposition 2.22(b)],

$$\tau^{w^{-1}}(w^{-1}) = \prod_{\nu \in w\Delta^- \cap \Delta^+} (1 - e^{\nu}). \tag{23}$$

Let $x_i := e^{\alpha_i}$, $1 \le i \le \ell$, where $\{\alpha_1, \ldots, \alpha_\ell\}$ are the simple roots. Then, $R := \mathbb{Z}[e^{\alpha_i}]_{\alpha_i \in \Pi}$ is the polynomial ring $\mathbb{Z}[x_1, \ldots, x_\ell]$, and

$$R \subset R(T') = \mathbb{Z}\left[x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}\right].$$

To prove the theorem, it suffices by (20) to show that $a_{u,v}^w \in R$. By (20)–(22), and induction,

$$a_{u,v}^w \, \tau^{w^{-1}}(w^{-1}) \in R. \tag{24}$$

Moreover, by (23), $\tau^{w^{-1}}(w^{-1})$ is in R and does not vanish at 0 (in the $\{x_i\}$ coordinates), and we have proved earlier that $a_{u,v}^w$ is in $R(T')$, so $a_{u,v}^w$ has no pole at 0. Hence,

$$a_{u,v}^w \in \mathbb{C}[x_1, \ldots, x_\ell].$$

We next show that the coefficient of each monomial in $a_{u,v}^w$ must be an integer, so that $a_{u,v}^w \in R$. Write

$$a_{u,v}^w = \sum_{\underline{d} \in \mathbb{Z}_+^\ell} c_{\underline{d}} \underline{x}^{\underline{d}}, \quad \text{for} \quad c_{\underline{d}} \in \mathbb{C},$$

where $\underline{x}^{\underline{d}} := x_1^{d_1} \cdots x_\ell^{d_\ell}$ for $\underline{d} = (d_1, \ldots, d_\ell)$. Choose, if possible, $c_{\underline{d}^o} \notin \mathbb{Z}$ so that $|\underline{d}^o| := d_1^o + \cdots + d_\ell^o$ is minimum with this property. Then, (24) implies that

$$\left(\sum_{|\underline{d}| \ge |\underline{d}^o|} c_{\underline{d}} \, \underline{x}^{\underline{d}} \right) \tau^{w^{-1}}(w^{-1}) \in R,$$

which is a contradiction, since the constant term of $\tau^{w^{-1}}(w^{-1})$ is 1. Hence, $a_{u,v}^w \in R$, as desired. ■

Corollary 5.2. For any $u, v, w \in W$, we have that $c_{u,v}^w \in \mathbb{Z}[e^{-\alpha_i} - 1]_{\alpha_i \in \Pi}$. □

Proof. This follows by combining Theorem 5.1 and Proposition 4.3. Alternatively, it can be deduced by an argument similar to the proof of Theorem 5.1, using the formula of [8] for the pullback of elements of the structure sheaf basis to T-fixed points. ■

6 Positivity in Equivariant K-Theory of $\mathbb{P}^n$

In this section, we prove an explicit formula (Theorem 6.5) for the structure constants in the dual structure sheaf basis in case $G/P = \mathbb{P}^n$. This explicit formula yields a recurrence relation (Theorem 6.6) that implies Conjecture 3.1 in this case. We give the analogous results for the structure sheaf basis. The proof of Theorem 6.5 given here was suggested by a referee and differs from our original proof. This section can be read independently of the previous sections, except for references to a few results.

6.1 Preliminary results on $K_T(\mathbb{P}^n)$

Let $X = \mathbb{P}^n$ with projective coordinates $[x_1, \ldots, x_{n+1}]$. Thus, $X = G/P$, where $G = SL_{n+1}$ and P is the stabilizer of $[1, 0, \ldots, 0]$. Let $T = \{(t_1, \ldots, t_{n+1}) \in (\mathbb{C}^*)^{n+1} | \prod t_i = 1\}$ be the maximal torus of G acting on X in the obvious way. Let q_i denote the point in X given by zero in each coordinate except the ith coordinate; then $X^T = \{q_1, \ldots, q_{n+1}\}$. Let B denote the set of upper triangular matrices in G. Let $\mathrm{e}^{\varepsilon_i} \in \hat{T}$ be defined by $\mathrm{e}^{\varepsilon_i}(t_1, \ldots, t_{n+1}) = t_i$, where $\hat{T}$ is the group of characters of T. Written additively, we denote $\mathrm{e}^{\varepsilon_i}$ by ε_i itself. Then, the set of positive roots is $\Delta^+ = \{\varepsilon_i - \varepsilon_j | 1 \leq i < j \leq n+1\}$. Let $\chi_i = \varepsilon_1 + \cdots + \varepsilon_i, i = 1, \ldots, n+1$; for $i \leq n$, these are the fundamental (dominant) weights. The elements of W^P can be identified with the set of integers $[n] := \{0, 1, \ldots, n\}$. In this section, we deviate from our convention in the rest of the paper and, for any $u \in [n]$, simply write X_u, X^u, ξ^u for X_u^P, X_P^u and ξ_P^u, respectively. Then, $X_u = \{[x_1, \ldots, x_{u+1}, 0, \ldots, 0]\}$ and $X^u = \{[0, \ldots, 0, x_{u+1}, \ldots, x_{n+1}]\}$. Note that $\dim X_u = \operatorname{codim} X^u = u$, and the intersections $X_u \cap X^u$ are transverse. We will write simply $p_{u,v}^w$ for the structure constants with respect to the basis $\{\xi^u\}$ of $K_T(\mathbb{P}^n)$, and $b_{u,v}^w$ for the structure constants with respect to the basis $\{[\mathcal{O}_{X_u}]\}$.

For any $n \in \mathbb{Z}$, the character $\mathrm{e}^{n\varepsilon_1}$ of T extends to a character of P. As earlier, let $\mathbb{C}_{n\varepsilon_1}$ denote the corresponding P-module and let $\mathcal{L}(n\varepsilon_1)$ denote the line bundle $G \times_P \mathbb{C}_{-n\varepsilon_1}$ on $X = G/P = \mathbb{P}^n$.

Lemma 6.1. For any $n \in \mathbb{Z}$, $\mathcal{L}(n\varepsilon_1) \cong \mathcal{O}_X(n)$ as G-equivariant line bundles, where $\mathcal{O}_X(1)$ denotes the dual of the tautological bundle. □

Proof. Both $\mathcal{L}(n\varepsilon_1)$ and $\mathcal{O}_X(n)$ are sheaves of sections of G-equivariant line bundles. Hence, the line bundles are determined by the character of P on the fiber over the P-fixed point [1,0, . . . ,0]. Since P acts by the character $e^{-n\varepsilon_1}$ on each line bundle, the line bundles are isomorphic. ∎

Let $Y_i \subset X$ be defined by the equation $x_i = 0$.

Lemma 6.2. $[\mathcal{O}_{Y_i}] = 1 - e^{-\varepsilon_i}[\mathcal{L}(-\varepsilon_1)]$ in $K_T(X)$. □

Proof. By Lemma 6.1, $\mathcal{L}(-\varepsilon_1) = \mathcal{O}_X(-1)$. Let $\mathcal{I}_{Y_i}$ denote the ideal sheaf of Y_i. Since $[\mathcal{O}_{Y_i}] = 1 - [\mathcal{I}_{Y_i}]$, it suffices to show that $\mathcal{I}_{Y_i} \cong e^{-\varepsilon_i}\,\mathcal{O}_X(-1)$ as T-equivariant coherent sheaves on X. On the open set $U_j : x_j \neq 0$, we have affine coordinates $\frac{x_k}{x_j}$ $(k \neq j)$. Now, $\mathcal{I}_{Y_i}(U_j)$ is generated by the section $\sigma_j = \frac{x_i}{x_j}$, which transforms under T by the weight $\varepsilon_j - \varepsilon_i$. Similarly, $\mathcal{O}_X(-1)(U_j)$ is generated by the section

$$\tau_j : [x_1, \ldots, x_{n+1}] \mapsto \left(\frac{x_1}{x_j}, \frac{x_2}{x_j}, \ldots, \frac{x_{n+1}}{x_j}\right),$$

which transforms under T by the weight ε_j. Since

$$\frac{\sigma_k}{\sigma_j} = \frac{\tau_k}{\tau_j} = \frac{x_j}{x_k},$$

it follows that the map $\mathcal{O}_X(-1) \otimes e^{-\varepsilon_i} \to \mathcal{I}_{Y_i}$ defined on U_j by $\tau_j \otimes 1 \mapsto \sigma_j$ is a T-equivariant sheaf isomorphism. ∎

Remark 6.3. More generally, the following is true. Let Y be any T-scheme and let $\mathcal{L}$ be a T-equivariant line bundle on Y. Given a section σ of weight λ and zero scheme $Z(\sigma)$, we have

$$[\mathcal{O}_{Z(\sigma)}] = 1 - e^{\lambda}[\mathcal{L}^*].$$

See Proposition 7.3 for a related result. □

Corollary 6.4.

$$\xi^v = e^{-\varepsilon_{v+1}}[\mathcal{L}(-\varepsilon_1)] \prod_{i=1}^{v} (1 - e^{-\varepsilon_i}\,[\mathcal{L}(-\varepsilon_1)]) \qquad (0 \leq v < n),$$

$$\xi^n = \prod_{i=1}^{n} (1 - e^{-\varepsilon_i}[\mathcal{L}(-\varepsilon_1)]).$$

□

Proof. $\xi^v = [\mathcal{O}_{X^v}(-\partial X^v)] = [\mathcal{O}_{X^v}] - [\mathcal{O}_{X^{v+1}}]$. For $1 \le v \le n$, since X^v is the transverse intersection of $Y_1, \ldots, Y_v$, we have by Lemma 6.2,

$$[\mathcal{O}_{X^v}] = \prod_{i=1}^{v} [\mathcal{O}_{Y_i}] = \prod_{i=1}^{v} (1 - \mathrm{e}^{-\varepsilon_i} [\mathcal{L}(-\varepsilon_1)]).$$

(For $v = 0$, this formula is interpreted as saying that $[\mathcal{O}_{X^0}] = 1$, which is true since $X^0 = X$.) A similar equation holds for $[\mathcal{O}_{X^{v+1}}]$ (with $[\mathcal{O}_{X^{n+1}}] = 0$, as X^{n+1} is empty). Subtracting the two formulas gives the result. ■

6.2 Structure constants with respect to the dual structure sheaf basis for $K_T(\mathbb{P}^n)$

Write $[\sum_i a_i t^i]_p = a_p$ and $[\sum_{i,j} b_{i,j} s^i t^j]_{p,q} = b_{p,q}$. We have the following explicit formula for the structure constants with respect to the dual structure sheaf basis.

Theorem 6.5. For any $0 \le u, v, w \le n$,

$$(-1)^{u+v+w} p_{u,v}^{w} = \mathrm{e}^{\chi_{w+1} - \chi_{u+1} - \chi_{v+1}} \left[\frac{\left(\prod_{i=1}^{u} (1 - t\mathrm{e}^{\varepsilon_i})\right) \left(\prod_{i=1}^{v} (1 - t\mathrm{e}^{\varepsilon_i})\right)}{\prod_{i=1}^{w+1} (1 - t\mathrm{e}^{\varepsilon_i})} \right]_{u+v-w+1}. \tag{25}$$

□

In proving this theorem, we will make use of the following recurrence relation, which is of interest in its own right. Let $\tilde{p}_{u,v}^{w}$ denote the right-hand side of (25) if $0 \le u, v, w \le n$, and 0 otherwise. (Of course, Theorem 6.5 implies that $\tilde{p}_{u,v}^{w} = (-1)^{u+v+w} p_{u,v}^{w}$, but we are not assuming this.) Let $\tilde{p}_{u,v}^{w}(\mu_1, \ldots, \mu_{n+1})$ denote the element of $R(T)$ obtained by replacing ε_i by μ_i in the expression for $\tilde{p}_{u,v}^{w}$, for $i = 1, \ldots, n+1$.

Theorem 6.6. If $n \ge v \ge 1$ and $0 \le u \le w \le n$, then

$$\tilde{p}_{u,v}^{w} = (\mathrm{e}^{\varepsilon_{u+1} - \varepsilon_1} - 1)\, \tilde{p}_{u-1,v-1}^{w-1}(\varepsilon_2, \ldots, \varepsilon_{n+1}) + \mathrm{e}^{\varepsilon_{u+2} - \varepsilon_1}\, \tilde{p}_{u,v-1}^{w-1}(\varepsilon_2, \ldots, \varepsilon_{n+1}).$$

□

Proof. Let

$$A = \frac{\left(\prod_{i=1}^{u} (1 - t\mathrm{e}^{\varepsilon_i})\right) \left(\prod_{i=1}^{v} (1 - t\mathrm{e}^{\varepsilon_i})\right)}{\prod_{i=1}^{w+1} (1 - t\mathrm{e}^{\varepsilon_i})}.$$

We can cancel common factors from the numerator and denominator to write

$$A = \frac{\prod_{i=1}^{v}(1 - t\mathrm{e}^{\varepsilon_i})}{\prod_{j=u+1}^{w+1}(1 - t\mathrm{e}^{\varepsilon_j})}.$$

Let

$$B = \frac{(1 - t\mathrm{e}^{\varepsilon_{u+1}})\prod_{i=2}^{v}(1 - t\mathrm{e}^{\varepsilon_i})}{\prod_{j=u+1}^{w+1}(1 - t\mathrm{e}^{\varepsilon_j})}.$$

By definition,

$$\begin{aligned}\tilde{p}_{u,v}^{w} &= \mathrm{e}^{\chi_{w+1}-\chi_{u+1}-\chi_{v+1}}[A]_{u+v+1-w}\\ &= \mathrm{e}^{\chi_{w+1}-\chi_{u+1}-\chi_{v+1}}[A - B + B]_{u+v+1-w}.\end{aligned}$$

We have

$$\mathrm{e}^{\chi_{w+1}-\chi_{u+1}-\chi_{v+1}}[A - B]_{u+v+1-w} = \mathrm{e}^{\chi_{w+1}-\chi_{u+1}-\chi_{v+1}}\left[\frac{t(\mathrm{e}^{\varepsilon_{u+1}} - \mathrm{e}^{\varepsilon_1})\prod_{i=2}^{v}(1 - t\mathrm{e}^{\varepsilon_i})}{\prod_{j=u+1}^{w+1}(1 - t\mathrm{e}^{\varepsilon_j})}\right]_{u+v+1-w}.$$

We can get rid of the factor t in the numerator by taking the part in degree $u + v - w$. Since $\mathrm{e}^{\varepsilon_{u+1}} - \mathrm{e}^{\varepsilon_1} = \mathrm{e}^{\varepsilon_1}(\mathrm{e}^{\varepsilon_{u+1}-\varepsilon_1} - 1)$, we see that the above expression equals

$$(\mathrm{e}^{\varepsilon_{u+1}-\varepsilon_1} - 1)\,\mathrm{e}^{\chi_{w+1}+\varepsilon_1-\chi_{u+1}-\chi_{v+1}}\left[\frac{\prod_{i=2}^{v}(1 - t\mathrm{e}^{\varepsilon_i})}{\prod_{j=u+1}^{w+1}(1 - t\mathrm{e}^{\varepsilon_j})}\right]_{u+v-w}.$$

Using the definition $\chi_k = \varepsilon_1 + \cdots + \varepsilon_k$, we see that the above expression equals

$$(\mathrm{e}^{\varepsilon_{u+1}-\varepsilon_1} - 1)\,\tilde{p}_{u-1,v-1}^{w-1}(\varepsilon_2, \ldots, \varepsilon_{n+1}).$$

Next, we have

$$\mathrm{e}^{\chi_{w+1}-\chi_{u+1}-\chi_{v+1}}[B]_{u+v+1-w} = \mathrm{e}^{\chi_{w+1}-\chi_{u+1}-\chi_{v+1}}\left[\frac{\prod_{i=2}^{v}(1 - t\mathrm{e}^{\varepsilon_i})}{\prod_{j=u+2}^{w+1}(1 - t\mathrm{e}^{\varepsilon_j})}\right]_{u+v+1-w}. \tag{26}$$

Now,

$$\begin{aligned}\chi_{w+1} - \chi_{u+1} - \chi_{v+1} &= (\varepsilon_1 + \cdots + \varepsilon_{w+1}) - (\varepsilon_1 + \cdots + \varepsilon_{u+1}) - (\varepsilon_1 + \cdots + \varepsilon_{v+1})\\ &= (\varepsilon_2 + \cdots + \varepsilon_{w+1}) - (\varepsilon_2 + \cdots + \varepsilon_{u+2}) - (\varepsilon_2 + \cdots + \varepsilon_{v+1}) + (\varepsilon_{u+2} - \varepsilon_1).\end{aligned}$$

Using this, we see that the expression (26) equals

$$e^{\varepsilon_{u+2}-\varepsilon_1}\ \tilde{p}^{w-1}_{u,v-1}(\varepsilon_2,\ldots,\varepsilon_{n+1}).$$

This proves the theorem. ∎

Proof of Theorem 6.5. We can assume that $u, v \le w$, since for either u or $v > w$, both the sides of equation (25) are zero. (The right side is zero for either u or $v > w$ by the proof of Corollary 6.9, and the vanishing of the left side in this case follows by combining Proposition 3.5 and Theorem 5.1.) We will first assume that $v < n$ (we separate the case $v = n$ because the formula for ξ^v is different for $v = n$ and $v < n$). Now, by Proposition 2.1,

$$p^w_{u,v} = \langle [\mathcal{O}_{X_w}], \xi^u\xi^v \rangle = \pi_{X*}\big(\nu_*[\mathcal{O}_{X_w}]\cdot \xi^u\xi^v\big), \tag{27}$$

where $\nu : X_w \hookrightarrow X$ is the inclusion, and for any space Y, π_Y is the map of Y to a point. Equation (27) implies that

$$p^w_{u,v} = \pi_{X*}\nu_*\big([\mathcal{O}_{X_w}]\cdot \nu^*(\xi^u\xi^v)\big) = \pi_{X_w*}\big([\mathcal{O}_{X_w}]\cdot \nu^*(\xi^u\xi^v)\big) = \pi_{X_w*}(\nu^*(\xi^u\xi^v)),$$

where we have used the projection formula and the fact that $[\mathcal{O}_{X_w}]$ is the identity in $K_T(X_w)$. For any $i \le w+1$, let ν_i denote the inclusion of the T-fixed point q_i in X_w. The cotangent space $T^*_{q_i}X_w$ has T-weights $-\varepsilon_j + \varepsilon_i$ for $j \le w+1$, $j \ne i$. Write $\lambda_{-1}(T^*_{q_i}X_w) = \prod_j (1 - e^{-\varepsilon_j+\varepsilon_i})$, where the product is over $j \le w+1$, $j \ne i$. The localization theorem in equivariant K-theory implies that if we localize $K_T(X_w)$ by inverting all the elements $\lambda_{-1}(T^*_{q_i}X_w) \in R(T)$, we have

$$\nu^*(\xi^u\xi^v) = \sum_{i=1}^{w+1} \nu_{i*}\left(\frac{\nu_i^*\nu^*(\xi^u\xi^v)}{\lambda_{-1}\big(T^*_{q_i}X_w\big)}\right).$$

As $\pi_{X_w} \circ \nu_i$ is the identity map of a point, we obtain

$$\pi_{X_w*}(\nu^*(\xi^u\xi^v)) = \sum_{i=1}^{w+1} \frac{\nu_i^*\nu^*(\xi^u\xi^v)}{\lambda_{-1}\big(T^*_{q_i}X_w\big)}. \tag{28}$$

Now, by Lemma 6.1, $\mathcal{L}(-\varepsilon_1) \cong \mathcal{O}_X(-1)$, so the pullback of this line bundle to the fixed point q_i is the 1-dimensional T-representation with weight ε_i. Combining this with the formula of Corollary 6.4, we see that $\nu_i^* \nu^*(\xi^u) = \mathrm{e}^{-\varepsilon_{u+1}+\varepsilon_i} \prod_{j=1}^{u}(1-\mathrm{e}^{-\varepsilon_j+\varepsilon_i})$, and similarly for $\nu_i^* \nu^*(\xi^v)$. Hence, equation (28) implies that

$$p_{u,v}^w = \sum_{i=1}^{w+1} \frac{\mathrm{e}^{-\varepsilon_{u+1}+\varepsilon_i}\left(\prod_{j=1}^{u}(1-\mathrm{e}^{-\varepsilon_j+\varepsilon_i})\right)\mathrm{e}^{-\varepsilon_{v+1}+\varepsilon_i}\prod_{j=1}^{v}(1-\mathrm{e}^{-\varepsilon_j+\varepsilon_i})}{\prod_{\substack{1\le j\le w+1\\ j\ne i}}(1-\mathrm{e}^{-\varepsilon_j+\varepsilon_i})}.$$

Hence,

$$(-1)^{u+v+w}\, p_{u,v}^w = \sum_{i=1}^{w+1} \frac{\mathrm{e}^{-\varepsilon_{u+1}+\varepsilon_i}\left(\prod_{j=1}^{u}(\mathrm{e}^{-\varepsilon_j+\varepsilon_i}-1)\right)\mathrm{e}^{-\varepsilon_{v+1}+\varepsilon_i}\prod_{j=1}^{v}(\mathrm{e}^{-\varepsilon_j+\varepsilon_i}-1)}{\prod_{\substack{1\le j\le w+1\\ j\ne i}}(\mathrm{e}^{-\varepsilon_j+\varepsilon_i}-1)}.$$

Write $\mathrm{e}^{\varepsilon_i} = Z_i$. The proof of Theorem 6.5 (for $v < n$) is reduced to the following lemma.

Lemma 6.7. For all integers $0 \le u, v \le w$,

$$\sum_{i=1}^{w+1} \frac{Z_{u+1}^{-1} Z_i\left(\prod_{j=1}^{u}(Z_j^{-1}Z_i-1)\right) Z_{v+1}^{-1} Z_i \prod_{j=1}^{v}(Z_j^{-1}Z_i-1)}{\prod_{\substack{1\le j\le w+1\\ j\ne i}}(Z_j^{-1}Z_i-1)}$$

is equal to

$$\left(\prod_{1\le i\le w+1} Z_i\right)\left(\prod_{1\le i\le u+1} Z_i^{-1}\right)\left(\prod_{1\le i\le v+1} Z_i^{-1}\right)\left[\frac{\left(\prod_{j=1}^{u}(1-tZ_j)\right)\left(\prod_{j=1}^{v}(1-tZ_j)\right)}{\prod_{j=1}^{w+1}(1-tZ_j)}\right]_{u+v-w+1}.$$

□

Proof. Denote by $A_{u,v}^w(Z_1,\ldots,Z_{w+1})$ the first expression of the lemma (set $A_{u,v}^w(Z_1,\ldots,Z_{w+1}) = 0$ if u, v or w is negative), and by $B_{u,v}^w(Z_1,\ldots,Z_{w+1})$ the second expression of the lemma (set $B_{u,v}^w(Z_1,\ldots,Z_{w+1}) = 0$ if u, v or w is negative).

We first verify the lemma in the case $u = v = 0$. We can write

$$A_{0,0}^w = \frac{1}{Z_1^2}\left(\prod_{1\le j\le w+1} Z_j\right)\sum_{i=1}^{w+1}\frac{Z_i}{\prod_{\substack{1\le j\le w+1\\ j\ne i}}(Z_i-Z_j)}.$$

Comparing this with $B^w_{0,0}$, we have to prove that

$$\sum_{i=1}^{w+1} \frac{Z_i}{\prod_{\substack{1\le j\le w+1\\ j\neq i}} (Z_i - Z_j)} \tag{29}$$

is equal to Z_1 if $w=0$; to 1 if $w=1$; and to 0 if $w>1$. This is obvious for $w=0$ and $w=1$. For $w>1$, note that the common denominator of the terms in the sum is the Vandermonde determinant in the variables $Z_1,\dots,Z_{w+1}$. If we rewrite the sum as a single fraction with this denominator, the numerator is an antisymmetric polynomial whose degree is strictly smaller than the degree of the denominator (since $w>1$). On the other hand, any antisymmetric polynomial in the variables Z_i is divisible by the Vandermonde determinant. This implies that (29) is 0, as desired.

We now prove the lemma by induction on w. For $w=0$, we have $u=v=0$, so we are in the previous case. Assume the result holds for $w-1$. Since both the expressions of the lemma are symmetric in u and v, by the preceding paragraph, we may assume $v\geq 1$. By Theorem 6.6, we know that

$$B^w_{u,v}(Z_1,\dots,Z_{w+1}) = \left(Z_{u+1}Z_1^{-1}-1\right)B^{w-1}_{u-1,v-1}(Z_2,\dots,Z_{w+1}) + Z_{u+2}Z_1^{-1}B^{w-1}_{u,v-1}(Z_2,\dots,Z_{w+1}). \tag{30}$$

Thus, it is enough to prove the analogous equation for $A^w_{u,v}$. After multiplying the numerator and denominator of $A^w_{u,v}(Z_1,\dots,Z_{w+1})$ by $\prod_{j=1}^{w+1} Z_j$ and simplifying (using the fact that all the summands vanish when the index i is less than $v+1$), we obtain

$$A^w_{u,v} = \frac{\prod_{j=u+2}^{w+1} Z_j}{\prod_{j=1}^{v+1} Z_j} \sum_{i=v+1}^{w+1} \frac{Z_i \prod_{j=1}^{u}(Z_i - Z_j)}{\prod_{\substack{v+1\le j\le w+1\\ j\neq i}} (Z_i - Z_j)}. \tag{31}$$

Using this expression, the verification of the recurrence relation for $A^w_{u,v}$ is straightforward. ■

Proof of Theorem 6.5 (continued). This proves Theorem 6.5 if $v<n$. Now, assume $v=n$. Since both the sides of equation (25) are symmetric in u and v, we may assume that $u=n$ as well. In this case, we will simply calculate both sides of (25). Since $\xi^n = [\mathcal{O}_{X^n}]$

and $X^n = \{[0,\dots,0,1]\}$, the self-intersection formula (cf. [17, Theorem 2.1]) implies that

$$\xi^n\xi^n = \left(\prod_{i=1}^{n}(1-\mathrm{e}^{\varepsilon_{n+1}-\varepsilon_i})\right)\xi^n.$$

This can also be seen by using Corollary 6.4 and localization at q_{n+1}.
So,

$$p^w_{n,n} = 0 \quad \text{if } w < n \text{ and} \tag{32}$$

$$(-1)^n\, p^n_{n,n} = (-1)^n \prod_{i=1}^{n}(1-\mathrm{e}^{\varepsilon_{n+1}-\varepsilon_i}) = \prod_{i=1}^{n}(\mathrm{e}^{\varepsilon_{n+1}-\varepsilon_i}-1). \tag{33}$$

Observe that the right side of the equation in the statement of the theorem is 0 unless $w = n$ (see the proof of Corollary 6.9 below). So, it suffices to calculate this side for $w = n$. The case $n = 1$ gives the correct result (we omit the calculation). Assume the result holds for $n-1$. We have

$$\begin{aligned}
&\mathrm{e}^{-\chi_{n+1}}\left[\frac{\prod_{i=1}^n(1-t\mathrm{e}^{\varepsilon_i})}{1-t\mathrm{e}^{\varepsilon_{n+1}}}\right]_{n+1}\\
&\quad= \mathrm{e}^{-\chi_{n+1}}\left[\frac{\prod_{i=1}^n(1-t\mathrm{e}^{\varepsilon_i})}{1-t\mathrm{e}^{\varepsilon_{n+1}}} - \frac{(1-t\mathrm{e}^{\varepsilon_{n+1}})\cdot\prod_{i=2}^n(1-t\mathrm{e}^{\varepsilon_i})}{1-t\mathrm{e}^{\varepsilon_{n+1}}} + \frac{(1-t\mathrm{e}^{\varepsilon_{n+1}})\cdot\prod_{i=2}^n(1-t\mathrm{e}^{\varepsilon_i})}{1-t\mathrm{e}^{\varepsilon_{n+1}}}\right]_{n+1}\\
&\quad= \mathrm{e}^{-\chi_{n+1}}\left[\frac{t(\mathrm{e}^{\varepsilon_{n+1}}-\mathrm{e}^{\varepsilon_1})\cdot\prod_{i=2}^n(1-t\mathrm{e}^{\varepsilon_i})}{1-t\mathrm{e}^{\varepsilon_{n+1}}} + \prod_{i=2}^{n}(1-t\mathrm{e}^{\varepsilon_i})\right]_{n+1}.
\end{aligned}$$

The second product is of degree $n-1$ in t and therefore contributes nothing to the degree $n+1$ part. So, the above expression equals

$$\begin{aligned}
\mathrm{e}^{-\chi_{n+1}}(\mathrm{e}^{\varepsilon_{n+1}}-\mathrm{e}^{\varepsilon_1})\left[\frac{\prod_{i=2}^n(1-t\mathrm{e}^{\varepsilon_i})}{1-t\mathrm{e}^{\varepsilon_{n+1}}}\right]_n &= (\mathrm{e}^{\varepsilon_{n+1}-\varepsilon_1}-1)\mathrm{e}^{-\chi_{n+1}+\varepsilon_1}\left[\frac{\prod_{i=2}^n(1-t\mathrm{e}^{\varepsilon_i})}{1-t\mathrm{e}^{\varepsilon_{n+1}}}\right]_n\\
&= (\mathrm{e}^{\varepsilon_{n+1}-\varepsilon_1}-1)\prod_{i=2}^{n}(\mathrm{e}^{\varepsilon_{n+1}-\varepsilon_i}-1),
\end{aligned}$$

where in the last step we have used our inductive hypothesis. The result follows from identity (33). ■

Remark 6.8. Letting $Z_i = \mathrm{e}^{\varepsilon_i}$, we obtain from equation (31) another expression for $\tilde{p}^w_{u,v} = A^w_{u,v}$. □

Corollary 6.9. If $p_{u,v}^w \neq 0$, then $u, v \leq w \leq u+v+1$. □

Proof. Let A be as in the proof of Theorem 6.6. Observe that the power series expansion of A contains no negative powers of t. Hence, if $p_{u,v}^w \neq 0$, then, by Theorem 6.5, $u+v-w+1 \geq 0$, i.e. $w \leq u+v+1$. Next, by the proof of Theorem 6.5, $u, v \leq w$. ■

Corollary 6.10.

$$\begin{aligned} p_{u,0}^w &= 0, \quad \text{if} \quad u \neq w-1, w, \\ p_{w-1,0}^w &= -e^{\varepsilon_{w+1}-\varepsilon_1}, \quad \text{for } w \geq 1 \\ p_{w,0}^w &= e^{\varepsilon_{w+1}-\varepsilon_1}. \end{aligned}$$ □

Proof. The first statement is immediate from Corollary 6.9. The other formulas are easy consequences of Theorem 6.5. ■

Corollary 6.11. If $n, m \geq w$, then the structure constants $p_{u,v}^w$ are the same for $\mathbb{P}^n$ and $\mathbb{P}^m$. □

Proof. This follows immediately from the expression for $p_{u,v}^w$ given by Theorem 6.5. ■

As an immediate consequence of these results, we can verify Conjecture 3.1 for the projective spaces $\mathbb{P}^n$:

Theorem 6.12. For all $0 \leq u, v, w \leq n$,

$$\tilde{p}_{u,v}^w \in \mathbb{Z}_+[e^{-\alpha_i} - 1]_{\alpha_i \in \Pi}.$$ □

Proof. This holds if $v = 0$ by Corollary 6.10. The general case follows by induction using the recurrence of Theorem 6.6. ■

Remark 6.13. (a) For $0 \leq u, v, w \leq n$, define

$$\tilde{q}_{u,v}^w = (-1)^{u+v+w} \chi(X_w \cap X^u, \xi^v).$$

Then, one can show that

$$\tilde{q}_{u,v}^w = (e^{\varepsilon_{u+1}-\varepsilon_1} - 1)\, \tilde{q}_{u-1,v-1}^{w-1}(\varepsilon_2, \ldots, \varepsilon_{n+1}) + e^{\varepsilon_{u+1}-\varepsilon_1}\, \tilde{q}_{u,v-1}^{w-1}(\varepsilon_2, \ldots, \varepsilon_{n+1}).$$

So, by induction on w, we get that

$$\tilde{q}^{w}_{u,v} \in \mathbb{Z}_{+}[\mathrm{e}^{-\alpha_i} - 1]_{\alpha_i \in \Pi}.$$

(b) As a consequence of Theorem 6.6 and Corollary 6.10, we get that in the nonequivariant K-theory $K(\mathbb{P}^n)$, we have

$$p^{w}_{u,v} = 0, \quad \text{for } u + v > w$$

and

$$p^{w}_{u,w-u} = 1, \text{ for any } u \le w; \; p^{w}_{u,w-u-1} = -1, \text{ for any } u \le w - 1.$$

□

6.3 Structure constants with respect to the structure sheaf basis of $K_T(\mathbb{P}^n)$

We give explicit formulas for the structure constants with respect to the structure sheaf basis. These are strikingly similar to the formulas for the structure constants in the dual structure sheaf basis, but they differ subtly. We will state these formulas here. We omit most details of the proofs, which are very similar to the proofs in the previous subsection.

Let w_o denote the longest element of the Weyl group of SL_{n+1}, so $w_o(\varepsilon_i) = \varepsilon_{n+2-i}$. For $u \in [n]$, let $\bar{u} = n - u$. To state our formulas, it will be convenient to introduce the notation $r^{w}_{u,v} = w_o(b^{\bar{w}}_{\bar{u},\bar{v}})$. Set $r^{w}_{u,v} = 0$ if u, v, or w is negative. Write $\tilde{r}^{w}_{u,v} = (-1)^{u+v+w} r^{w}_{u,v}$.

Theorem 6.14. For any $0 \le u, v, w \le n$,

$$(-1)^{u+v+w} r^{w}_{u,v} = \mathrm{e}^{\chi_w - \chi_u - \chi_v} \left[\frac{\left(\prod_{i=1}^{u}(1 - t\mathrm{e}^{\varepsilon_i})\right)\left(\prod_{i=1}^{v}(1 - t\mathrm{e}^{\varepsilon_i})\right)}{\prod_{i=1}^{w+1}(1 - t\mathrm{e}^{\varepsilon_i})} \right]_{u+v-w}.$$

□

Proof. Arguing as in the proof of Theorem 6.5, we get that

$$b^{\bar{w}}_{\bar{u},\bar{v}} = \langle [\mathcal{O}_{X_{\bar{u}}}][\mathcal{O}_{X_{\bar{v}}}], \xi^{\bar{w}} \rangle = \pi_{X_{\bar{u}*}}\left(\nu^{*}\left([\mathcal{O}_{X_{\bar{v}}}]\xi^{\bar{w}}\right)\right),$$

where $\nu : X_{\bar{u}} \hookrightarrow X$ is the inclusion. This can be calculated using the localization theorem as in the proof of Theorem 6.5. The necessary formula for $[\mathcal{O}_{X_{\bar{v}}}]$ can be obtained from

Lemma 6.2 as in the proof of Corollary 6.4. The proof is completed as for Theorem 6.5, making use of the recurrence formula given by the next theorem. ■

Theorem 6.15. If $v \geq 1$, then

$$\tilde{r}^w_{u,v} = (e^{\varepsilon_{u+1}-\varepsilon_1} - 1)\, \tilde{r}^{w-1}_{u-1,v-1}(\varepsilon_2, \ldots, \varepsilon_{n+1}) + e^{\varepsilon_{u+1}-\varepsilon_1}\, \tilde{r}^{w-1}_{u,v-1}(\varepsilon_2, \ldots, \varepsilon_{n+1}).$$

□

Proof. This is similar to the proof of Theorem 6.6; we omit the details. ■

Remark 6.16. As in the case of the dual basis (cf. Remark 6.8), we obtain from the localization theorem another formula for $r^w_{u,v}$: Set $Z_i = e^{\varepsilon_i}$. Then, for $0 \leq u, v \leq w$,

$$\tilde{r}^w_{u,v} = \frac{\prod_{j=v+1}^{w} Z_j}{\prod_{j=1}^{u} Z_j} \sum_{i=u+1}^{n+1} \frac{\prod_{j=1}^{v}(Z_i - Z_j)}{\prod_{\substack{u+1 \leq j \leq w+1 \\ j \neq i}} (Z_i - Z_j)}.$$

□

Arguing as in the proof of Corollary 6.9, we get the following result.

Corollary 6.17. If $r^w_{u,v} \neq 0$, then $u, v \leq w \leq u + v$. □

The next result gives “initial conditions” for $r^w_{u,v}$.

Proposition 6.18. $r^w_{u,0} = \delta_{u,w}$. □

Proof. This follows because $[\mathcal{O}_{X_n}]$ is the identity element in $K_T(\mathbb{P}^n)$. (Alternatively, the proposition can be deduced from Theorem 6.14.) ■

Theorem 6.15 and Proposition 6.18 imply that Conjecture 3.10 holds for projective spaces (cf. Remark 3.11).

Theorem 6.19. For all $0 \leq u, v, w \leq n$,

$$\tilde{r}^w_{u,v} \in \mathbb{Z}_+[e^{-\alpha_i} - 1]_{\alpha_i \in \Pi}.$$

Hence,

$$(-1)^{n+u+v+w} b^w_{u,v} \in \mathbb{Z}_+[e^{\alpha_i} - 1]_{\alpha_i \in \Pi}.$$

□

Proof. The first result follows from Theorem 6.15 and Proposition 6.18. The second result follows from the first, since w_o takes negative roots to positive roots. ■

7 A More General Positivity Conjecture

We revert to the notation and assumptions of Section 3. The following conjecture is an equivariant generalization of [3, Theorem 1]. By Proposition 3.6, this conjecture, with $G \times G$ in place of G and T' equal to the diagonal torus in $T \times T$, would imply Conjecture 3.1.

Conjecture 7.1. Let T' be a subtorus of T and let $Y \subset G/P$ be a T'-stable irreducible subvariety with rational singularities. Express, in $K_{T'}(G/P)$,

$$[\mathcal{O}_Y] = \sum_{w \in W^P} a_w^Y [\mathcal{O}_{X_w^P}].$$

Then,

$$(-1)^{\text{codim } Y + \text{codim } X_w^P} a_w^Y \in \mathbb{Z}_+[\mathrm{e}^{-\alpha_i}{}_{|T'} - 1]_{\alpha_i \in \Pi}.$$

□

Remark 7.2.

1) By (a subsequent) Proposition 7.6 and Remark 7.7(a), the above conjecture is true for $Y = X_P^v \subset G/P$.
2) We have verified the above conjecture by an explicit calculation for $G = SL_3$, $P = B$, and $Y = X_w \cap X^v$ for any $v, w \in W$. □

In the next proposition, we view $\mathbb{P}^1$ as having projective coordinates $[x_0 : x_1]$, so $\frac{x_0}{x_1}$ is a rational function on $\mathbb{P}^1$. We write $0 = [0 : 1]$ and $\infty = [1 : 0]$.

Proposition 7.3. Let T be any torus. Suppose T acts on $\mathbb{P}^1$ such that 0 and ∞ are T-fixed and $\frac{x_0}{x_1}$ is a T-weight vector with any weight $-\alpha$. Let X be an irreducible T-variety and $\phi : X \to \mathbb{P}^1$ a dominant T-equivariant morphism. Then, in $K_T(X)$,

$$[\mathcal{O}_{\phi^{-1}(\infty)}] = (1 - \mathrm{e}^{\alpha})[\mathcal{O}_X] + \mathrm{e}^{\alpha}[\mathcal{O}_{\phi^{-1}(0)}].$$

□

Proof. Since ϕ is a flat morphism (see [10, Ch. III, Prop. 9.7]), $\phi^*[\mathcal{O}_{\{0\}}] = [\mathcal{O}_{\phi^{-1}(0)}]$ and similarly for $\mathcal{O}_{\{\infty\}}$. Hence, it suffices to show that on $\mathbb{P}^1$,

$$[\mathcal{O}_{\{\infty\}}] = (1 - \mathrm{e}^{\alpha})[\mathcal{O}_{\mathbb{P}^1}] + \mathrm{e}^{\alpha}[\mathcal{O}_{\{0\}}],$$

since applying ϕ^* gives the desired equation. We have exact sequences

$$0 \to \mathcal{I}_0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\{0\}} \to 0,$$

and

$$0 \to \mathcal{I}_\infty \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\{\infty\}} \to 0,$$

where $\mathcal{I}_0$ and $\mathcal{I}_\infty$ are the ideal sheaves of $\{0\}$ and $\{\infty\}$, respectively. Nonequivariantly, $\mathcal{I}_0 = \mathcal{I}_\infty = \mathcal{O}_{\mathbb{P}^1}(-1)$, so $\mathcal{I}_0 \otimes \mathcal{I}_\infty^*$ is nonequivariantly isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. Near $0 = [0:1]$, the sheaf $\mathcal{I}_0$ is generated by x_0/x_1, which has weight $-\alpha$, and $\mathcal{I}_\infty$ near 0 is generated by 1. Hence, as T-equivariant sheaves, $\mathcal{I}_0 \simeq \mathrm{e}^{-\alpha}\mathcal{I}_\infty$. So,

$$\begin{aligned}[\mathcal{O}_{\{\infty\}}] &= [\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{I}_{\{\infty\}}] = [\mathcal{O}_{\mathbb{P}^1}] - \mathrm{e}^{\alpha}[\mathcal{I}_0] = [\mathcal{O}_{\mathbb{P}^1}] - \mathrm{e}^{\alpha}([\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{O}_{\{0\}}]) \\ &= (1 - \mathrm{e}^{\alpha})[\mathcal{O}_{\mathbb{P}^1}] + \mathrm{e}^{\alpha}[\mathcal{O}_{\{0\}}]\end{aligned}$$

as desired. ■

Let $w \in W$. If s is a simple reflection with $sw < w$, then $sX_w = X_w$, and hence $[\mathcal{O}_{sX_w}] = [\mathcal{O}_{X_w}]$. On the other hand if $sw > w$, then we have the following result.

Proposition 7.4. If s is a simple reflection with $sw > w$, then

$$\left[\mathcal{O}_{sX_w}\right] = \mathrm{e}^{-\alpha}\left[\mathcal{O}_{X_w}\right] - (\mathrm{e}^{-\alpha} - 1)\left[\mathcal{O}_{X_{sw}}\right],$$

where α is the simple root corresponding to s. □

Proof. Let P_s be the minimal parabolic corresponding to s. Consider

$$\begin{array}{ccc} P_s \times^B X_w & \overset{\mu}{\twoheadrightarrow} & X_{sw} \\ \big\downarrow \pi & & \\ P_s/B = \mathbb{P}^1, & & \end{array}$$

where μ takes the B-orbit $[p, x] \mapsto px$ and π takes $[p, x] \mapsto p \bmod B$. Then,

$$\pi^{-1}(0) = \{1\} \times X_w, \ \pi^{-1}(\infty) = \{s\} \times X_w.$$

So, by Proposition 7.3,

$$\big[\mathcal{O}_{\{s\}\times X_w}\big] = \mathrm{e}^{-\alpha}\big[\mathcal{O}_{\{1\}\times X_w}\big] + (1 - \mathrm{e}^{-\alpha})\big[\mathcal{O}_{P_s \times^B X_w}\big].$$

Push forward the above identity to X_{sw} via μ to get the result. (Here we have used [1, Proposition 3.2.1].) ■

Lemma 7.5. For any T-stable closed subscheme $Y \subset G/P$, write in $K_T(G/P)$,

$$[\mathcal{O}_Y] = \sum_{w \in W^P} P_w\big[\mathcal{O}_{X_w^P}\big], \text{ for some (unique) } P_w \in R(T). \tag{34}$$

Then, for any $v \in W$,

$$[\mathcal{O}_{v^{-1}Y}] = \sum_{w \in W^P} (v^{-1}P_w)\big[\mathcal{O}_{v^{-1}X_w^P}\big].$$

□

Proof. Let $f : T \to T'$ be any homomorphism. If X is any smooth scheme with T'-action, and T acts on X via f, then there is a ring homomorphism $f^* : K_{T'}(X) \to K_T(X)$ extending the natural pullback map $R(T') \to R(T)$. For any T'-stable closed subscheme Y of X, f^* takes the class of $\mathcal{O}_Y$ in $K_{T'}(X)$ to the class of $\mathcal{O}_Y$ in $K_T(X)$. We now apply this to $X = G/P$ and $f : T \to T$ given by $f(t) = vtv^{-1}$. Since $f^*r = v^{-1}r$ for $r \in R(T)$, we get from (34) the equation

$$[\mathcal{O}_Y] = \sum_{w \in W^P} (v^{-1}P_w)\big[\mathcal{O}_{X_w^P}\big], \tag{35}$$

where in this equation T is viewed as acting on G/P through f. Write $(G/P, \odot)$ to indicate G/P with this new action of T.

Consider the isomorphism

$$\phi_v : G/P \to (G/P, \odot), \ gP \mapsto \dot{v}gP,$$

where $\dot{v}$ is a representative of v in $N(T)$, $N(T)$ being the normalizer of T in G. This is T-equivariant, where T acts on the source G/P by the standard action. Then,

$\phi_v^*[\mathcal{O}_{X_w^P}] = [\mathcal{O}_{v^{-1}X_w^P}]$ and $\phi_v^*[\mathcal{O}_Y] = [\mathcal{O}_{v^{-1}Y}]$. Since ϕ_v^* is $R(T)$-linear, applying ϕ_v^* to (35) proves the result. ∎

Proposition 7.6. Write $[\mathcal{O}_{X^w}] = \sum_u f_{w,u}[\mathcal{O}_{X_u}]$. Then,

$$(-1)^{\text{codim } X^w + \text{codim } X_u} f_{w,u} \in \mathbb{Z}_+[e^{-\alpha_i} - 1]_{\alpha_i \in \Pi}.$$

□

Proof. For any $v, w \in W$, write

$$[\mathcal{O}_{vX_w}] = \sum f^v_{w,u}[\mathcal{O}_{X_u}].$$

We prove by induction on $\ell(v)$, that for any $u, w \in W$,

$$(-1)^{\text{codim } X_w + \text{codim } X_u} f^v_{w,u} \in \mathbb{Z}_+[e^{-\alpha_i} - 1]_{\alpha_i \in \Pi}. \tag{36}$$

Of course, (36) is true for $v = 1$. Now, take vs_i with $\ell(vs_i) > \ell(v)$. If $s_i w < w$, then $[\mathcal{O}_{vX_w}] = [\mathcal{O}_{vs_iX_w}]$ and we are done. If $s_i w > w$, then, by Proposition 7.4,

$$[\mathcal{O}_{s_iX_w}] = e^{-\alpha_i}[\mathcal{O}_{X_w}] - (e^{-\alpha_i} - 1)[\mathcal{O}_{X_{s_iw}}].$$

Thus, by Lemma 7.5,

$$[\mathcal{O}_{vs_iX_w}] = e^{-v\alpha_i}[\mathcal{O}_{vX_w}] - (e^{-v\alpha_i} - 1)[\mathcal{O}_{vX_{s_iw}}].$$

Since $vs_i > v$, $v\alpha_i \in \Delta^+$. Moreover, by induction, for any $u \in W$, $(-1)^{\text{codim}X_w + \text{codim}X_u} f^v_{w,u}$ and $(-1)^{\text{codim}X_w - 1 + \text{codim}X_u} f^v_{s_iw,u}$ are in $\mathbb{Z}_+[e^{-\alpha_i} - 1]_{\alpha_i \in \Pi}$. Hence, $(-1)^{\text{codim}X_w + \text{codim}X_u} f^{vs_i}_{w,u} \in \mathbb{Z}_+[e^{-\alpha_i} - 1]_{\alpha_i \in \Pi}$. This completes the induction, and hence (36) is proved for any $u, v, w \in W$. Since $X^w = w_o X_{w_o w}$, the proposition follows. ∎

Remark 7.7. (a) For any standard parabolic P and any closed T-stable subvariety $Z \subset G/P$, since $\pi^*[\mathcal{O}_Z] = [\mathcal{O}_{\pi^{-1}(Z)}]$ (cf. the proof of Lemma 3.13), where $\pi : G/B \to G/P$ is the standard projection, the above proposition and (36) remain true for the Schubert varieties in G/P.

(b) Since any T-stable closed irreducible subvariety of $\mathbb{P}^n$ (under the standard action of the maximal torus T of $SL(n+1)$) is a W-translate of a Schubert variety of $\mathbb{P}^n$, Conjecture 7.1 is true for any T-stable closed irreducible subvariety of $\mathbb{P}^n$ (by virtue of (36) and the above remark). □

8 Appendix: Products for A_2 and B_2

In this appendix, we explicitly give the product in the dual basis $\{\xi^w\}_{w\in W}$ of $K_T(G/B)$ for G of types A_2 and B_2. In particular, in these cases, Conjecture 3.1 holds. These were computed using the localization theorem in equivariant K-theory. The pullbacks of the elements of the dual basis to the fixed points were computed by hand. Using this, the products were computed using Maple. We have also computed these products for G of type G_2, but because of the size of the expressions (a single structure constant in type G_2 can take several lines to write), we have not included them in the paper.

In what follows, we write $\mu^w = (-1)^{\ell(w)}\xi^w$. Also, α_1 and α_2 are the simple roots, and $u := \mathrm{e}^{-\alpha_1} - 1$ and $v := \mathrm{e}^{-\alpha_2} - 1$. In the case of B_2, α_2 is the short root.

The products for G of type A_2 are as follows:

$$\begin{aligned}
\mu^1\mu^1 &= \mu^1 + (u+1)\mu^{s_1} + (v+1)\mu^{s_2} + (v+2)(u+1)(v+1)\mu^{s_2s_1}\\
&\quad + (u+2)(u+1)(v+1)\mu^{s_1s_2} + (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{s_1}\mu^1 &= (u+1)\mu^{s_1} + (v+2)(u+1)(v+1)\mu^{s_2s_1}\\
&\quad + (u+1)^2(v+1)\mu^{s_1s_2} + (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{s_1}\mu^{s_1} &= u\mu^{s_1} + (v+1)(uv+2u+v+1)\mu^{s_2s_1}\\
&\quad + u(u+1)(v+1)\mu^{s_1s_2} + (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{s_2}\mu^1 &= (v+1)\mu^{s_2} + (v+1)^2(u+1)\mu^{s_2s_1}\\
&\quad + (u+2)(u+1)(v+1)\mu^{s_1s_2} + (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{s_2}\mu^{s_1} &= (v+1)^2(u+1)\mu^{s_2s_1} + (u+1)^2(v+1)\mu^{s_1s_2} + (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{s_2}\mu^{s_2} &= v\mu^{s_2} + (u+1)(v+1)v\mu^{s_2s_1}\\
&\quad + (u+1)(uv+u+2v+1)\mu^{s_1s_2} + (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{s_2s_1}\mu^1 &= (v+1)^2(u+1)\mu^{s_2s_1} + (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{s_2s_1}\mu^{s_1} &= (v+1)(uv+u+v)\mu^{s_2s_1} + (u+1)(v+1)(uv+u+v)\mu^{w_o}\\
\mu^{s_2s_1}\mu^{s_2} &= (u+1)(v+1)v\mu^{s_2s_1} + (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{s_2s_1}\mu^{s_2s_1} &= v(uv+u+v)\mu^{s_2s_1} + (u+1)v(uv+u+v)\mu^{w_o}\\
\mu^{s_1s_2}\mu^1 &= (u+1)^2(v+1)\mu^{s_1s_2} + (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{s_1s_2}\mu^{s_1} &= u(u+1)(v+1)\mu^{s_1s_2} + (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{s_1s_2}\mu^{s_2} &= (u+1)(uv+u+v)\mu^{s_1s_2} + (u+1)(v+1)(uv+u+v)\mu^{w_o}\\
\mu^{s_1s_2}\mu^{s_2s_1} &= (u+1)(v+1)(uv+u+v)\mu^{w_o}\\
\mu^{s_1s_2}\mu^{s_1s_2} &= u(uv+u+v)\mu^{s_1s_2} + (v+1)u(uv+u+v)\mu^{w_o}
\end{aligned}$$

$$
\begin{aligned}
\mu^{w_o}\mu^{1} &= (u+1)^2(v+1)^2\mu^{w_o}\\
\mu^{w_o}\mu^{s_1} &= (u+1)(v+1)(uv+u+v)\mu^{w_o}\\
\mu^{w_o}\mu^{s_2} &= (u+1)(v+1)(uv+u+v)\mu^{w_o}\\
\mu^{w_o}\mu^{s_2s_1} &= (u+1)v(uv+u+v)\mu^{w_o}\\
\mu^{w_o}\mu^{s_1s_2} &= (v+1)u(uv+u+v)\mu^{w_o}\\
\mu^{w_o}\mu^{w_o} &= uv(uv+u+v)\mu^{w_o}
\end{aligned}
$$

The products for G of type B_2 are as follows:

$$
\begin{aligned}
\mu^1\mu^1 &= \mu^1 + (u+1)\mu^{s_1} + (v+1)\mu^{s_2} + (v^2+3v+3)(u+1)(v+1)\mu^{s_2s_1}\\
&\quad + (u+2)(u+1)(v+1)\mu^{s_1s_2} + (v+1)^2(u+1)^2(uv+u+2v+3)\mu^{s_2s_1s_2}\\
&\quad + (v+1)^2(u+1)(uv^2+3uv+3u+v^2+3v+4)\mu^{s_1s_2s_1} + (u+1)^3(v+1)^4\mu^{w_o}\\
\mu^1\mu^{s_1} &= (u+1)\mu^{s_1} + (v^2+3v+3)(u+1)(v+1)\mu^{s_2s_1} + (u+1)^2(v+1)\mu^{s_1s_2}\\
&\quad + (v+1)^2(u+1)^2(uv+u+2v+3)\mu^{s_2s_1s_2} + (v+1)^2(u+1)^2(v^2+3v+3)\mu^{s_1s_2s_1}\\
&\quad + (u+1)^3(v+1)^4\mu^{w_o}\\
\mu^1\mu^{s_2} &= (v+1)\mu^{s_2} + (u+1)(v+1)^3\mu^{s_2s_1} + (u+2)(u+1)(v+1)\mu^{s_1s_2}\\
&\quad + (v+1)^3(u+1)^2(u+2)\mu^{s_2s_1s_2} + (v+1)^2(u+1)(uv^2+3uv\\
&\quad + 3u+v^2+3v+4)\mu^{s_1s_2s_1} + (u+1)^3(v+1)^4\mu^{w_o}\\
\mu^1\mu^{s_2s_1} &= (u+1)(v+1)^3\mu^{s_2s_1} + (v+1)^3(u+1)^2(u+2)\mu^{s_2s_1s_2}\\
&\quad + (u+1)^2(v+1)^4\mu^{s_1s_2s_1} + (u+1)^3(v+1)^4\mu^{w_o}\\
\mu^1\mu^{s_1s_2} &= (u+1)^2(v+1)\mu^{s_1s_2} + (u+1)^3(v+1)^3\mu^{s_2s_1s_2}\\
&\quad + (v+1)^2(u+1)^2(v^2+3v+3)\mu^{s_1s_2s_1} + (u+1)^3(v+1)^4\mu^{w_o}\\
\mu^1\mu^{s_2s_1s_2} &= (u+1)^3(v+1)^3\mu^{s_2s_1s_2} + (u+1)^3(v+1)^4\mu^{w_o}\\
\mu^1\mu^{s_1s_2s_1} &= (u+1)^2(v+1)^4\mu^{s_1s_2s_1} + (u+1)^3(v+1)^4\mu^{w_o}\\
\mu^1\mu^{w_o} &= (u+1)^3(v+1)^4\mu^{w_o}\\
\mu^{s_1}\mu^{s_1} &= u\mu^{s_1} + (uv^2+3uv+3u+v^2+3v+2)(v+1)\mu^{s_2s_1} + (u+1)(v+1)u\mu^{s_1s_2}\\
&\quad + (v+1)^2(u+1)(u^2v+u^2+3uv+4u+2v+3)\mu^{s_2s_1s_2}\\
&\quad + (v+1)^2(u+1)(uv^2+3uv+3u+v^2+3v+2)\mu^{s_1s_2s_1}\\
&\quad + (v+1)^3(u+1)^2(uv+u+v+1)\mu^{w_o}\\
\mu^{s_1}\mu^{s_2} &= (u+1)(v+1)^3\mu^{s_2s_1} + (u+1)^2(v+1)\mu^{s_1s_2} + (v+1)^3(u+1)^2(u+2)\mu^{s_2s_1s_2}\\
&\quad + (v+1)^2(u+1)^2(v^2+3v+3)\mu^{s_1s_2s_1} + (u+1)^3(v+1)^4\mu^{w_o}
\end{aligned}
$$

$$
\begin{aligned}
\mu^{s_1}\mu^{s_2s_1} &= (v+1)(uv^2+2uv+u+v^2+2v)\mu^{s_2s_1}\\
&\quad + (v+1)(u+1)(2v^2+4v+1+u^2v^2+2u^2v+u^2+3uv^2+6uv+3u)\mu^{s_2s_1s_2}\\
&\quad + (u+1)(v+1)^2(uv^2+2uv+u+v^2+2v)\mu^{s_1s_2s_1}\\
&\quad + (v+1)^2(u+1)^2(uv^2+2uv+u+v^2+2v+1)\mu^{w_o}\\
\mu^{s_1}\mu^{s_1s_2} &= (u+1)(v+1)u\mu^{s_1s_2} + (u+1)^3(v+1)^3\mu^{s_2s_1s_2}\\
&\quad + (v+1)^2(u+1)(uv^2+3uv+3u+v^2+3v+2)\mu^{s_1s_2s_1}\\
&\quad + (v+1)^3(u+1)^2(uv+u+v+1)\mu^{w_o}\\
\mu^{s_1}\mu^{s_2s_1s_2} &= (uv+u+v)(uv+u+v+2)(u+1)(v+1)\mu^{s_2s_1s_2}\\
&\quad + (v+1)^2(uv+u+v+2)(u+1)(uv+u+v)\mu^{w_o}\\
\mu^{s_1}\mu^{s_1s_2s_1} &= (u+1)(v+1)^2(uv^2+2uv+u+v^2+2v)\mu^{s_1s_2s_1}\\
&\quad + (v+1)^2(u+1)^2(uv^2+2uv+u+v^2+2v+1)\mu^{w_o}\\
\mu^{s_1}\mu^{w_o} &= (v+1)^2(uv+u+v+2)(u+1)(uv+u+v)\mu^{w_o}\\
\mu^{s_2}\mu^{s_2} &= v\mu^{s_2} + (u+1)(v+1)^2v\mu^{s_2s_1} + (uv+u+2v+1)(u+1)\mu^{s_1s_2}\\
&\quad + (v+1)^2(u+1)^2(uv+u+2v+1)\mu^{s_2s_1s_2}\\
&\quad + (v+1)(u+1)(3u+uv^3+4uv^2+6uv+v^3+4v^2+7v+3)\mu^{s_1s_2s_1}\\
&\quad + (u+1)^3(v+1)^4\mu^{w_o}\\
\mu^{s_2}\mu^{s_2s_1} &= (u+1)(v+1)^2v\mu^{s_2s_1} + (v+1)^2(u+1)^2(uv+u+2v+1)\mu^{s_2s_1s_2}\\
&\quad + (u+1)^2(v+1)^4\mu^{s_1s_2s_1} + (u+1)^3(v+1)^4\mu^{w_o}\\
\mu^{s_2}\mu^{s_1s_2} &= (u+1)(uv+u+v)\mu^{s_1s_2} + (u+1)^2(v+1)^2(uv+u+v)\mu^{s_2s_1s_2}\\
&\quad + (v+1)(u+1)(3u+1+uv^3+4uv^2+6uv+v^3+4v^2+5v)\mu^{s_1s_2s_1}\\
&\quad + (v+1)^3(u+1)^2(uv+u+v+1)\mu^{w_o}\\
\mu^{s_2}\mu^{s_2s_1s_2} &= (u+1)^2(v+1)^2(uv+u+v)\mu^{s_2s_1s_2} + (v+1)^3(u+1)^2(uv+u+v+1)\mu^{w_o}\\
\mu^{s_2}\mu^{s_1s_2s_1} &= (u+1)(v+1)^2(uv^2+2uv+u+v^2+2v)\mu^{s_1s_2s_1}\\
&\quad + (u+1)^2(v+1)^2(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_2}\mu^{w_o} &= (u+1)^2(v+1)^2(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_2s_1}\mu^{s_2s_1} &= v(uv^2+2uv+u+v^2+2v)\mu^{s_2s_1}\\
&\quad + (uv^2+2uv+u+v^2+2v)(uv+u+2v+1)(u+1)\mu^{s_2s_1s_2}\\
&\quad + (u+1)(v+1)v(uv^2+2uv+u+v^2+2v)\mu^{s_1s_2s_1}\\
&\quad + (u+1)^2(v+1)^2(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_2s_1}\mu^{s_1s_2} &= (u+1)^2(v+1)^2(uv+u+v)\mu^{s_2s_1s_2}
\end{aligned}
$$

$$
\begin{aligned}
&+ (u+1)(v+1)^2(uv^2+2uv+u+v^2+2v)\mu^{s_1s_2s_1}\\
&+ (v+1)^2(u+1)^2(uv^2+2uv+u+v^2+2v+1)\mu^{w_o}\\
\mu^{s_2s_1}\mu^{s_2s_1s_2} &= (u+1)(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{s_2s_1s_2}\\
&+ (u+1)(v+1)(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_2s_1}\mu^{s_1s_2s_1} &= (u+1)(v+1)v(uv^2+2uv+u+v^2+2v)\mu^{s_1s_2s_1}\\
&+ (u+1)^2(v+1)^2(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_2s_1}\mu^{w_o} &= (u+1)(v+1)(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_1s_2}\mu^{s_1s_2} &= u(uv+u+v)\mu^{s_1s_2} + (u+1)(v+1)^2u(uv+u+v)\mu^{s_2s_1s_2}\\
&+ (uv+u+v)(uv^2+3uv+3u+v^2+3v+2)(v+1)\mu^{s_1s_2s_1}\\
&+ (v+1)^2(uv+u+v+2)(u+1)(uv+u+v)\mu^{w_o}\\
\mu^{s_1s_2}\mu^{s_2s_1s_2} &= (u+1)(v+1)^2u(uv+u+v)\mu^{s_2s_1s_2}\\
&+ (v+1)^2(uv+u+v+2)(u+1)(uv+u+v)\mu^{w_o}\\
\mu^{s_1s_2}\mu^{s_1s_2s_1} &= (v+1)(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{s_1s_2s_1}\\
&+ (u+1)(v+1)(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_1s_2}\mu^{w_o} &= (u+1)(v+1)(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_2s_1s_2}\mu^{s_2s_1s_2} &= u(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{s_2s_1s_2}\\
&+ (v+1)u(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_2s_1s_2}\mu^{s_1s_2s_1} &= (u+1)(v+1)(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_2s_1s_2}\mu^{w_o} &= (v+1)u(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_1s_2s_1}\mu^{s_1s_2s_1} &= v(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{s_1s_2s_1}\\
&+ (u+1)v(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{s_1s_2s_1}\mu^{w_o} &= (u+1)v(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{w_o}\\
\mu^{w_o}\mu^{w_o} &= uv(uv+u+v)(uv^2+2uv+u+v^2+2v)\mu^{w_o}
\end{aligned}
$$

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