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EIGENCONE, SATURATION AND HORN PROBLEMS FOR SYMPLECTIC AND ODD ORTHOGONAL GROUPS

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Abstract

In this paper we consider the eigenvalue problem, intersection theory of homogeneous spaces (in particular, the Horn problem) and the saturation problem for the symplectic and odd orthogonal groups. The classical embeddings of these groups in the special linear groups play an important role. We deduce properties for these classical groups from the known properties for the special linear groups.

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1. Introduction

Let G be a semisimple, connected, complex algebraic group with a maximal compact subgroup K. We fix a Borel subgroup B and a maximal torus $H \subset B$

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and let $\mathfrak{h}$ be the Lie algebra of H. Let $\mathfrak{h}_+ \subset \mathfrak{h}$ be the positive Weyl chamber of G. Then, there is a bijection $C : \mathfrak{h}_+ \to \mathfrak{k}/K$, $x \mapsto \overline{ix}$, where $\mathfrak{k}$ is the Lie algebra of K, K acts on $\mathfrak{k}$ by the adjoint representation, $i = \sqrt{-1}$ and $\overline{ix}$ denotes the K-conjugacy class of ix. Define the *eigencone* (for any $s \geq 1$):

$$\Gamma(s, K) = \{(\bar{k}_1, \dots, \bar{k}_s) \in (\mathfrak{k}/K)^s : \exists k_j \in \bar{k}_j \text{ with } \sum_{j=1}^{s} k_j = 0\}.$$

For an algebraic group homomorphism $G \to G'$ which takes $K \to K'$, there is an induced map $\Gamma(s, K) \to \Gamma(s, K')$. Therefore, $\Gamma(s, K)$ is functorial. In the case of $\mathrm{Sp}(2n) \hookrightarrow \mathrm{SL}(2n)$ (similarly $\mathrm{SO}(2n+1) \hookrightarrow \mathrm{SL}(2n+1)$), we have the stronger properties described below:

Let $\mathfrak{h}^{\mathrm{Sp}(2n)}$ be the Cartan subalgebra for $\mathrm{Sp}(2n)$ and similarly for $SO(2n{+}1)$, $\mathrm{SL}(2n)$ and $\mathrm{SL}(2n+1)$. There are natural (linear) embeddings (see Sections 2.2 and 2.3):

$$\mathfrak{h}_+^{\mathrm{Sp}(2n)} \hookrightarrow \mathfrak{h}_+^{\mathrm{SL}(2n)}, \text{ and } \mathfrak{h}_+^{\mathrm{SO}(2n+1)} \hookrightarrow \mathfrak{h}_+^{\mathrm{SL}(2n+1)}.$$

Identify $\mathfrak{k}/K$ with $\mathfrak{h}_+$ under C as above. We state our first main theorem (cf. Theorem 15):

Theorem 1. (a) *For $h_1, \dots, h_s \in \mathfrak{h}_+^{\mathrm{Sp}(2n)}$,*

$$(h_1, \dots, h_s) \in \Gamma(s, \mathrm{Sp}(2n)) \Leftrightarrow (h_1, \dots, h_s) \in \Gamma(s, SU(2n)).$$

(b) *For $h_1, \dots, h_s \in \mathfrak{h}_+^{\mathrm{SO}(2n+1)}$,*

$$(h_1, \dots, h_s) \in \Gamma(s, \mathrm{SO}(2n+1)) \Leftrightarrow (h_1, \dots, h_s) \in \Gamma(s, SU(2n+1)).$$

Our proof of this theorem is a consequence of a surprising result in intersection theory described below.

1.1. Intersection theory. Let V be a $2n$-dimensional complex vector space equipped with a nondegenerate symplectic form $\langle\, , \rangle$. Recall that if $A = \{a_1 < a_2 < \cdots < a_m\}$ is a subset of $[2n] := \{1, \dots, 2n\}$ of cardinality m and $F_\bullet$ a complete flag on $\mathbb{C}^{2n}$, then, by definition, the corresponding Schubert cell $\Omega_A(F_\bullet) := \{X \in \mathrm{Gr}(m, \mathbb{C}^{2n}) : \dim X \cap F_u = \ell \text{ for } a_\ell \leq u < a_{\ell+1}, \ell = 0, \dots, m\}$, where $\mathrm{Gr}(m, \mathbb{C}^{2n})$ is the ordinary Grassmannian of m-dimensional subspaces of $\mathbb{C}^{2n}$ and $a_0 = 0$, $a_{m+1} = 2n$.

Let $A^1, \dots, A^s$ be subsets of $[2n]$ each of cardinality m. Let $E^1_\bullet, \dots, E^s_\bullet$ be complete *isotropic* flags on V in general position. The following second main theorem (cf. Theorem 10) is a key technical result that underlies the proof of the first main theorem.

Theorem 2. *The intersection $\bigcap_{j=1}^{s} \Omega_{A^j}(E^j_\bullet)$ of subvarieties of $\mathrm{Gr}(m, V)$ is proper (possibly empty) for complete isotropic flags $E^j_\bullet$ on V in general position.*

We have the same result for $\mathrm{SO}(2n+1)$ (cf. Theorem 12). Let V' be a vector space of dimension $2n+1$ equipped with a nondegenerate symmetric bilinear form. Let $A^1, \dots, A^s$ be subsets of $[2n+1]$ each of cardinality m. Let $E^1_\bullet, \dots, E^s_\bullet$ be isotropic flags on V' in general position.

Theorem 3. *The intersection $\bigcap_{j=1}^s \Omega_{A^j}(E^j_\bullet)$ of subvarieties of $\mathrm{Gr}(m, V')$ is proper.*

We only give a very brief outline of our original proof of Theorem 2 (in Section 3.1), since F. Sottile has recently obtained a shorter proof of a stronger version of our theorem (cf. [So]). Our original proof of Theorem 3 was similar to that of Theorem 2.

We remark that the above intersection result is false for $\mathrm{SO}(2n)$, since Corollary 11 can easily be seen to be violated in this case even for $m = 1$.

1.2. The saturation problem. Any dominant weight λ of $\mathrm{SL}(2n)$ restricts to a dominant weight λ_C of the symplectic group $\mathrm{Sp}(2n)$. Similarly, any dominant weight λ of $\mathrm{SL}(2n+1)$ restricts to a dominant weight λ_B of the orthogonal group $\mathrm{SO}(2n+1)$. The following theorem (cf. Theorem 23) is proved geometrically by the method of "theta sections". We obtain invariants in tensor products by constructing divisors in the products of isotropic flag varieties ("the theta divisor"). This technique originates in our context in [B_2]. Theorems 4 and 5 follow from our basic transversality result (Theorem 16). Our proof of Theorem 16 crucially requires some ideas of Schofield [S], in addition to Theorems 2 and 3.

Theorem 4. *Let $V_{\lambda^1}, \dots, V_{\lambda^s}$ be irreducible representations of $\mathrm{SL}(2n)$ (with highest weights $\lambda^1, \dots, \lambda^s$ respectively) such that their tensor product has a nonzero $\mathrm{SL}(2n)$-invariant. Then, the tensor product of the irreducible representations of $\mathrm{Sp}(2n)$ with highest weights $\lambda^1_C, \dots, \lambda^s_C$ has a nonzero $\mathrm{Sp}(2n)$-invariant, where λ^j_C is the restriction of λ^j to the Cartan subalgebra of $\mathrm{Sp}(2n)$.*

A similar property holds for the odd orthogonal group $\mathrm{SO}(2n+1)$.

Theorem 5. *Let $V_{\lambda^1}, \dots, V_{\lambda^s}$ be irreducible representations of $\mathrm{SL}(2n+1)$ such that their tensor product has a nonzero $\mathrm{SL}(2n+1)$-invariant. Then, the tensor product of the irreducible representations of $\mathrm{SO}(2n+1)$ with highest weights $\lambda^1_B, \dots, \lambda^s_B$ has a nonzero $\mathrm{SO}(2n+1)$-invariant, where λ^j_B is the restriction of λ^j to the Cartan subalgebra of $\mathrm{SO}(2n+1)$.*

Using the saturation theorem of Knutson-Tao [KT], together with our Theorems 4 and 1, and results of Kumar-Stembridge [KS], we obtain the following improvement of the general Kapovich-Millson saturation theorem in the cases of $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ (cf. Theorems 25 and 26 and Remark 27). The general saturation result proved by Kapovich-Millson requires that $\mu^1 + \cdots + \mu^s$ belongs to the root lattice, and in the cases of $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ they require a saturation factor of 4, whereas our results do not have these restrictions.

Theorem 6. *Given dominant integral weights $\mu^1, \dots, \mu^s$ of* $\mathrm{Sp}(2n)$*, the following are equivalent:*

(1) *For some $N \geq 1$, the tensor product of irreducible representations of* $\mathrm{Sp}(2n)$ *with highest weights $N\mu^1, \dots, N\mu^s$ has a nonzero* $\mathrm{Sp}(2n)$*-invariant.*
(2) *The tensor product of irreducible representations with highest weights $2\mu^1, \dots, 2\mu^s$ has a nonzero* $\mathrm{Sp}(2n)$*-invariant.*

Theorem 7. *Given dominant integral weights $\nu^1, \dots, \nu^s$ of* $\mathrm{SO}(2n+1)$*, the following are equivalent:*

(1) *For some $N \geq 1$, the tensor product of irreducible representations with highest weights $N\nu^1, \dots, N\nu^s$ has a nonzero* $\mathrm{SO}(2n+1)$*-invariant.*
(2) *The tensor product of irreducible representations with highest weights $2\nu^1, \dots, 2\nu^s$ has a nonzero* $\mathrm{SO}(2n+1)$*-invariant.*

1.3. Horn's problem for symplectic and odd orthogonal groups. There are two senses in which the term "Horn's problem" is generally applied:

(A) Determination of a system of inequalities for the eigencone $\Gamma(s, K)$, which is "cohomology free" (but perhaps recursive).
(B) A system of inequalities characterizing nonvanishing structure coefficients in the cohomology $H^*(G/P)$ with the standard cup product and also $(H^*(G/P), \odot_0)$, where we recall that $\odot_0$ is a deformation of the cup product in the cohomology of the flag variety G/P introduced in [BK]. In this paper, we only consider the question in the case of $(H^*(G/P), \odot_0)$. The system of inequalities can reasonably be expected to be parameterized by the cohomology $(H^*(L/Q), \odot_0)$, where L is a Levi subgroup of G and $Q \subset L$ is a parabolic subgroup. It was suggested in [BK] that nonvanishing structure coefficients in the deformed cohomology can be characterized recursively. Recently, Richmond [R] showed that the cohomology $(H^*(SL_n/P), \odot_0)$ of partial flag varieties is strongly recursive (not just nonvanishing of structure coefficients, but also the actual structure coefficients).

We consider (A) and (B) for the groups $G = \mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$. Theorem 1 implies (A), because the eigenvalue problem for the special linear groups is recursive by Horn's original conjecture, proved by the combined works of Klyachko and Knutson-Tao (see [F_4] for a survey).

It follows from [BK] that (B) implies (A). We reduce the Horn problem (B) for $(H^*(G/P), \odot_0)$ for the groups $G = \mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ and maximal parabolic subgroups P to the corresponding problem for the Lagrangian

Grassmannians together with the problem of existence of suitable non-zero $SL(r)$ invariants, in Sections 8 and 9 (cf. Theorems 30 and 41). As a basic ingredient we need our basic transversality result (cf. Theorem 16). Now, the Horn problem for the Lagrangian Grassmannians was solved by Purbhoo-Sottile [PS]. It should be noted that the solution of (A) obtained through the solution for (B) is different from the one obtained from Theorem 1.

We make a conjecture (cf. Conjecture 29) generalizing many of the results in the paper for a diagram automorphism of G.

We establish the basic notation in Section 2 *and use them through the paper often without further explanation.*

2. Notation and preliminaries

Let G be a connected semisimple complex algebraic group. We fix a Borel subgroup B and a maximal torus $H \subset B$. Their Lie algebras are denoted by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}$ respectively. Let $W = N(H)/H$ be the Weyl group of G, $N(H)$ being the normalizer of H in G. Let $R^+ \subset \mathfrak{h}^*$ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$) and $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset R^+$ the set of simple roots. Let $\{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ be the set of corresponding simple coroots and $\{s_1, \dots, s_n\} \subset W$ be the set of corresponding simple reflections. Let $\mathfrak{h}_+ \subset \mathfrak{h}$ be the dominant chamber defined by

$$\mathfrak{h}_+ = \{x \in \mathfrak{h} : \alpha_i(x) \in \mathbb{R}_+, \forall \alpha_i\},$$

where $\mathbb{R}_+$ is the set of nonnegative real numbers. For any $1 \le i \le n$, let $\omega_i \in \mathfrak{h}^*$ denote the i-th fundamental weight defined by

$$\omega_i(\alpha_j^\vee) = \delta_{i,j}.$$

Let $B \subset P$ be a (standard) parabolic subgroup with the unique Levi subgroup L containing H. We denote by W^P the set of minimal-length coset representatives in the W_P-cosets W/W_P, where W_P is the Weyl group of P (which is, by definition, the Weyl group of the Levi subgroup L). For any $w \in W^P$, we have the Bruhat cell

$$\Lambda_w^P := BwP/P \subset G/P.$$

This is a locally closed subset of the flag variety G/P isomorphic to the affine space $\mathbb{A}^{\ell(w)}$, $\ell(w)$ being the length of w. Its closure is denoted by $\bar{\Lambda}_w^P$, which is an irreducible projective variety (of dimension $\ell(w)$). We denote by

$$[\bar{\Lambda}_w^P] \in H^{2(\dim_{\mathbb{C}}(G/P)-\ell(w))}(G/P)$$

the cycle class of the subvariety $\bar{\Lambda}^P_w$, where $H^*(G/P)$ is the singular cohomology of G/P with integral coefficients. Then, by the Bruhat decomposition, $\{[\bar{\Lambda}^P_w]\}_{w\in W^P}$ is an integral basis of $H^*(G/P)$. Define the basis $\{x_1,\dots,x_n\}$ of $\mathfrak{h}$ dual to the basis $\{\alpha_1,\dots,\alpha_n\}$ of $\mathfrak{h}^*$, i.e.,

$$\alpha_j(x_i)=\delta_{i,j}.$$

Let $X(H)_+$ be the set of dominant characters of H. For any $\lambda\in X(H)_+$, let V_λ be the finite dimensional irreducible G-module with highest weight λ. This sets up a bijective correspondence between $X(H)_+$ and the set of isomorphism classes of finite dimensional irreducible G-modules. Taking the derivative, we get an embedding $X(H)_+\hookrightarrow D$, where

$$D:=\{\lambda\in\mathfrak{h}^*:\lambda(\alpha_i^\vee)\in\mathbb{R}_+,\forall\alpha_i^\vee\}$$

is the set of dominant weights. If G is simply-connected, $X(H)_+$ can be identified with

$$D_{\mathbb{Z}}:=\{\lambda\in\mathfrak{h}^*:\lambda(\alpha_i^\vee)\in\mathbb{Z}_+,\forall\alpha_i^\vee\},$$

where $\mathbb{Z}_+$ is the set of nonnegative integers.

We now give more specific details about the groups $\mathrm{SL}(n+1),\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ below (see, e.g., [BL, Chapter 3]), since they will be of special interest to us in the paper.

2.1. Special linear group $\mathrm{SL}(n+1)$**.** In this case we take B to be the (standard) Borel subgroup consisting of upper triangular matrices of determinant 1, and H to be the subgroup consisting of diagonal matrices (of determinant 1). Then,

$$\mathfrak{h}=\{\mathbf{t}=\mathrm{diag}(t_1,\dots,t_{n+1}):\sum t_i=0\},$$

and (by the expression of α_i given below)

$$\mathfrak{h}_+=\{\mathbf{t}\in\mathfrak{h}:t_i\in\mathbb{R}\,\mathrm{and}\,t_1\geq\cdots\geq t_{n+1}\}.$$

For any $1\leq i\leq n$,

$$\alpha_i(\mathbf{t})=t_i-t_{i+1};\alpha_i^\vee=\mathrm{diag}(0,\dots,0,1,-1,0,\dots,0);\omega_i(\mathbf{t})=t_1+\cdots+t_i,$$

where 1 is placed in the i-th place.

The Weyl group W can be identified with the symmetric group S_{n+1} which acts via the permutation of the coordinates of $\mathbf{t}$. Let $\{r_1,\dots,r_n\}\subset S_{n+1}$ be the (simple) reflections corresponding to the simple roots $\{\alpha_1,\dots,\alpha_n\}$ respectively. Then,

$$r_i=(i,i+1).$$

For any $1\leq m\leq n$, let $P_m\supset B$ be the (standard) maximal parabolic subgroup of $\mathrm{SL}(n+1)$ such that its unique Levi subgroup L_m containing H has for its simple roots $\{\alpha_1, \dots, \hat{\alpha}_m, \dots, \alpha_n\}$. Then, $\mathrm{SL}(n+1)/P_m$ can be identified with the Grassmannian $\mathrm{Gr}(m, n+1) = \mathrm{Gr}(m, \mathbb{C}^{n+1})$ of m-dimensional subspaces of $\mathbb{C}^{n+1}$. Moreover, the set of minimal coset representatives W^{P_m} of W/W_{P_m} can be identified with the set of m-tuples

$$S(m, n+1) = \{A := 1 \le a_1 < \cdots < a_m \le n+1\}.$$

Any such m-tuple A represents the permutation

$$v_A = (a_1, \dots, a_m, a_{m+1}, \dots, a_{n+1}),$$

where $\{a_{m+1} < \cdots < a_{n+1}\} = [n+1] \setminus \{a_1, \dots, a_m\}$ and

$$[n+1] := \{1, \dots, n+1\}.$$

For a complete flag $E_\bullet : 0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{n+1} = \mathbb{C}^{n+1}$, and $A \in S(m, n+1)$, define the corresponding *shifted Schubert cell* inside $\mathrm{Gr}(m, n+1)$:

$$\Omega_A(E_\bullet) = \{M \in \mathrm{Gr}(m, n+1) : \text{for any}\, 0 \le \ell \le m \,\text{and any} \\ a_\ell \le b < a_{\ell+1}, \dim M \cap E_b = \ell\},$$

where we set $a_0 = 0$ and $a_{m+1} = n+1$. Then, $\Omega_A(E_\bullet) = g(E_\bullet)\Lambda_{v_A}^{P_m}$, where $g(E_\bullet)$ is an element of $\mathrm{SL}(n+1)$ which takes the standard flag $E_\bullet^o$ to the flag $E_\bullet$. (Observe that $g(E_\bullet)$ is determined up to the right multiplication by an element of B.) Its closure in $\mathrm{Gr}(m, n+1)$ is denoted by $\bar{\Omega}_A(E_\bullet)$ and its cycle class in $H^*(\mathrm{Gr}(m, n+1))$ by $[\bar{\Omega}_A]$. (Observe that the cohomology class $[\bar{\Omega}_A]$ does not depend upon the choice of $E_\bullet$.) For the standard flag $E_\bullet = E_\bullet^o$, we thus have $\Omega_A(E_\bullet) = \Lambda_{v_A}^{P_m}$.

Remark 8. Note that, in the literature, it is more common to denote the Schubert cell by $\Omega_A^o(E_\bullet)$ and its closure by $\Omega_A(E_\bullet)$. For notational uniformity we have denoted the Schubert cell by $\Omega_A(E_\bullet)$, and its closure by $\bar{\Omega}_A(E_\bullet)$.

2.2. Symplectic group $\mathrm{Sp}(2n)$**.** Let $V = \mathbb{C}^{2n}$ be equipped with the nondegenerate symplectic form $\langle\, , \rangle$ so that its matrix $\big(\langle e_i, e_j\rangle\big)_{1 \le i,j \le 2n}$ in the standard basis $\{e_1, \dots, e_{2n}\}$ is given by

$$E = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix},$$

where J is the anti-diagonal matrix $(1, \dots, 1)$ of size n. Let

$$\mathrm{Sp}(2n) := \{g \in \mathrm{SL}(2n) : g \,\text{leaves the form}\, \langle\, , \rangle \,\text{invariant}\}$$

be the associated symplectic group. Clearly, $\mathrm{Sp}(2n)$ can be realized as the fixed point subgroup G^σ under the involution $\sigma : G \to G$ defined by $\sigma(A) = E(A^t)^{-1}E^{-1}$, where $G = \mathrm{SL}(2n)$. The involution σ keeps both of B and H stable, where B and H are as in the $\mathrm{SL}(2n)$ case. Moreover, B^σ (respectively, H^σ) is a Borel subgroup (respectively, a maximal torus) of $\mathrm{Sp}(2n)$. We denote

B^σ, H^σ by $B^C = B^{C_n}, H^C = H^{C_n}$ respectively and (when confusion is likely) B, H by $B^{A_{2n-1}}, H^{A_{2n-1}}$ respectively (for $\mathrm{SL}(2n)$). Then, the Lie algebra of H^C (the Cartan subalgebra $\mathfrak{h}^C$),

$$\mathfrak{h}^C = \{\mathrm{diag}(t_1, \dots, t_n, -t_n, \dots, -t_1) : t_i \in \mathbb{C}\}.$$

Let $\Delta^C = \{\beta_1, \dots, \beta_n\}$ be the set of simple roots. Then, for any $1 \leq i \leq n$, $\beta_i = \alpha_{i|\mathfrak{h}^C}$, where $\{\alpha_1, \dots, \alpha_{2n-1}\}$ are the simple roots of $\mathrm{SL}(2n)$. The corresponding (simple) coroots $\{\beta_1^\vee, \dots, \beta_n^\vee\}$ are given by

$$\beta_i^\vee = \alpha_i^\vee + \alpha_{2n-i}^\vee, \ \text{for}\, 1 \leq i < n,$$

and

$$\beta_n^\vee = \alpha_n^\vee.$$

Thus,

$$\mathfrak{h}_+^C = \{\mathrm{diag}(t_1, \dots, t_n, -t_n, \dots, -t_1) : \text{each } t_i \text{ is real and}\, t_1 \geq \cdots \geq t_n \geq 0\}.$$

Moreover, $\mathfrak{h}_+^{A_{2n-1}}$ is σ-stable and

$$\left(\mathfrak{h}_+^{A_{2n-1}}\right)^\sigma = \mathfrak{h}_+^C.$$

Let $\{s_1, \dots, s_n\}$ be the (simple) reflections in the Weyl group $W^C = W^{C_n}$ of $\mathrm{Sp}(2n)$ corresponding to the simple roots $\{\beta_1, \dots, \beta_n\}$ respectively. Since $H^{A_{2n-1}}$ is σ-stable, there is an induced action of σ on the Weyl group S_{2n} of $\mathrm{SL}(2n)$. The Weyl group W^C can be identified with the subgroup of S_{2n} consisting of σ-invariants:

$$\{(a_1, \dots, a_{2n}) \in S_{2n} : a_{2n+1-i} = 2n + 1 - a_i \,\forall 1 \leq i \leq 2n\}.$$

In particular, $w = (a_1, \dots, a_{2n}) \in W^C$ is determined from $(a_1, \dots, a_n)$.

Under the inclusion $W^C \subset S_{2n}$, we have

$$\begin{aligned} s_i &= r_i r_{2n-i}, \ \text{ if } \ 1 \leq i \leq n-1, \\ &= r_n, \ \text{ if } \ i = n. \end{aligned} \tag{1}$$

Moreover, for any $u, v \in W^C$ such that $\ell^C(uv) = \ell^C(u) + \ell^C(v)$, we have

$$\ell^{A_{2n-1}}(uv) = \ell^{A_{2n-1}}(u) + \ell^{A_{2n-1}}(v), \tag{2}$$

where $\ell^C(w)$ denotes the length of w as an element of the Weyl group W^C of $\mathrm{Sp}(2n)$ and similarly for $\ell^{A_{2n-1}}$.

For $1 \leq r \leq n$, we let $\mathrm{IG}(r, 2n) = \mathrm{IG}(r, V)$ to be the set of r-dimensional isotropic subspaces of V with respect to the form $\langle\, ,\, \rangle$, i.e.,

$$\mathrm{IG}(r, 2n) := \{M \in \mathrm{Gr}(r, 2n) : \langle v, v' \rangle = 0, \ \forall\, v, v' \in M\}.$$

Then, it is the quotient $\mathrm{Sp}(2n)/P_r^C$ of $\mathrm{Sp}(2n)$ by the standard maximal parabolic subgroup P_r^C with $\Delta^C \setminus \{\beta_r\}$ as the set of simple roots of its Levi component L_r^C. (Again we take L_r^C to be the unique Levi subgroup of P_r^C

containing H^C.) It can be easily seen that the set W_r^C of minimal-length coset representatives of $W^C/W_{P_r^C}$ is identified with the set

$$\mathfrak{S}(r, 2n) = \{I := 1 \leq i_1 < \cdots < i_r \leq 2n \text{ and } I \cap \bar{I} = \emptyset\},$$

where

$$\bar{I} := \{2n+1-i_1, \ldots, 2n+1-i_r\}. \tag{3}$$

Any such I represents the permutation $w_I = (i_1, \ldots, i_n) \in W^C$ by taking $\{i_{r+1} < \cdots < i_n\} = [n] \setminus (I \sqcup \bar{I})$.

Definition 9. A complete flag

$$E_\bullet : 0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{2n} = V$$

is called an isotropic flag if $E_a^\perp = E_{2n-a}$, for $a = 1, \ldots, 2n$. (In particular, E_n is a maximal isotropic subspace of V.)

For an isotropic flag $E_\bullet$ as above, there exists an element $k(E_\bullet) \in \mathrm{Sp}(2n)$ which takes the standard flag $E_\bullet^o$ to the flag $E_\bullet$. (Observe that $k(E_\bullet)$ is determined up to the right multiplication by an element of B^C.)

For any $I \in \mathfrak{S}(r, 2n)$ and any isotropic flag $E_\bullet$, we have the corresponding *shifted Schubert cell* inside $\mathrm{IG}(r, V)$:

$$\Phi_I(E_\bullet) = \{M \in \mathrm{IG}(r, V) : \text{for any } 0 \leq \ell \leq r \text{ and any } i_\ell \leq a < i_{\ell+1}, \dim M \cap E_a = \ell\},$$

where we set $i_0 = 0$ and $i_{r+1} = 2n$. Clearly, set theoretically,

$$\Phi_I(E_\bullet) = \Omega_I(E_\bullet) \cap \mathrm{IG}(r, V); \tag{4}$$

this is also a scheme theoretic equality (cf. Proposition 36(4)). Moreover, $\Phi_I(E_\bullet) = k(E_\bullet)\Lambda_{w_I}^{P_r^C}$. Denote the closure of $\Phi_I(E_\bullet)$ inside $\mathrm{IG}(r, V)$ by $\bar{\Phi}_I(E_\bullet)$ and its cycle class in $H^*(\mathrm{IG}(r, V))$ (which does not depend upon the choice of the isotropic flag $E_\bullet$) by $[\bar{\Phi}_I]$. For the standard flag $E_\bullet = E_\bullet^o$, we have $\Phi_I(E_\bullet) = \Lambda_{w_I}^{P_r^C}$.

2.3. Special orthogonal group $\mathrm{SO}(2n+1)$. Let $V' = \mathbb{C}^{2n+1}$ be equipped with the nondegenerate symmetric form $\langle\, , \rangle$ so that its matrix $E = \big(\langle e_i, e_j\rangle\big)_{1 \leq i,j \leq 2n+1}$ (in the standard basis $\{e_1, \ldots, e_{2n+1}\}$) is the $(2n+1) \times (2n+1)$ antidiagonal matrix with 1's all along the antidiagonal except at the $(n+1, n+1)$-th place where the entry is 2. Note that the associated quadratic form on V' is given by

$$Q(\sum t_i e_i) = t_{n+1}^2 + \sum_{i=1}^n t_i t_{2n+2-i}.$$

Let

$$\mathrm{SO}(2n+1) := \{g \in \mathrm{SL}(2n+1) : g \,\text{leaves the quadratic form}\; Q \;\text{invariant}\}$$

be the associated special orthogonal group. Clearly, $\mathrm{SO}(2n+1)$ can be realized as the fixed point subgroup G^θ under the involution $\theta : G \to G$ defined by $\theta(A) = E^{-1}(A^t)^{-1}E$, where $G = \mathrm{SL}(2n+1)$. The involution θ keeps both B and H stable. Moreover, B^θ (respectively, H^θ) is a Borel subgroup (respectively, a maximal torus) of $\mathrm{SO}(2n+1)$. We denote B^θ, H^θ by $B^B = B^{B_n}, H^B = H^{B_n}$ respectively. Then, the Lie algebra of H^B (the Cartan subalgebra $\mathfrak{h}^B$),

$$\mathfrak{h}^B = \{\mathrm{diag}(t_1, \dots, t_n, 0, -t_n, \dots, -t_1) : t_i \in \mathbb{C}\}.$$

This allows us to identify $\mathfrak{h}^C$ with $\mathfrak{h}^B$ under the map

$$\mathrm{diag}(t_1, \dots, t_n, -t_n, \dots, -t_1) \mapsto \mathrm{diag}(t_1, \dots, t_n, 0, -t_n, \dots, -t_1).$$

Let $\Delta^B = \{\delta_1, \dots, \delta_n\}$ be the set of simple roots. Then, for any $1 \le i \le n$, $\delta_i = \alpha_{i|\mathfrak{h}^B}$, where $\{\alpha_1, \dots, \alpha_{2n}\}$ are the simple roots of $\mathrm{SL}(2n+1)$. The corresponding (simple) coroots $\{\delta_1^\vee, \dots, \delta_n^\vee\}$ are given by

$$\delta_i^\vee = \alpha_i^\vee + \alpha_{2n+1-i}^\vee, \;\text{ for}\, 1 \le i < n,$$

and

$$\delta_n^\vee = 2(\alpha_n^\vee + \alpha_{n+1}^\vee).$$

Thus, under the above identification,

$$\mathfrak{h}_+^B = \mathfrak{h}_+^C.$$

Moreover, $\mathfrak{h}_+^{A_{2n}}$ is θ-stable and

$$\left(\mathfrak{h}_+^{A_{2n}}\right)^\theta = \mathfrak{h}_+^B.$$

Let $\{s'_1, \dots, s'_n\}$ be the (simple) reflections in the Weyl group $W^B = W^{B_n}$ of $\mathrm{SO}(2n+1)$ corresponding to the simple roots $\{\delta_1, \dots, \delta_n\}$ respectively. Since $H^{A_{2n}}$ is θ-stable, there is an induced action of θ on the Weyl group S_{2n+1} of $\mathrm{SL}(2n+1)$. The Weyl group W^B can be identified with the subgroup of S_{2n+1} consisting of θ-invariants:

$$\{(a_1, \dots, a_{2n+1}) \in S_{2n+1} : a_{2n+2-i} = 2n+2-a_i \,\forall 1 \le i \le 2n+1\}.$$

In particular, $w = (a_1, \dots, a_{2n+1}) \in W^B$ is determined from $(a_1, \dots, a_n)$. (Observe that $a_{n+1} = n+1$.) We can identify the Weyl groups $W^C \simeq W^B$ under the map $(a_1, \dots, a_{2n}) \mapsto (b_1, \dots, b_n, n+1, b_{n+1}, \dots, b_{2n})$, where $b_i = a_i$ if $a_i \le n$ and $b_i = a_{i+1}$ if $a_i > n$.

Under the inclusion $W^B \subset S_{2n+1}$, we have

$$
\begin{aligned}
s'_i &= r_i r_{2n+1-i}, \text{ if } 1 \le i \le n-1, \\
&= r_n r_{n+1} r_n, \text{ if } i = n.
\end{aligned} \tag{5}
$$

For $1 \le r \le n$, we let $\mathrm{OG}(r, 2n+1) = \mathrm{OG}(r, V')$ to be the set of r-dimensional isotropic subspaces of V' with respect to the quadratic form Q, i.e.,

$$\mathrm{OG}(r, 2n+1) := \{M \in \mathrm{Gr}(r, V') : Q(v) = 0, \ \forall\, v \in M\}.$$

Then, it is the quotient $\mathrm{SO}(2n+1)/P_r^B$ of $\mathrm{SO}(2n+1)$ by the standard maximal parabolic subgroup P_r^B with $\Delta^B \setminus \{\delta_r\}$ as the set of simple roots of its Levi component L_r^B. (Again we take L_r^B to be the unique Levi subgroup of P_r^B containing H^B.) It can be easily seen that the set W_r^B of minimal-length coset representatives of $W^B/W_{P_r^B}$ is identified with the set

$$\mathfrak{S}'(r, 2n+1) = \{J := 1 \le j_1 < \cdots < j_r \le 2n+1, j_p \ne n+1 \\ \text{for any } p \text{ and } J \cap \bar{J}' = \emptyset\},$$

where

$$\bar{J}' := \{2n+2-j_1, \ldots, 2n+2-j_r\}.$$

Any such J represents the permutation $w'_J = (j_1, \ldots, j_n) \in W^B$ by taking $\{j_{r+1} < \cdots < j_n\} = [n] \setminus (J \sqcup \bar{J}')$.

Similar to the Definition 9 of isotropic flags on V, we have the notion of isotropic flags on V'. Then, for an isotropic flag $E'_\bullet$, there exists an element $k(E'_\bullet) \in \mathrm{SO}(2n+1)$ which takes the standard flag ${E'}^o_\bullet$ to the flag $E'_\bullet$. (Observe that $k(E'_\bullet)$ is determined up to the right multiplication by an element of B^B.)

For any $J \in \mathfrak{S}'(r, 2n+1)$ and any isotropic flag $E'_\bullet$, we have the corresponding *shifted Schubert cell* inside $\mathrm{OG}(r, V')$:

$$\Psi_J(E'_\bullet) = \{M \in \mathrm{OG}(r, V') : \text{ for any } 0 \le \ell \le r \text{ and} \\ \text{any } j_\ell \le a < j_{\ell+1}, \dim M \cap E'_a = \ell\},$$

where we set $j_0 = 0$ and $j_{r+1} = 2n+1$. Clearly, set theoretically,

$$\Psi_J(E'_\bullet) = \Omega_J(E'_\bullet) \cap \mathrm{OG}(r, V'); \tag{6}$$

this is also a scheme theoretic equality. Moreover, $\Psi_J(E'_\bullet) = k(E'_\bullet)\Lambda_{w'_J}^{P_r^B}$. Denote the closure of $\Psi_J(E'_\bullet)$ inside $\mathrm{OG}(r, V')$ by $\bar{\Psi}_J(E'_\bullet)$ and its cycle class in $H^*(\mathrm{OG}(r, V'))$ (which does not depend upon the choice of the isotropic flag $E'_\bullet$) by $[\bar{\Psi}_J]$. For the standard flag $E'_\bullet = {E'}^o_\bullet$, we have $\Psi_J(E'_\bullet) = \Lambda_{w'_J}^{P_r^B}$.

3. Isotropic flags and proper intersections of Schubert cells in $\mathrm{Gr}(m, V)$

Fix a positive integer s. Let $V = \mathbb{C}^{2n}$ be equipped with the nondegenerate symplectic form $\langle\, , \rangle$ as in Section 2, and let $1 \le m \le n$ be a positive integer. Let $A^1, \dots, A^s \in S(m, 2n)$. The following theorem is a key technical result that underlies the proof of our theorem on the comparison of eigencone for $\mathrm{Sp}(2n)$ with that of $\mathrm{SL}(2n)$.

Theorem 10. *Let $E^1_\bullet, \dots, E^s_\bullet$ be isotropic flags on V in general position. Then, the intersection of subvarieties $\bigcap_{j=1}^s \Omega_{A^j}(E^j_\bullet)$ inside $\mathrm{Gr}(m, V)$ is proper (possibly empty). Thus, for general isotropic flags $\{E^j_\bullet\}$, the intersection $\bigcap_{j=1}^s \bar{\Omega}_{A^j}(E^j_\bullet)$ inside $Gr(m, V)$ is also proper.*

After our paper was circulated with the original proof of this theorem, F. Sottile gave a shorter proof. In fact, he proved a stronger result that the above intersection is transverse. The essential ingredient in his proof of transversality is a recent paper of Mukhin-Tarasov-Varchenko [MTV]. Since his proof is shorter (cf. [So]), we have removed the details of our proof and only outline a brief idea below in Section 3.1.

As an immediate consequence of the above theorem, we get the following:

Corollary 11. *Let $1 \le m \le n$ and let $I^1, \dots, I^s \in \mathfrak{S}(m, 2n)$ be such that*

$$\prod_{j=1}^{s} [\bar{\Phi}_{I^j}] \neq 0 \in H^*(\mathrm{IG}(m, 2n)).$$

Then, $\prod_{j=1}^s [\bar{\Omega}_{I^j}] \neq 0 \in H^*(\mathrm{Gr}(m, 2n))$.

Proof. Observe that by [F_2, Proposition 7.1 and Section 12.2],

$$\prod_{j=1}^{s} [\bar{\Phi}_{I^j}] \neq 0 \text{ if and only if } \bigcap_{j=1}^{s} \bar{\Phi}_{I^j}(E^j_\bullet) \neq \emptyset \tag{7}$$

for isotropic flags $\{E^j_\bullet\}$ such that the above intersection is proper. Let $\{E^j_\bullet\}$ be general isotropic flags on V. Thus, by assumption, $\bigcap_{j=1}^s \bar{\Phi}_{I^j}(E^j_\bullet) \neq \emptyset$ for such flags $\{E^j_\bullet\}$. From Equation (4), we conclude that $\bigcap_{j=1}^s \bar{\Omega}_{I^j}(E^j_\bullet) \neq \emptyset$. Thus, by the above theorem, and using Equation (7) for $\mathrm{Gr}(m, V)$, the corollary follows. □

3.1. Sketch of the proof of Theorem 10. Choose isotropic flags $\{E^j_\bullet\}_{1 \le j \le s}$ such that the intersection $\bigcap_{j=1}^s \Phi_{I^j}(E^j_\bullet)$ is transverse and dense in $\bigcap_{j=1}^s \bar{\Phi}_{I^j}(E^j_\bullet)$ for all $I^j \in \mathfrak{S}(r, 2n)$ and all $1 \le r \le m \le n$ (cf. [BK, Proposition 3]). For any irreducible component C of $\bigcap_{j=1}^s \Omega_{A^j}(E^j_\bullet)$, there exists a dense open subset U of C and an r together with subsets $\{I^j\}_{1 \le j \le s} \subset \mathfrak{S}(r, 2n)$

such that $U \subset \mathcal{L}(m,r,2n)$ and $\pi(U) \subset \bigcap_{j=1}^s \Phi_{I^j}(E_\bullet^j)$, where

$$\mathcal{L}(m,r,2n) = \{X \in \mathrm{Gr}(m,V) : \dim(X \cap X^\perp) = r\},$$

and $\pi : \mathcal{L}(m,r,2n) \to \mathrm{IG}(r,2n)$ is the map $X \mapsto X \cap X^\perp$.

It suffices to show that the dimension of any irreducible component of $\bigcap_{j=1}^s \Phi_{I^j,A^j}(E_\bullet^j)$ is no greater than $\dim \mathrm{Gr}(m,2n) - \sum_{j=1}^s \mathrm{codim}(\Omega_{A^j})$, where, for an isotropic flag $E_\bullet$ on V, and subsets $A \in S(m,2n)$ and $I \in \mathfrak{S}(r,2n)$,

$$\Phi_{I,A}(E_\bullet) = \{X \in \mathrm{Gr}(m,V) :\ X \in \Omega_A(E_\bullet), X \cap X^\perp \in \Phi_I(E_\bullet)\} \subset \mathcal{L}(m,r,2n).$$

Take (generic) isotropic flags $\{E_\bullet^j\}_{1\le j\le s}$ such that the intersection $\bigcap_{j=1}^s \Phi_{I^j,A^j}(E_\bullet^j)$ in $\mathcal{L}(m,r,2n)$ is proper for all the choices of $I^j \in \mathfrak{S}(r,2n)$ and $A^j \in S(m,2n)$. This is possible since $\mathrm{Sp}(2n)$ acts transitively on $\mathcal{L}(m,r,2n)$.

Further, any irreducible component of $\bigcap_{j=1}^s \Phi_{I^j,A^j}(E_\bullet^j)$ has dimension no greater than

$$\begin{aligned}
&\dim \mathrm{Gr}(m,2n) - (r(r-1)/2)\\
&\qquad - \sum_{j=1}^s \Big(\dim \mathrm{Gr}(m,2n) - r(r-1)/2 - (\dim \Omega_{A^j} - \textstyle\bigwedge^2(I^j))\Big)\\
&\quad = \dim \mathrm{Gr}(m,2n) - \big(\sum_{j=1}^s \mathrm{codim}\, \Omega_{A^j}\big) - \Big(r(r-1)/2 - \sum_{j=1}^s \mathrm{co} \textstyle\bigwedge^2(I^j)\Big),
\end{aligned}$$

where $\wedge^2(I^j)$ is defined in Section 8 and $\mathrm{co} \wedge^2 (I^j) := \frac{r(r-1)}{2} - \wedge^2(I^j)$. Moreover, this last quantity is no greater than

$$\dim \mathrm{Gr}(m,2n) - \sum_{j=1}^s \mathrm{codim}\, \Omega_{A^j},$$

as desired. *Both the above assertions are nontrivial, but we do not include a proof.* □

3.2. Analogue of Theorem 10 for $\mathrm{SO}(2n+1)$**.** We originally obtained the following result by a proof similar to that of Theorem 10. Again, similar to the $\mathrm{Sp}(2n)$ case, a shorter proof is given by Sottile [So].

Theorem 12. *Let $A^1, \dots, A^s$ be subsets of $[2n+1]$ each of cardinality m. Let ${E'}_\bullet^1, \dots, {E'}_\bullet^s$ be isotropic flags on $V' = \mathbb{C}^{2n+1}$ in general position. Then, the intersection $\bigcap_{j=1}^s \Omega_{A^j}({E'}_\bullet^j)$ of subvarieties of $\mathrm{Gr}(m,V')$ is proper.*

As an immediate consequence of the above theorem (just as in the case of $\mathrm{Sp}(2n)$), we get the following:

Corollary 13. *Let $1 \le m \le n$ and let $J^1, \dots, J^s \in \mathfrak{S}'(m, 2n+1)$ be such that*

$$\prod_{j=1}^{s} [\bar{\Psi}_{J^j}] \neq 0 \in H^*(\mathrm{OG}(m, 2n+1)).$$

Then, $\prod_{j=1}^{s} [\bar{\Omega}_{J^j}] \neq 0 \in H^*(\mathrm{Gr}(m, 2n+1))$.

4. Comparison of eigencone for $\mathrm{Sp}(2n)$ with that of $\mathrm{SL}(2n)$

Let G be a connected (complex) semisimple group. Choose a maximal compact subgroup K of G with Lie algebra $\mathfrak{k}$. Then, there is a natural homeomorphism $C : \mathfrak{k}/K \to \mathfrak{h}_+$, where K acts on $\mathfrak{k}$ by the adjoint representation and $\mathfrak{h}_+$ is the positive Weyl chamber in $\mathfrak{h}$ as in Section 2. The inverse map C^{-1} takes any $h \in \mathfrak{h}_+$ to the K-conjugacy class of $\sqrt{-1}h$.

For a positive integer s, the *eigencone* is defined as the cone:

$$\Gamma(s, K) := \{(h_1, \dots, h_s) \in \mathfrak{h}_+^s \mid \exists (k_1, \dots, k_s) \in \mathfrak{k}^s \colon \sum_{j=1}^{s} k_j = 0 \text{ and } C(k_j) = h_j \forall j = 1, \dots, s\}.$$

Given a standard maximal parabolic subgroup P, let ω_P denote the corresponding fundamental weight, i.e., $\omega_P(\alpha_i^\vee) = 1$, if $\alpha_i \in \Delta \setminus \Delta(P)$ and 0 otherwise, where $\Delta(P)$ is the set of simple roots for the Levi subgroup L of P containing H. Then, ω_P is invariant under the Weyl group W_P of P.

We recall the following theorem from [BeSj].

Theorem 14. *Let $(h_1, \dots, h_s) \in \mathfrak{h}_+^s$. Then, the following are equivalent:*

(a) $(h_1, \dots, h_s) \in \Gamma(s, K)$.

(b) *For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \dots, w_s) \in (W^P)^s$ such that*

$$[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] = d[\bar{\Lambda}_e^P] \in H^*(G/P), \text{ for some nonzero } d,$$

the following inequality holds:

$$\omega_P(\sum_{j=1}^{s} w_j^{-1} h_j) \le 0.$$

In fact, assume that (a) *is satisfied, i.e., $(h_1, \dots, h_s) \in \Gamma(s, K)$. Then, for every standard maximal parabolic subgroup P in G and every choice of*

s-tuples $(w_1, \dots, w_s) \in (W^P)^s$ *such that*

$$[\bar{\Lambda}^P_{w_1}] \cdot \ \dots \ \cdot [\bar{\Lambda}^P_{w_s}] \neq 0 \in H^*(G/P),$$

the following inequality holds:

$$\omega_P(\sum_{j=1}^{s} w_j^{-1} h_j) \leq 0.$$

Recall that $\mathfrak{h}^C_+$ (respectively, $\mathfrak{h}^B_+$) is the dominant chamber in the Cartan subalgebra of $\mathrm{Sp}(2n)$ (respectively, $\mathrm{SO}(2n+1)$) as in Section 2.

The following theorem is our main result on the comparison of the eigencone for $\mathrm{Sp}(2n)$ with that of $\mathrm{SL}(2n)$ (and also for $\mathrm{SO}(2n+1)$ with that of $\mathrm{SL}(2n+1)$).

Theorem 15. (a) *For* $h_1, \dots, h_s \in \mathfrak{h}^C_+$,

$$(h_1, \dots, h_s) \in \Gamma(s, \mathrm{Sp}(2n)) \Leftrightarrow (h_1, \dots, h_s) \in \Gamma(s, SU(2n)).$$

(b) *For* $h_1, \dots, h_s \in \mathfrak{h}^B_+$,

$$(h_1, \dots, h_s) \in \Gamma(s, \mathrm{SO}(2n+1)) \Leftrightarrow (h_1, \dots, h_s) \in \Gamma(s, SU(2n+1)).$$

(Observe that by Section 2, $\mathfrak{h}^C_+ \subset \mathfrak{h}^{A_{2n-1}}_+$ *and* $\mathfrak{h}^B_+ \subset \mathfrak{h}^{A_{2n}}_+$*.)*

Proof. Clearly, $\Gamma(s, \mathrm{Sp}(2n)) \subseteq \Gamma(s, SU(2n))$. Conversely, we need to show that if $\mathbf{h} = (h_1, \dots, h_s) \in (\mathfrak{h}^C_+)^s$ is such that $\mathbf{h} \in \Gamma(s, SU(2n))$, then $\mathbf{h} \in \Gamma(s, \mathrm{Sp}(2n))$. Take any $1 \leq m \leq n$ and any $I^1, \dots, I^s \in \mathfrak{S}(m, 2n)$ such that

$$[\bar{\Phi}_{I^1}] \cdot \ \dots \ \cdot [\bar{\Phi}_{I^s}] = d[\bar{\Phi}_e] \in H^*(\mathrm{IG}(m, 2n)) \text{ for some nonzero } d.$$

By Corollary 11,

$$[\bar{\Omega}_{I^1}] \cdot \ \dots \ \cdot [\bar{\Omega}_{I^s}] \neq 0 \in H^*(\mathrm{Gr}(m, 2n)).$$

In particular, by Theorem 14 applied to $SU(2n)$,

$$\omega_m(\sum_{j=1}^{s} v_{I^j}^{-1} h_j) \leq 0,$$

where ω_m is the m-th fundamental weight of $\mathrm{SL}(2n)$ and $v_{I^j} \in S_{2n}$ is the element associated to I^j as in Subsection 2.1. It is easy to see that the m-th fundamental weight ω^C_m of $\mathrm{Sp}(2n)$ is the restriction of ω_m to $\mathfrak{h}^C$. Moreover, even though the elements $v_{I^j} \in S_{2n}$ and $w_{I^j} \in W^C$ are, in general, different, we still have

$$\omega_m(v_{I^j}^{-1} h_j) = \omega^C_m(w_{I^j}^{-1} h_j).$$

Applying Theorem 14 for $\mathrm{Sp}(2n)$, we get the (a)-part of the theorem.

The proof for $\mathrm{SO}(2n+1)$ is similar. (Apply Corollary 13 instead of Corollary 11.) □

5. A basic transversality result

Let M be an r-dimensional space and $\mathcal{F} = (F^1_\bullet, \dots, F^s_\bullet)$ an s-tuple of complete flags on M. As earlier, let V be a $2n$-dimensional vector space equipped with a nondegenerate symplectic form, and $\mathcal{G} = (G^1_\bullet, \dots, G^s_\bullet)$ an s-tuple of (complete) isotropic flags on V. Let $\mu = (\mu^1, \dots, \mu^s)$, where μ^j is an ordered sequence with r elements $2n \geq \mu^j_1 \geq \cdots \geq \mu^j_r \geq 0, j \in [s]$. We fix μ in this section once and for all.

Make the definition:

$$\mathcal{H}_\mu(\mathcal{F}, \mathcal{G}) = \{\phi \in \operatorname{Hom}(M, V) : \phi(F^j_a) \subset G^j_{2n-\mu^j_a}, a \in [r], j \in [s]\}.$$

(In the next section, we will need to use $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$ in the case when $G^j_\bullet$ is an arbitrary complete flag on V, not necessarily an isotropic flag.)

Now assume that $(\mathcal{F}, \mathcal{G})$ is a generic point of $\mathrm{Fl}(M)^s \times \mathrm{IFl}(V)^s$, where $\mathrm{IFl}(V)$ is the full isotropic flag variety of V and $\mathrm{Fl}(M)$ is the full flag variety of M. Then,

Theorem 16. *The following are equivalent:*

(A) $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$ *is of the expected dimension* $2nr - \sum_{j=1}^s |\mu^j|$, *where* $|\mu^j| := \sum_{a=1}^r \mu^j_a$.

(B) *For any* $1 \leq d \leq r$ *and subsets* $B^1, \dots, B^s$ *of* $[r]$ *each of cardinality* d *such that the product* $\prod_{j=1}^s [\bar{\Omega}_{B^j}] \neq 0 \in H^*(\mathrm{Gr}(d, r))$, *the inequality* $\sum_{j=1}^s \sum_{a \in B^j} \mu^j_a \leq 2dn$ *holds.*

We record the following corollary.

Corollary 17. *Suppose that* $2nr - \sum_{j=1}^s |\mu^j| = 0$. *Then* (A) *(or* (B)*) is equivalent to the condition*

$$\left(V_{\mu^1} \otimes V_{\mu^2} \otimes \cdots \otimes V_{\mu^s}\right)^{SL(r)} \neq 0,$$

where V_{μ^j} *is the irreducible representation of* $\mathrm{SL}(r)$ *as in Section* 6.2.

Proof. The equivalence of the condition in the corollary and condition (B) in Theorem 16 is a consequence of the Knutson-Tao saturation theorem for $\mathrm{SL}(n)$ [KT], together with Klyachko's work [K]: these works together characterize the existence of invariants in a tensor product of $\mathrm{SL}(r)$ representations by a system of inequalities, which is (B) (cf. [F_4]). □

5.1. (A) **implies** (B) **in Theorem 16.** This follows from [B_1, Proposition 2.8(1)]. The idea is that if (B) fails, pick an $S \in \bigcap_{j=1}^s \Omega_{B^j}(F^j_\bullet) \subset \mathrm{Gr}(d, M)$, compute the expected dimension of the vector space $\{\phi \in \mathcal{H}_\mu(\mathcal{F}, \mathcal{G}) : \phi(S) = 0\}$, and find it to be greater than the expected dimension of $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$.

5.2. (B) **implies** (A) **in Theorem 16.** We proceed by induction on r. To show that (B) implies (A), suppose ϕ^o is a general member of $\mathcal{H}_\mu(\mathcal{F},\mathcal{G})$, $S=\ker(\phi^o)$ and assume $S\in\bigcap_{j=1}^s \Omega_{B^j}(F^j_\bullet)\subset \mathrm{Gr}(d,M)$, for $B^j=\{1\le b^j_1<\cdots<b^j_d\le r\}$.

We will use ideas of Schofield [S] in the proof. We replace the Ext groups in [S] by the cohomology of suitable (2-term) complexes. There are two other ingredients required (beyond the technique of Schofield).

- A critical dimension count (Proposition 18), for which we need Theorem 10.
- An idea from [B$_1$] on genericity of induced structures (we could have used another technique of Schofield instead as well).

We were unable to use the strategy of [B$_1$, Section 5] (essentially because we could not show that the induced flags on S and M/S were "mutually generic" in the situation here).

5.3. The setup, and the key inequality. Let $L=\mathrm{im}(\phi^o)$. Let

$$\mathcal{A}^0(\mathrm{Hom}(M,V))=\mathrm{Hom}(M,V),$$

and

$$\mathcal{A}^1(\mathrm{Hom}(M,V))=\bigoplus_{j=1}^s \mathrm{Hom}(M,V)/P^j,$$

where

$$P^j=\{\tau\in\mathrm{Hom}(M,V):\tau(F^j_a)\subset G^j_{2n-\mu^j_a},a=1,\dots,r\}.$$

There is a natural differential (direct sum of projections)

$$d:\mathcal{A}^0(\mathrm{Hom}(M,V))\to\mathcal{A}^1(\mathrm{Hom}(M,V)).$$

Call this (2-term) complex $\mathcal{A}^*(\mathrm{Hom}(M,V))$. Clearly, $H^0(\mathcal{A}^*(\mathrm{Hom}(M,V)))$ is the same as $\mathcal{H}_\mu(\mathcal{F},\mathcal{G})$ and $\chi(\mathcal{A}^*(\mathrm{Hom}(M,V)))$ is the "expected dimension" of $\mathcal{H}_\mu(\mathcal{F},\mathcal{G})$, which is $2nr-\sum_{j=1}^s|\mu^j|$. To make the dependence on $(\mathcal{F},\mathcal{G})$ and μ precise, we sometimes write this complex as $\mathcal{A}^*(M,V,\mathcal{F},\mathcal{G},\mu)$.

Hypothesis (A) is clearly equivalent to

(C) $h^1(\mathcal{A}^*(\mathrm{Hom}(M,V)))=0$.

We have surjections

$$\mathrm{Hom}(M,V)\overset{\tau}{\twoheadrightarrow}\mathrm{Hom}(S,V)\overset{\pi}{\twoheadrightarrow}\mathrm{Hom}(S,V/L).$$

Let

$$\mathcal{A}^0(\mathrm{Hom}(S,V))=\mathrm{Hom}(S,V),\ \mathcal{A}^0(\mathrm{Hom}(S,V/L))=\mathrm{Hom}(S,V/L)$$

and

$$\mathcal{A}^1(\mathrm{Hom}(S,V)) = \bigoplus_{j=1}^{s} \mathrm{Hom}(S,V)/\tau(P^j),$$
$$\mathcal{A}^1(\mathrm{Hom}(S,V/L)) = \mathrm{Hom}(S,V/L)/\pi\circ\tau(P^j).$$

There are natural differentials $d:\mathcal{A}^0\to\mathcal{A}^1$ in each case. It is easy to see that

- $\tau(P^j) = \{\phi\in\mathrm{Hom}(S,V) : \phi(F^j(S)_x)\subset G^j_{2n-\mu^j_{b^j_x}}, x\in[d]\}$, where $F^j(S)_\bullet$ is the induced complete flag on S with the changed label by the dimension and $1\le b^j_x\le r$ is the smallest integer such that $F^j(S)_x = S\cap F^j_{b^j_x}$. So, if we let $\gamma^j_x = \mu^j_{b^j_x}$, the (2-term) complex $\mathcal{A}^*(\mathrm{Hom}(S,V))$ is the same as $\mathcal{A}^*(S,V,\mathcal{F}(S),\mathcal{G},\gamma)$, where $\mathcal{F}(S)$ is the induced s-tuple of flags on S.
- $\pi\circ\tau(P^j) = \{\phi\in\mathrm{Hom}(S,V/L) : \phi(F^j(S)_x)\subset (G^j_{2n-\gamma^j_x}+L)/L, x\in[d]\}$.

Define numbers δ^j_x by

$$2n-\dim(L)-\delta^j_x = \dim((G^j_{2n-\gamma^j_x}+L)/L) = 2n-\gamma^j_x-\dim(G^j_{2n-\gamma^j_x}\cap L),$$

and hence, $\mathcal{A}^*(\mathrm{Hom}(S,V/L)) = \mathcal{A}^*(S,V/L,\mathcal{F}(S),\mathcal{G}(V/L),\delta)$.

The following critical result will be proved in Section 5.6 by a dimension count.

Proposition 18. $h^1(\mathcal{A}^*(\mathrm{Hom}(M,V)))\le -\chi(\mathcal{A}^*(\mathrm{Hom}(S,V/L)))$.

5.4. Proof of (B) implies (A) in Theorem 16 assuming Proposition 18. The map of (2-term) complexes $\mathcal{A}^*(\mathrm{Hom}(M,V))\to\mathcal{A}^*(\mathrm{Hom}(S,V/L))$ is surjective in each degree. Therefore, the map

$$\theta : H^1(\mathcal{A}^*(\mathrm{Hom}(M,V)))\to H^1(\mathcal{A}^*(\mathrm{Hom}(S,V/L))) \tag{8}$$

is a surjection.

Assume Proposition 18. Hence,

$$h^1(\mathcal{A}^*(\mathrm{Hom}(M,V)))\le -\chi(\mathcal{A}^*(\mathrm{Hom}(S,V/L)))\le h^1(\mathcal{A}^*(\mathrm{Hom}(S,V/L))). \tag{9}$$

The surjection (8) and the inequality (9) imply that θ is an isomorphism. Thus, equality holds in all the inequalities in (9); in particular,

$$h^0(\mathcal{A}^*(\mathrm{Hom}(S,V/L))) = 0.$$

The isomorphism θ factors as

$$H^1(\mathcal{A}^*(\mathrm{Hom}(M,V)))\xrightarrow{\bar\tau} H^1(\mathcal{A}^*(\mathrm{Hom}(S,V)))\to H^1(\mathcal{A}^*(\mathrm{Hom}(S,V/L))),$$

where the map $\bar{\tau}$ is induced from the surjective map of complexes

$$\mathcal{A}^*(\mathrm{Hom}(M,V)) \to \mathcal{A}^*(\mathrm{Hom}(S,V)),$$

and is hence surjective. We conclude that $\bar{\tau}$ is an isomorphism.

It follows from [B$_1$] that the induced flags $(\mathcal{F}(S),\mathcal{G})$ can also be assumed to be suitably generic (see Section 5.5). Now, by induction on the dimension of M, assume the validity of Theorem 16 with M replaced by S and μ^j replaced by γ^j. The hypothesis (B) holds because of Lemma 19 below.

By conclusion (C) (valid by induction), we find that $h^1(\mathcal{A}^*(\mathrm{Hom}(S,V))) = 0$. Hence, $h^1(\mathcal{A}^*(\mathrm{Hom}(M,V))) = 0$ as desired. This completes the proof of Theorem 16.

Lemma 19. *Let $1 \le d' \le d$. In the above situation, suppose $C^1,\dots,C^s$ are subsets of $[d]$ each of cardinality d' such that $\prod_{j=1}^s [\bar{\Omega}_{C^j}] \neq 0 \in H^*(\mathrm{Gr}(d',d))$. Then, if $T^j = \{b^j_a : a \in C^j\}$,*

(i) $\prod_{j=1}^s [\bar{\Omega}_{T^j}] \neq 0 \in H^*(\mathrm{Gr}(d',r))$.

(ii) $\sum_{j=1}^s \sum_{a\in C^j} \gamma^j_a = \sum_{j=1}^s \sum_{b\in T^j} \mu^j_b \le 2d'n$.

Proof. Since $S \in \bigcap_{j=1}^s \Omega_{B^j}(F^j_\bullet) \subset \mathrm{Gr}(d,M)$, and F^j are generic flags on M, the product $\prod_{j=1}^s [\bar{\Omega}_{B^j}] \neq 0 \in H^*(\mathrm{Gr}(d,r))$. Now, using the hypothesis $\prod_{j=1}^s [\bar{\Omega}_{C^j}] \neq 0 \in H^*(\mathrm{Gr}(d',d))$, and [F$_3$, Proposition 1], we conclude that (i) holds. Because of our assumption (B), (i) implies the inequality in (ii). The equality in (ii) is trivial. □

5.5. Genericity of induced structures. Let $U \subseteq \mathrm{Fl}(M)^s \times \mathrm{IFl}(V)^s$ be a nonempty open subset consisting of the pairs of flags which are generic for our dimension calculations. There is a nonempty open subset $\hat{U} \subseteq \mathrm{Fl}(S)^s \times \mathrm{IFl}(V)^s$ of points $(\mathcal{H},\mathcal{G}')$ such that $(\mathcal{H},\mathcal{G}')$ is generic for the application of induction on $\mathcal{A}^*(S,V,\mathcal{H},\mathcal{G}',\gamma)$. The projection of $\hat{U}$ to $\mathrm{IFl}(V)^s$ is a (diagonal) $\mathrm{Sp}(V)$-invariant open subset of $\mathrm{IFl}(V)^s$ which does not depend upon the choice of $S \in \mathrm{Gr}(d,M)$.

We obtain a point $(\mathcal{F}(S),\mathcal{G}) \in \mathrm{Fl}(S)^s \times \mathrm{IFl}(V)^s$ as in Subsection 5.3. Since $\mathcal{G}$ is generic, it may be assumed to be in the projection of $\hat{U}$ to $\mathrm{IFl}(V)^s$. Let $\mathcal{H} \in \mathrm{Fl}(S)^s$ be close to $\mathcal{F}(S)$ (in the complex topology) such that $(\mathcal{H},\mathcal{G}) \in \hat{U}$. Now, find $\mathcal{F}' \in \mathrm{Fl}(M)^s$ so that $\mathcal{F}'(S) = \mathcal{H}$, $\mathcal{F}'(M/S) = \mathcal{F}(M/S)$ and $S \in \bigcap_{j=1}^s \Omega_{B^j}(F'^j_\bullet)$ (see Lemma 2.4 in [B$_1$]). Clearly, $(\mathcal{F}',\mathcal{G})$ is close to $(\mathcal{F},\mathcal{G})$ and therefore is generic in $\mathrm{Fl}(M)^s \times \mathrm{IFl}(V)^s$ (and hence belongs to U). Now $\phi^o \in H^0(\mathcal{A}^*(M,V,\mathcal{F}',\mathcal{G},\mu))$ and we replace $(\mathcal{F},\mathcal{G})$ by $(\mathcal{F}',\mathcal{G})$.

5.6. Proof of Proposition 18. Since we already know the Euler characteristic of $\mathcal{A}^*(\mathrm{Hom}(M,V))$, we want to give a formula for $h^0(\mathcal{A}^*(\mathrm{Hom}(M,V))) = \mathcal{H}_\mu(\mathcal{F},\mathcal{G})$.

If we drop the isotropy condition on the flags $\mathcal{G}$, then we are in a situation where Schofield's original argument can be applied. We will compute

$\mathcal{H}_\mu(\mathcal{F},\mathcal{G})$ by a procedure that "essentially" does not see the isotropy condition on the flags $\mathcal{G}$, and hence Schofield's set up generalizes.

To calculate the dimension of $\mathcal{H}_\mu(\mathcal{F},\mathcal{G})$, we can fiber the universal parameter space of triples $(\phi,\mathcal{F},\mathcal{G})$, with ϕ generic in $\mathcal{H}_\mu(\mathcal{F},\mathcal{G})$, over the parameter space of pairs $(\mathrm{im}(\phi),\mathcal{G})$ so that $\mathrm{im}(\phi)$ is in its generic Schubert state with respect to the flag $\mathcal{G}$. The dimension of the choices of such pairs is the expected one, thanks to our main transversality result Theorem 10.

5.7. Calculation of $\chi(\mathcal{A}^*(\mathrm{Hom}(S,V/L)))$. First, as in Subsection 5.3,

$$\pi\circ\tau(P^j)=\{\theta\in\mathrm{Hom}(S,V/L):\theta(F^j(S)_x)\subset(G^j_{2n-\gamma^j_x}+L)/L,\ x=1,\dots,d\},$$

which has dimension $\sum_{x=1}^d(2n-\gamma^j_x-\dim(L\cap G^j_{2n-\gamma^j_x}))$. Therefore, $\chi(\mathcal{A}^*(\mathrm{Hom}(S,V/L)))$ equals

$$d(2n-r+d)-\sum_{j=1}^s\Big(d(2n-r+d)-\sum_{x=1}^d(2n-\gamma^j_x-\dim(L\cap G^j_{2n-\gamma^j_x}))\Big)$$

$$=d(2n-r+d)+\sum_{j=1}^s\sum_{x=1}^d\big(r-d-\gamma^j_x-\dim(L\cap G^j_{2n-\gamma^j_x})\big). \tag{10}$$

5.8. The dimension of $\mathcal{H}_\mu(\mathcal{F},\mathcal{G})$: Assume that

$$L\in\bigcap_{j=1}^s\Omega_{A^j}(G^j_\bullet)\subset\mathrm{Gr}(r-d,V).$$

Let $U_A(V,s)\subset\mathrm{IFl}(V)^s$ be the open subset consisting of isotropic flags $E^1_\bullet,\dots,E^s_\bullet$ such that the intersection $\bigcap_{j=1}^s\Omega_{A^j}(E^j_\bullet)$ inside $\mathrm{Gr}(r-d,V)$ is proper. By Theorem 10, this is nonempty. Let $\mathcal{U}$ denote the parameter space

$$\{(\mathcal{F},\mathcal{G},\phi,S,L):\mathcal{F}\in\mathrm{Fl}(M)^s,\mathcal{G}\in U_A(V,s),L\in\bigcap_{j=1}^s\Omega_{A^j}(G^j_\bullet),$$
$$S\in\bigcap_{j=1}^s\Omega_{B^j}(F^j_\bullet),\phi\in\mathcal{H}_\mu(\mathcal{F},\mathcal{G}),S=\ker(\phi),L=\mathrm{im}(\phi)\}.$$

Now, the dimension of $\mathcal{H}_\mu(\mathcal{F},\mathcal{G})$ equals the dimension of a generic fiber of $\mathcal{U}\to\mathrm{FL}(M)^s\times U_A(V,s)$. We will therefore need to give an upper bound for the dimension of irreducible components of $\mathcal{U}$.

Clearly, $\mathcal{U}$ maps to the parameter space

$$\mathcal{V} = \{(\mathcal{G}, \phi, S, L) : \mathcal{G} \in U_A(V, s), L \in \bigcap_{j=1}^{s} \Omega_{A^j}(G^j_\bullet), S \in \mathrm{Gr}(d, M),\\ \phi \in \mathrm{Hom}(M, V), S = \ker(\phi), L = \mathrm{im}(\phi)\}.$$

5.8.1. The dimension of $\mathcal{V}$. Clearly $\mathcal{V}$ fibers over the space

$$\mathcal{W} = \{(L, \mathcal{G}) : \mathcal{G} \in U_A(V, s), L \in \bigcap_{j=1}^{s} \Omega_{A^j}(G^j_\bullet)\}$$

with irreducible fibers of dimension $d(r-d) + (r-d)^2$ (the first term is the dimension of the space of choices of S and the second term is the dimension of the space of isomorphisms $M/S \xrightarrow{\phi} L$).

Now, the fiber dimension of $\mathcal{W} \to U_A(V, s)$ is $((r-d)(2n-r+d) - \sum_{j=1}^{s} \mathrm{codim}(\Omega_{A^j}))$. Therefore, the dimension of any irreducible component of $\mathcal{V}$ is no more than

$$\dim(\mathrm{IFl}(V)^s) + \big((r-d)(2n-r+d) - \sum_{j=1}^{s} \mathrm{codim}(\Omega_{A^j})\big) + d(r-d) + (r-d)^2.$$

By simplifying the above expression, we find

$$\dim \mathcal{V} \leq 2nr + (r-d-2n)d + \dim(\mathrm{IFl}(V)^s) - \sum_{j=1}^{s} \mathrm{codim}(\Omega_{A^j}). \tag{11}$$

5.8.2. Codimension of Ω_{A^j}.

Lemma 20. $\dim(\Omega_{A^j}) \leq \sum_{a \notin B^j} (2n - \mu^j_a - \dim(L \cap G^j_{2n-\mu^j_a}))$*, and hence* $\mathrm{codim}(\Omega_{A^j}) \geq \sum_{a \notin B^j} (\mu^j_a + \dim(L \cap G^j_{2n-\mu^j_a}) - (r-d))$.

Proof. Let $A^j = \{a_1 < \cdots < a_{r-d}\}$. Therefore, $\dim(\Omega_{A^j})$ equals $\sum_{b=1}^{r-d}(a_b - b)$. Let $[r] \setminus B^j = \{x_1 < \cdots < x_{r-d}\}$. It therefore suffices to show that, for any $1 \leq b \leq r-d$,

$$a_b - b \leq 2n - \mu^j_{x_b} - \dim(L \cap G^j_{2n-\mu^j_{x_b}}),$$

i.e.,

$$a_b \leq b + 2n - \mu^j_{x_b} - \dim(L \cap G^j_{2n-\mu^j_{x_b}}). \tag{12}$$

Set $\dim(L \cap G^j_{2n-\mu^j_{x_b}}) = b + y$, where $y \geq 0$ (since $L \cap G^j_{2n-\mu^j_{x_b}}$ contains $\phi(F^j_{x_b})$ which is b-dimensional).

Hence, to prove Equation (12), we need to show that $L \cap G^j_{2n-\mu^j_{x_b}-y}$ is at least b-dimensional, which is now immediate because $L \cap G^j_{2n-\mu^j_{x_b}}$ is $(b+y)$-dimensional. ☐

5.8.3. The fiber dimension of $\mathcal{U} \to \mathcal{V}$. We need to do the following basic calculation. Let $S \subset M$ be a d-dimensional subspace, L a subspace of V of dimension $r-d$ and $B \in S(d,r)$.

Lemma 21. *Let $G_\bullet$ be a fixed flag on V, and $\phi : M \to V$ a map with kernel S and image $L \in \Omega_A(G_\bullet)$. The dimension of the space of flags $F_\bullet \in \mathrm{Fl}(M)$, so that $S \in \Omega_B(F_\bullet)$ and $\phi(F_a) \subset G_{2n-\mu_a}$ for $a = 1, \ldots, r$, equals*

$$\dim(\mathrm{Fl}(M)) - \big(\mathrm{codim}(\Omega_B) + \sum_{a \notin B} (r - d - \dim(L \cap G_{2n-\mu_a}))\big).$$

Proof. There are two sets of conditions imposed on $F_\bullet$. The first one is that $S \in \Omega_B(F_\bullet)$; the second is that $\phi(F_a) \subset G_{2n-\mu_a} \cap L$ for $a \notin B$. These conditions are independent. □

Therefore the fiber dimension of $\mathcal{U} \to \mathcal{V}$ equals

$$\dim(\mathrm{Fl}(M)^s) - \sum_{j=1}^{s} \big(\mathrm{codim}(\Omega_{B^j}) + \sum_{a \notin B^j} (r - d - \dim(L \cap G^j_{2n-\mu^j_a}))\big). \tag{13}$$

5.9. Conclusion of the proof of Proposition 18. The dimension of any irreducible component of $\mathcal{U}$ is less than or equal to the sum of the fiber dimension of $\mathcal{U} \to \mathcal{V}$ (given by (13)) and the right hand side of inequality (11). This sum equals

$$\begin{aligned} &\dim(\mathrm{Fl}(M)^s) - \sum_{j=1}^{s} \big(\mathrm{codim}(\Omega_{B^j}) + \sum_{a \notin B^j} (r - d - \dim(L \cap G^j_{2n-\mu^j_a}))\big) \\ &+ 2nr + (r-d-2n)d + \dim(\mathrm{IFl}(V)^s) - \sum_{j=1}^{s} \mathrm{codim}(\Omega_{A^j}). \end{aligned} \tag{14}$$

The dimension of $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$, which equals the dimension of the generic fiber of the map $\mathcal{U} \to \mathrm{Fl}(M)^s \times U_A(V,s)$, is therefore less than or equal to the integer (14) minus the dimension of $\mathrm{Fl}(M)^s \times U_A(V,s)$. Therefore, using Lemma 20,

$$\begin{aligned} &\dim(\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})) \\ &\quad \le -\sum_{j=1}^{s} \big(\mathrm{codim}(\Omega_{B^j}) + \sum_{a \notin B^j} (r - d - \dim(L \cap G^j_{2n-\mu^j_a}))\big) \\ &\qquad + 2nr + d(r-d-2n) - \sum_{j=1}^{s} \sum_{a \notin B^j} (\mu^j_a + \dim(L \cap G^j_{2n-\mu^j_a}) - (r-d)) \\ &\quad = 2nr + d(r-d-2n) - \sum_{j=1}^{s} \mathrm{codim}(\Omega_{B^j}) - \sum_{j=1}^{s} \sum_{a \notin B^j} \mu^j_a. \end{aligned}$$

Now, $h^1(\mathcal{A}^*(M,V))$ equals the dimension of $\mathcal{H}_\mu(\mathcal{F},\mathcal{G})$ minus the Euler characteristic which is $2nr - \sum_{j=1}^{s}\sum_{a=1}^{r}\mu_a^j$. This yields that $h^1(\mathcal{A}^*(M,V))$ is less than or equal to

$$d(r-d-2n) - \sum_{j=1}^{s}\operatorname{codim}(\Omega_{B^j}) + \sum_{j=1}^{s}\sum_{a\in B^j}\mu_a^j, \tag{15}$$

which can be written in an illuminating (and expected) form

$$(\dim \operatorname{Gr}(d,r) - \sum_j \operatorname{codim}(\Omega_{B^j})) + (\sum_{j=1}^{s}\sum_{a\in B^j}\mu_a^j) - 2nd.$$

Comparing Equation (10) with Equation (15), it suffices to show (for each j),

$$\sum_{x=1}^{d}(d-r+b_x^j-x) + \sum_{a\in B^j}\mu_a^j \leq -\sum_{x=1}^{d}((r-d)-\gamma_x^j - \dim(L\cap G^j_{2n-\gamma_x^j})),$$

which rearranges to (since $\gamma_x^j = \mu^j_{b_x^j}$ by definition)

$$\sum_{x=1}^{d}\dim(L\cap G^j_{2n-\gamma_x^j}) \geq \sum_{x=1}^{d}(b_x^j - x).$$

But, this is clear because $L\cap G^j_{2n-\gamma_x^j}$ contains the $b_x^j - x$ dimensional subspace $\phi(F^j_{b_x^j})$. This completes the proof of Proposition 18 and hence Theorem 16 is completely proved. □

6. Tensor product decomposition for SL(r) versus Sp($2n$)

Let G, B be as in the beginning of Section 2.

6.1. Line bundles on G/B. For a character $\chi : B \to \mathbb{C}^*$, let $\mathcal{L}_\chi$ be the corresponding G-equivariant (homogeneous) line bundle on G/B associated to the principal B-bundle $G \to G/B$ via the character χ^{-1}. Recall that its total space is the set of all pairs $\{(g,c) : g\in G, c\in\mathbb{C}\}$ modulo the equivalence

$$(g,c) \sim (gb, \chi(b)c).$$

If χ is a dominant weight, then

$$H^0(G/B, \mathcal{L}_\chi) = V_\chi^*,$$

where (as in Section 2) V_χ is the irreducible representation of G with highest weight χ.

Let $f : G' \to G$ be a morphism of algebraic groups so that $f(B') \subset B$, where B' and B are Borel subgroups of G' and G respectively. Then, f

induces a map $\bar{f} : G'/B' \to G/B$. It can be checked that $f^*\mathcal{L}_\chi = \mathcal{L}_{\chi\circ f}$, where $\chi \circ f$ is the composition of χ with $f_{|B'} : B' \to B$.

Specific identification in the $\mathrm{SL}(r)$ **case:** Let B be the standard Borel subgroup of $\mathrm{SL}(r)$ consisting of upper triangular matrices as in Subsection 2.1. Then, the full flag variety $\mathrm{Fl}(r) = \mathrm{SL}(r)/B$ can be identified with the space of full flags $F_\bullet$ on an r-dimensional vector space M. As such it receives natural line bundles $\mathcal{L}_a = \det(F_a)^*$ $(1 \le a \le r)$. The line bundle $\mathcal{L}_a$ is isomorphic with the line bundle $\mathcal{L}_{\omega_a}$, where ω_a is the character of B which takes the diagonal matrix $(t_1, \dots, t_r)$ to the product $\prod_{i=1}^a t_i$. Recall that ω_a is the a^{th} fundamental weight.

6.2. Representations of $\mathrm{GL}(r)$**.** Irreducible polynomial representations of the general linear group $\mathrm{Gl}(r)$ correspond to the sequences of integers $\mu = (\mu_1 \ge \dots \ge \mu_r \ge 0)$ (also called Young diagrams or partitions). Given a sequence of integers $\mu = (\mu_1 \ge \mu_2 \ge \dots \ge \mu_r \ge 0)$, form the line bundle

$$\mathcal{L}_\mu = \prod_{a=1}^{r} \mathcal{L}_a^{\mu_a - \mu_{a+1}},$$

where $\mu_{r+1} := 0$. Then, $H^0(\mathrm{Fl}(r), \mathcal{L}_\mu) = V_\mu^*$, where V_μ is the irreducible representation of $\mathrm{Gl}(r)$ with highest weight μ. To any such μ, we associate the character $\bar{\mu}$ of B by

$$\bar{\mu}(t) = \prod_{i=1}^{r} t_i^{\mu_i}, \text{ where } t = \mathrm{diag}(t_1, \dots, t_r).$$

Then, clearly, $\mathcal{L}_\mu \simeq \mathcal{L}_{\bar{\mu}}$. From now on, we will identify the partition μ with the character $\bar{\mu}$.

6.3. A basic calculation. As earlier in Subsection 2.2, let $V = \mathbb{C}^{2n}$ be equipped with the symplectic form $\langle\, ,\, \rangle$ and let M be a vector space of dimension r.

We will now make a basic determinantal calculation (which appeared originally in a weaker form in [B_2, Lemma 4.5]). Let $P := \mathrm{Hom}(M, V)$. Fix an integer $m \le 2n$. For a full flag $F_\bullet$ on M, a full flag $G_\bullet$ on V, and a partition $\mu = (m \ge \mu_1 \ge \dots \ge \mu_r \ge 0)$, let $P_m(\mu, F_\bullet, G_\bullet)$ denote the subspace

$$\{\phi \in \mathrm{Hom}(M, V) : \phi(F_a) \subset G_{m-\mu_a}, \forall a \in [r]\}.$$

Consider the vector bundle $T_m(\mu)$ (respectively, $P_m(\mu)$) on $\mathrm{Fl}(M) \times \mathrm{Fl}(V)$ whose fiber over $(F_\bullet, G_\bullet)$ is $T_m(\mu, F_\bullet, G_\bullet) := \mathrm{Hom}(M, V)/P_m(\mu, F_\bullet, G_\bullet)$ (respectively, $P_m(\mu, F_\bullet, G_\bullet)$). We now calculate the determinant line bundles of $P_m(\mu)$ and $T_m(\mu)$. We begin the computation by setting (for $1 \le b \le r$),

$$P_m^b = \{\phi \in \mathrm{Hom}(M, V) : \phi(F_a) \subset G_{m-\mu_a}, a = 1, \dots, b\}.$$

There exist exact sequences for $0 \le b \le r-1$:

$$0 \to P_m^{b+1} \to P_m^b \to \mathrm{Hom}(F_{b+1}/F_b, V/G_{m-\mu_{b+1}}) \to 0,$$

where $P_m^0 := \mathrm{Hom}(M, V)$. This permits us to write ($\mathcal{L}_a := \det(F_a)^*$, $\mathcal{K}_b := \det(G_b)^*$),

$$\det T_m(\mu) = \Big((\prod_{a=1}^{r} \mathcal{L}_a^{\mu_a - \mu_{a+1}}) \otimes (\det M^*)^{2n-m} \Big) \boxtimes \big((\det V)^r \otimes \prod_{a=1}^{r} \mathcal{K}_{m-\mu_a} \big),$$

where $\mu_{r+1} := 0$. (Here we have used the fact that for vector spaces $V_1, V_2, \det(\mathrm{Hom}(V_1, V_2)) \simeq ((\det V_1)^*)^{\dim V_2} \otimes (\det V_2)^{\dim V_1}$.) Let λ be the partition conjugate to the partition $\lambda' := (m - \mu_r \ge \cdots \ge m - \mu_1 \ge 0)$. We call λ to be obtained from μ by *m-flip*. Then, we may rewrite the above formula as

$$\det T_m(\mu) = \Big(\mathcal{L}_\mu \otimes (\det M^*)^{2n-m} \Big) \boxtimes \Big((\det V)^r \otimes \mathcal{L}_\lambda \Big). \tag{16}$$

6.4. The theta section. Let M and V be as in Section 6.3. We take $m = 2n$. Let $\{\mu^j\}_{1 \le j \le s}$ be partitions:

$$2n \ge \mu_1^j \ge \cdots \ge \mu_r^j \ge 0$$

satisfying

$$\sum_{j=1}^{s} |\mu^j| = 2nr,$$

where $|\mu^j| := \sum_{i=1}^{r} \mu_i^j$.

Consider two $\mathrm{GL}(M) \times \mathrm{Gl}(V)$-equivariant bundles $\mathcal{P}$ and $\mathcal{Q}$ on $\mathrm{Fl}(M)^s \times \mathrm{Fl}(V)^s$ (with the diagonal actions of $\mathrm{Gl}(M)$ and $\mathrm{Gl}(V)$ on $\mathrm{Fl}(M)^s$ and $\mathrm{Fl}(V)^s$ respectively). The fiber of $\mathcal{P}$ over $(\mathcal{F}, \mathcal{G})$ is just the constant vector space $\mathrm{Hom}(M, V)$ (with the natural action of the group $\mathrm{GL}(M) \times \mathrm{Gl}(V)$), where $\mathcal{F} = (F_\bullet^1, \ldots, F_\bullet^s)$ is an s-tuple of full flags on M and similarly for $\mathcal{G}$. The fiber of $\mathcal{Q}$ over $(\mathcal{F}, \mathcal{G})$ is the direct sum

$$\bigoplus_{j=1}^{s} T(\mu^j, F_\bullet^j, G_\bullet^j),$$

where $T(\mu^j, F_\bullet^j, G_\bullet^j) := T_{2n}(\mu^j, F_\bullet^j, G_\bullet^j)$. Consider the canonical $\mathrm{GL}(M) \times \mathrm{Gl}(V)$-equivariant map $S : \mathcal{P} \to \mathcal{Q}$ between vector bundles (of the same rank) on $\mathrm{Fl}(M)^s \times \mathrm{Fl}(V)^s$ induced via the quotient maps. Thus, taking its determinant, we find a $\mathrm{GL}(M) \times \mathrm{Gl}(V)$-equivariant global section $\Theta \in \det(\mathcal{Q}) \otimes (\det \mathcal{P})^{-1}$ such that Θ vanishes at $(\mathcal{F}, \mathcal{G})$ if and only if $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G}) \neq 0$, where $\mathcal{H}_\mu(\mathcal{F}, \mathcal{G})$ is as defined in the beginning of Section 5 (extended for any, not necessarily isotropic, complete flags $\mathcal{G} = (G^1_\bullet, \dots, G^s_\bullet))$. By Equation (16),

$$\det(\mathcal{Q}) = \mathcal{L}_{\mu^1} \boxtimes \cdots \boxtimes \mathcal{L}_{\mu^s} \boxtimes \big(\det(V)^r \otimes \mathcal{L}_{\lambda^1}\big) \boxtimes \cdots \boxtimes \big(\det(V)^r \otimes \mathcal{L}_{\lambda^s}\big).$$

Here λ^i is obtained from μ^i by $2n$-flip.

Fixing a point $\mathcal{F} \in \mathrm{Fl}(M)^s$, on restriction, we get (as in [B$_2$]) an $\mathrm{SL}(V)$-invariant section $\Theta_{\mathcal{F}}$ in

$$H^0(\mathrm{Fl}(V)^s, \mathcal{L}_{\lambda^1} \boxtimes \cdots \boxtimes \mathcal{L}_{\lambda^s})^{\mathrm{SL}(V)}.$$

Restricting this to $\mathrm{IFl}(V)^s$ we find a section in

$$H^0(\mathrm{IFl}(V)^s, \mathcal{L}_{\lambda^1_C} \boxtimes \cdots \boxtimes \mathcal{L}_{\lambda^s_C})^{\mathrm{Sp}(V)} = \big(M^*_{\lambda^1_C} \otimes \cdots \otimes M^*_{\lambda^s_C}\big)^{\mathrm{Sp}(2n)},$$

where $\mathrm{IFl}(V)$ is the space of (full) isotropic flags on V, λ^i_C is the weight of $\mathrm{Sp}(2n)$ obtained by the restriction of λ^i to the Cartan of $\mathrm{Sp}(2n)$ and M_ν is the irreducible representation of $\mathrm{Sp}(2n)$ with highest weight ν. (Observe that for a dominant weight λ of $\mathrm{SL}(2n)$, its restriction λ_C is dominant for $\mathrm{Sp}(2n)$.)

We can therefore record the following consequence of Corollary 17.

Corollary 22. *Let $\{\mu^j\}_{1 \le j \le s}$ be partitions: $(2n \ge \mu^j_1 \ge \cdots \ge \mu^j_r \ge 0)$ such that $\sum_j |\mu^j| = 2nr$. Assume further that*

$$\big(V_{\mu^1} \otimes \cdots \otimes V_{\mu^s}\big)^{SL(r)} \neq 0.$$

Then,

$$\big(M_{\nu^1} \otimes \cdots \otimes M_{\nu^s}\big)^{\mathrm{Sp}(2n)} \neq 0,$$

where $\nu^j := \lambda^j_C$ and λ^j is obtained from μ^j by $2n$-flip.

We now come to the main theorem of this section.

Theorem 23. *Let $V_{\lambda^1}, \dots, V_{\lambda^s}$ be irreducible representations of* $\mathrm{SL}(2n)$ *(with highest weights $\lambda^1, \dots, \lambda^s$ respectively) such that their tensor product has a nonzero* $\mathrm{SL}(2n)$*-invariant. Then, the tensor product of the irreducible representations of* $\mathrm{Sp}(2n)$ *with highest weights $\lambda^1_C, \dots, \lambda^s_C$ has a nonzero* $\mathrm{Sp}(2n)$*-invariant, where λ^j_C is the restriction of λ^j to the Cartan subalgebra $\mathfrak{h}^C$ of* $\mathrm{Sp}(2n)$.

Proof. We would like to find some r and partitions $\mu^1, \dots, \mu^s$ such that

$$2n \ge \mu^j_1 \ge \cdots \ge \mu^j_r \ge 0$$

with $\sum_{j=1}^s |\mu^j| = 2nr$, and such that λ^j is obtained from μ^j by $2n$-flip.

To achieve this, express λ^j as sequences of integers (with r large)

$$r \ge \lambda^j_1 \ge \cdots \ge \lambda^j_{2n} \ge 0.$$

Since the tensor product of the corresponding representations of SL($2n$) has a nonzero SL($2n$)-invariant,

$$\sum_{j=1}^{s}\sum_{a=1}^{2n}\lambda_a^j = 2nb' \text{ for some integer } b' \geq 0,$$

i.e.,

$$\sum_{j=1}^{s}\sum_{a=1}^{2n}(r-\lambda_a^j) = 2nr + 2nb,$$

where $b := r(s-1) - b'$. Taking $r \geq \frac{s}{s-1}\lambda_1^j$ (for every j), we can assume that $b \geq 0$. Rewrite the above equality as $\sum_{j=1}^{s}\sum_{a=1}^{2n}(r+b-(b+\lambda_a^j)) = 2n(r+b)$. So, we replace r by $r+b$ and λ_a^j by $b+\lambda_a^j$ to assume $b=0$. Let μ^j be obtained from λ^j by r-flip. Since, by assumption,

$$\big(V_{\lambda^1}\otimes\cdots\otimes V_{\lambda^s}\big)^{\mathrm{SL}(2n)} \neq 0,$$

we obtain (using ordinary duality)

$$\big(V_{\lambda^{1\prime}}\otimes\cdots\otimes V_{\lambda^{s\prime}}\big)^{\mathrm{SL}(2n)} \neq 0,$$

where $\lambda^{j\prime} = (r-\lambda_{2n}^j, \dots, r-\lambda_1^j)$.

Now using Grassmann duality (see the next Section 6.5),

$$\big(V_{\mu^1}\otimes\cdots\otimes V_{\mu^s}\big)^{SL(r)} \neq 0.$$

Now, apply Corollary 22. (Observe that $2n$-flip of μ^j is the partition $\lambda_1^j - \lambda_{2n}^j \geq \cdots \geq \lambda_{2n-1}^j - \lambda_{2n}^j \geq 0$.) This completes the proof of the theorem. ☐

6.5. Grassmann duality. Let r and k be positive integers and let $m = r+k$. Given a Young diagram $\mu = (k \geq \mu_1 \geq \cdots \geq \mu_r \geq 0)$, one naturally obtains a cohomology class ω_μ of $\mathrm{Gr}(r,m)$, which is the cycle class of $\bar{\Omega}_{A(\mu)}$ with $A(\mu) = \{m-r+a-\mu_a | a \in [r]\}$. It is known that if $\mu^1, \dots, \mu^s$ are such Young diagrams with $\sum_{j=1}^{s}|\mu^j| = kr$, then the dimension of $\big(V_{\mu^1}\otimes\cdots\otimes V_{\mu^s}\big)^{SL(r)}$ equals the coefficient of the cycle class of a point in the cup product $\prod_{j=1}^{s}\omega_{\mu^j} \in H^{2rk}(\mathrm{Gr}(r,m))$ (see, e.g., [F$_1$, Chapter 9]).

Clearly, under the natural duality map $\mathrm{Gr}(r,\mathbb{C}^m) = \mathrm{Gr}(k,(\mathbb{C}^m)^*)$, the class ω_μ goes to $\omega_{\tilde{\mu}}$, where $\tilde{\mu}$ is the conjugate diagram to μ (see, e.g., [F$_1$, Exercise 20, page 152]). We therefore obtain the following consequence, which we call the *Grassmann duality*.

Lemma 24. *Suppose* $\sum_{j=1}^{s}|\mu^j| = kr$. *Then,*

$$\dim\big(V_{\mu^1}\otimes\cdots\otimes V_{\mu^s}\big)^{SL(r)} = \dim\big(V_{\tilde{\mu}^1}\otimes\cdots\otimes V_{\tilde{\mu}^s}\big)^{SL(k)}.$$

We clearly have another "ordinary duality". Let ν^j be the dual partition to μ^j (so that $\nu^j = (k-\mu^j_r \geq \cdots \geq k-\mu^j_1)$). Then, clearly, V_{ν^j} is the SL(r)-dual of V_{μ^j} and hence

$$\dim\bigl(V_{\mu^1}\otimes\cdots\otimes V_{\mu^s}\bigr)^{SL(r)} = \dim\bigl(V_{\nu^1}\otimes\cdots\otimes V_{\nu^s}\bigr)^{SL(r)}.$$

7. Saturation for Sp($2n$) and SO($2n+1$)

Let $X(H^C)_+$ be the set of dominant characters of H^C (for the group Sp($2n$)) and similarly let $X(H^B)_+$ be the set of dominant characters of H^B (for the group SO($2n+1$)). We have the following saturation theorems for the groups Sp($2n$) and SO($2n+1$) respectively.

Theorem 25. *Given $\nu^1,\dots,\nu^s \in X(H^C)_+$, the following are equivalent:*

(1) *For some positive integer N, the tensor product of irreducible* Sp($2n$)-*representations with highest weights $N\nu^1,\dots,N\nu^s$ has a nonzero* Sp($2n$)-*invariant.*

(2) *The tensor product of irreducible representations with highest weights $2\nu^1,\dots,2\nu^s$ has a nonzero* Sp($2n$)-*invariant.*

Proof. For any simple algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, we identify $\kappa:\mathfrak{h}\xrightarrow{\sim}\mathfrak{h}^*$ via the normalized invariant bilinear form $\langle\,,\,\rangle$ on $\mathfrak{h}^*$, normalized so that $\langle\theta,\theta\rangle = 2$ for the highest root θ. Under this identification, $\mathfrak{h}_+ \xrightarrow{\sim} D$ and, moreover, $x_i \mapsto 2\omega_i/\langle\alpha_i,\alpha_i\rangle$.

For any $s\geq 1$ and G as in the beginning of Section 2, define

$$\hat{\Gamma}(s,G) = \bigl\{(\lambda^1,\dots,\lambda^s)\in\bigl(X(H)_+\bigr)^s : \text{ for some } N\geq 1,\ \bigl(V_{N\lambda^1}\otimes\cdots\otimes V_{N\lambda^s}\bigr)^G \neq 0\bigr\}.$$

Then, under the identification induced by κ (and still denoted by) $\kappa:\mathfrak{h}^s_+\to D^s$, we have (cf., e.g., [Sj, Theorem 7.6])

$$\kappa^{-1}\bigl(\hat{\Gamma}(s,G)\bigr) = \Gamma(s,K)\cap\kappa^{-1}\bigl((X(H)_+)^s\bigr),$$

where the eigencone $\Gamma(s,K)$ is defined in Section 4.

Recall from Subsection 2.2 the inclusion $i:\mathfrak{h}^C \hookrightarrow \mathfrak{h}^{A_{2n-1}}$ and hence we get the dual (surjective) map $i^*:\left(\mathfrak{h}^{A_{2n-1}}\right)^* \twoheadrightarrow (\mathfrak{h}^C)^*$. We have the following commutative diagram:

$$(*)\qquad \begin{array}{ccc} \mathfrak{h}^C & \xrightarrow{\ i\ } & \mathfrak{h}^{A_{2n-1}} \\ \big\downarrow{\scriptstyle 2\kappa_C} & & \big\downarrow{\scriptstyle \kappa_A} \\ (\mathfrak{h}^C)^* & \xleftarrow{\ i^*\ } & \bigl(\mathfrak{h}^{A_{2n-1}}\bigr)^*, \end{array}$$

where κ_C (respectively, κ_A) denotes κ for Sp($2n$) (respectively, SL($2n$)).

We only need to prove (1) $\Rightarrow$ (2). By assumption, $\nu := (\nu^1, \cdots, \nu^s) \in \hat{\Gamma}(s, \mathrm{Sp}(2n))$ and hence $\kappa_C^{-1}\nu \in \Gamma(s, \mathrm{Sp}(2n))$. But, clearly, $\Gamma(s, \mathrm{Sp}(2n)) \subset \Gamma(s, SU(2n))$ and hence $\kappa_A\, i\, \kappa_C^{-1}(\nu) \in \hat{\Gamma}(s, \mathrm{SL}(2n))$. (Observe that $\kappa_A\, i\, \kappa_C^{-1}(X(H^C)_+) \subset Q^{A_{2n-1}} \cap X(H^{A_{2n-1}})_+$, since $\kappa_C(\beta_j^\vee) = \beta_j$ for $1 \leq j < n$ and $\kappa_C(\beta_n^\vee) = \frac{1}{2}\beta_n$, where $Q^{A_{2n-1}}$ is the root lattice of $\mathrm{SL}(2n)$.) Moreover, since $\kappa_A\, i\, \kappa_C^{-1}(\nu^j) \in Q^{A_{2n-1}}$, by the saturation theorem of Knutson-Tao [KT],

$$\left(V_{\lambda^1} \otimes \cdots \otimes V_{\lambda^s}\right)^{\mathrm{SL}(2n)} \neq 0,$$

where $\lambda^j := \kappa_A\, i\, \kappa_C^{-1}(\nu^j)$ and V_{λ^j} is the irreducible representation of $\mathrm{SL}(2n)$ with highest weight λ^j.

Applying Theorem 23, we get

$$\left(M_{\lambda_C^1} \otimes \cdots \otimes M_{\lambda_c^s}\right)^{\mathrm{Sp}(2n)} \neq 0,$$

where $\lambda_C^j := \lambda^j_{|_{\mathfrak{h}^C}}$ and $M_{\lambda_C^j}$ is the irreducible representation of $\mathrm{Sp}(2n)$ with highest weight λ_C^j. Moreover, by the commutativity of the diagram $(*)$, we get $\lambda_C^j = 2\nu^j$. This proves the theorem. $\square$

By using [KS, Section 4] and the above theorem, we get the corresponding result for the odd orthogonal groups.

Theorem 26. *Given $\nu^1, \ldots, \nu^s \in X(H^B)_+$, the following are equivalent:*

(1) *For some positive integer N, the tensor product of irreducible $\mathrm{SO}(2n+1)$-representations with highest weights $N\nu^1, \ldots, N\nu^s$ has a nonzero $\mathrm{SO}(2n+1)$-invariant.*
(2) *The tensor product of irreducible representations with highest weights $2\nu^1, \ldots, 2\nu^s$ has a nonzero $\mathrm{SO}(2n+1)$-invariant.*

Remark 27. (a) The general saturation result proved by Kapovich-Millson [KM] specialized to the cases of $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ requires the saturation factor of 4. Moreover, they require that $\nu^1 + \cdots + \nu^s$ belongs to the root lattice. So, our saturation theorems 25 and 26 in these cases provide an improvement over their result.

(b) We can prove an analogue of Theorem 16 for $\mathrm{SO}(2n+1)$ by a similar method and use it to prove an analogue of Theorem 23 for $\mathrm{SO}(2n+1)$ (cf. Theorem 5 of Section 1). The latter can be used to give a proof of Theorem 26 analogous to that of Theorem 25.

As an immediate consequence of Theorem 26, we get the following special case of the general saturation result due to Kapovich-Millson [KM].

Corollary 28. *Let $\nu^1, \ldots, \nu^s$ be dominant characters for* $\mathrm{Spin}(2n+1)$ *(whose sum is not necessarily in the root lattice). Then, the following are equivalent:*

(1) *For some positive integer N, the tensor product of irreducible representations of* $\mathrm{Spin}(2n+1)$ *with highest weights $N\nu^1, \ldots, N\nu^s$ has a nonzero* $\mathrm{Spin}(2n+1)$*-invariant.*
(2) *The tensor product of irreducible representations with highest weights $4\nu^1, \ldots, 4\nu^s$ has a nonzero* $\mathrm{Spin}(2n+1)$*-invariant.*

We would like to conjecture the following generalization of Theorems 10, 12 and 23.

Let G be a connected simply-connected, semisimple complex algebraic group and let σ be a diagram automorphism of G with fixed subgroup K.

Conjecture 29. (a) *For any standard parabolic subgroup P and any Bruhat cells $\Lambda^P_{w_1}, \cdots, \Lambda^P_{w_s}$ in G/P, there exist elements $k_1, \cdots, k_s \in K$ such that the intersection $\bigcap_{i=1}^s k_i \Lambda^P_{w_i}$ is transversal.*

(b) *Let $V_{\lambda^1}, \cdots, V_{\lambda^s}$ be irreducible representations of G (with highest weights $\lambda^1, \cdots, \lambda^s$ respectively) such that their tensor product has a nonzero G-invariant. Then, the tensor product of irreducible K-modules with highest weights $\lambda^1_K, \cdots, \lambda^s_K$ has a nonzero K-invariant, where λ^i_K is the restriction of λ^i to the Cartan subalgebra $\mathfrak{h}_K := \mathfrak{h}^\sigma$ of K.*

(Observe that λ^i_K is dominant for K for any dominant weight λ of G with respect to the Borel subgroup $B^K := B^\sigma$ of K.)

8. Horn's problem for symplectic groups

In this section we prove an analogue of Horn's problem for symplectic groups. We follow the notation from Subsection 2.2. In particular, $V = \mathbb{C}^{2n}$.

Fix $1 \le r \le n$ and let $I^1, \ldots, I^s \in \mathfrak{S}(r, 2n)$ be such that

$$\sum_{j=1}^{s} \mathrm{codim}(\Phi_{I^j}) = \dim \mathrm{IG}(r, 2n). \tag{17}$$

Given subsets I and K of $[2n]$, we denote by $|I > K|$ the number of pairs (i,k) with $i \in I$, $k \in K$ and $i > k$. (We set $|I > \emptyset| = 0$.) For K a singleton $\{k\}$, we abbreviate $|I > K|$ by $|I > k|$. For an $I \in \mathfrak{S}(r, 2n)$, we define

$$\tilde{I} = [2n] \setminus (I \sqcup \bar{I}).$$

(See Equation (3) for the definition of $\bar{I}$.) We also set (for any $I \in \mathfrak{S}(r, 2n)$),

$$\mathrm{sym}^2(I) = \frac{1}{2}(|I > \bar{I}| + \mu(w_I)), \qquad \mathrm{cosym}^2(I) = \frac{r(r+1)}{2} - \mathrm{sym}^2(I),$$

and

$$\bigwedge^2(I) = |I > \bar{I}| - \mathrm{sym}^2(I),$$

where $\mu(w_I)$ represents the number of times the simple reflection s_n appears in any reduced decomposition of w_I.

Let β_j be the order preserving bijection of $I^j \sqcup \bar{I}^j$ with $[2r]$ and let I^j_o be the subset $\beta_j(I^j)$ of $[2r]$ (of cardinality r). For any $a \in [r]$ and $j \in [s]$, let $\lambda^j_a := |i^j_a \geq \tilde{I}^j|$ and $\mu^j_a := 2n - 2r - \lambda^j_a$, where $I^j = \{i^j_1 < \cdots < i^j_r\}$. Recall the definition of the deformed product $\odot_0$ in the cohomology $H^*(G/P)$ from [BK, Definition 18].

Theorem 30. *Under the assumption* (17)*, the following conditions* (α) *and* (β) *are equivalent:*

(α) $\prod_{j=1}^s [\bar{\Phi}_{I^j}] \neq 0 \in H^*(\mathrm{IG}(r,2n), \odot_0)$.
(β) (β_1) $\dim \mathrm{IG}(r,2r) = \sum_{j=1}^s \mathrm{cosym}^2(I^j)$.
(β_2) $\left(V_{\mu^1} \otimes \cdots \otimes V_{\mu^s}\right)^{SL(r)} \neq 0$.
(β_3) $\prod_{j=1}^s [\bar{\Phi}_{I^j_o}] \neq 0 \in H^*(\mathrm{IG}(r,2r))$.

Remark 31. (1) Assume that the condition (β_1) is satisfied. Then, the condition (β_2) is equivalent to the following condition.

For any $1 \leq d \leq r$ and subsets $B^1, \dots, B^s$ of $[r]$ each of cardinality d such that the product $\prod_{j=1}^s [\bar{\Omega}_{B^j}] \neq 0 \in H^*(\mathrm{Gr}(d,r))$, the following inequality holds

$$\sum_{j=1}^s \sum_{a \in B^j} \mu^j_a \leq 2d(n-r).$$

This follows from Corollary 17 applied to a space M of dimension r and $V = M^\perp/M$ of dimension $2n-2r$ and identity (28) (in Section 8). In fact, we only need the equivalence of the assertion in Corollary 17 and the (B)-part of Theorem 16, which is due to Klyachko and Knutson-Tao (cf., the proof of Corollary 17).

(2) The condition (β_3) in the above theorem is recursive by the work of Purbhoo and Sottile [PS]. The Purbhoo-Sottile recursion says that (β_2) is equivalent to a system of inequalities indexed by nonvanishing s-fold intersections of Schubert cycle classes in the Grassmannians $\mathrm{Gr}(d,r)$ where $0 < d < r$.

Before we can prove the above theorem, we need the following preliminary work.

Proposition 32. *For any* $I \in \mathfrak{S}(r,2n)$ *and any isotropic flag* $E_\bullet$*,*

$$\dim \Phi_I(E_\bullet) = |I > \tilde{I}| + \mathrm{sym}^2(I).$$

Proof. Under the canonical inclusion $W^C \hookrightarrow S_{2n}$,

$$w_I \mapsto \hat{w}_I = (i_1, \dots, i_r, j_1, \dots, j_{n-r}, 2n+1-j_{n-r}, \dots, 2n+1-j_1, \\ 2n+1-i_r, \dots, 2n+1-i_1),$$

where $\{j_1 < \dots < j_{n-r}\} := [n] \setminus (I \sqcup \bar{I})$. Now, by Equations (1), (2) and [F_1, §10.2],

$$\begin{aligned}\ell^C(w_I) &= \frac{1}{2}\big(\ell^{A_{2n-1}}(\hat{w}_I) + \mu(w_I)\big) \\ &= \frac{1}{2}\big(|I > \tilde{I}| + |I > \bar{I}| + |\tilde{I} > \bar{I}| + \mu(w_I)\big) \\ &= \frac{1}{2}\big(|I > \tilde{I}| + |I > \bar{I}| + |I > \tilde{I}| + \mu(w_I)\big) \\ &= |I > \tilde{I}| + \operatorname{sym}^2(I).\end{aligned}$$

This proves the proposition. □

8.1. Tangent space of isotropic Grassmannians. We calculate the tangent space $T(X)_M$ to $X = \mathrm{IG}(r, V)$ at a point M. Because of the natural embedding $\mathrm{IG}(r, V) \subseteq \mathrm{Gr}(r, V)$, we have $T(X)_M \subseteq T\,\mathrm{Gr}(r, V)_M = \mathrm{Hom}(M, V/M)$.

Clearly, $M^\perp$ is a $2n - r$ dimensional subspace of V that contains M and there is a canonical isomorphism $V/M^\perp \simeq M^*$ (induced from the symplectic form). Hence, we have an exact sequence

$$0 \to \mathrm{Hom}(M, M^\perp/M) \to \mathrm{Hom}(M, V/M) \xrightarrow{\phi} \mathrm{Hom}(M, V/M^\perp) \\ = \mathrm{Hom}(M, M^*) \to 0,$$

obtained from the inclusions $M \subset M^\perp \subset V$. It is clear that $M^\perp/M$ is a $2n - 2r$ dimensional space that possesses a nondegenerate symplectic form. Let $P_M \subset \mathrm{Sp}(2n)$ be the stabilizer of M and $\operatorname{sym}^2 M^*$ the space of symmetric bilinear forms on M.

Lemma 33. $T(X)_M = \phi^{-1}(\operatorname{sym}^2 M^*)$ *and hence there is an exact sequence of* P_M*-modules.*

$$0 \to \mathrm{Hom}(M, M^\perp/M) \xrightarrow{\xi} T(X)_M \xrightarrow{\phi} \operatorname{sym}^2 M^* \to 0. \tag{18}$$

Proof. Let $\psi : M \to V/M$ be a linear map (viewed as a deformation of the trivial map). The deformed M (obtained from ψ) needs to be isotropic. So, up to the first order, we have

$$\langle v + \epsilon\psi(v), v' + \epsilon\psi(v')\rangle = 0, \forall\, v, v' \in M.$$

Hence,

$$\langle v, \psi(v')\rangle + \langle \psi(v), v'\rangle = 0,$$

or that

$$\langle v, \psi(v')\rangle = \langle v', \psi(v)\rangle.$$

This gives us the symmetric bilinear form $\phi(\psi)(v, v') = \langle v, \psi(v')\rangle$. Hence, $T(X)_M \subseteq \phi^{-1}(\mathrm{sym}^2 M^*)$. But,

$$\dim T(X)_M = \dim \mathrm{IG}(r, V) = \frac{r}{2}(4n - 3r + 1) = \dim \phi^{-1}(\mathrm{sym}^2 M^*),$$

and hence $T(X)_M = \phi^{-1}(\mathrm{sym}^2 M^*)$. □

Let $E_\bullet$ be an isotropic flag on V. This induces flags on M, $M^\perp$ and hence on $M^\perp/M$.[1]

Lemma 34. *For any isotropic flag $E_\bullet$ on V, the induced flag on $M^\perp/M$ is isotropic with respect to the nondegenerate symplectic form on $M^\perp/M$.*

Proof. Consider an element of the induced flag

$$E_a \cap M^\perp / E_a \cap M = (E_{2n-a} + M)^\perp / E_a \cap M.$$

Its perpendicular (in $M^\perp/M$) is therefore $((E_{2n-a} + M) \cap M^\perp)/M = E_{2n-a} \cap M^\perp / E_{2n-a} \cap M$, which is again a member of the induced flag on $M^\perp/M$. □

8.2. Tangent space of the Schubert cells. For any $I \in \mathfrak{S}(r, 2n)$, isotropic flag $E_\bullet$ on V and $M \in \Phi_I(E_\bullet)$, we now calculate the tangent space $T(\Phi_I(E_\bullet))_M \subseteq T(\Omega_I(E_\bullet))_M$ (with the notation as in Section 2).

Lemma 35. $M^\perp \in \Omega_{I \sqcup \tilde{I}}(E_\bullet)$.

Proof. By definition, $I \sqcup \tilde{I} = [2n] \setminus \bar{I}$. Now, $\dim(E_a \cap M^\perp) = \dim((E_{2n-a} + M)^\perp) = 2n - (r + 2n - a - \dim(E_{2n-a} \cap M)) = a - r + \dim(E_{2n-a} \cap M)$. Hence, $E_a \cap M^\perp \neq E_{a-1} \cap M^\perp$ if and only if $E_{2n-a} \cap M = E_{2n+1-a} \cap M$, i.e., $2n + 1 - a \notin I \Leftrightarrow a \notin \bar{I}$. □

For any $a \in [r]$, let $\lambda_a := |i_a \geq \tilde{I}|$, where $I = \{1 \leq i_1 < \cdots < i_r \leq 2n\}$. Let $X = \mathrm{IG}(r, V)$.

Proposition 36.

(1)

$$\mathrm{Hom}(M, M^\perp/M) \cap T(\Omega_I(E_\bullet))_M = \{\gamma \in \mathrm{Hom}(M, M^\perp/M) : \gamma(E(M)_a) \subset E(M^\perp/M)_{\lambda_a}, \forall a \in [r]\},$$

where $E(M)_\bullet$ and $E(M^\perp/M)_\bullet$ are the induced complete flags on M and $M^\perp/M$ respectively with the changed labels $\bullet$ by the dimension. The dimension of this vector space is $|I \geq \tilde{I}|$.

(2) $\mathrm{Hom}(M, M^\perp/M) \cap T(\Phi_I(E_\bullet))_M = \mathrm{Hom}(M, M^\perp/M) \cap T(\Omega_I(E_\bullet))_M$.

[1] There is exactly one way of inducing a complete flag on $M^\perp/M$ from $E_\bullet$. The elements of the flag are $(M^\perp/M) \cap (E_a + M/M)$. This can be written as $((E_a + M) \cap M^\perp)/M = E_a \cap M^\perp / E_a \cap M$.

(3)

$$\phi(T(\Phi_I(E_\bullet))_M) = \phi\big(T(\Omega_I(E_\bullet))_M \cap T(X)_M\big)$$
$$= \{\gamma \in \operatorname{sym}^2 M^* : \gamma(E(M)_a, E(M)_{t_a}) = 0, \forall a \in [r]\},$$

where $t_a = |\bar{I} \geq i_a|$. *Moreover, the dimension of this vector space is* $\operatorname{sym}^2(I)$.

(4) *We have an equality of schemes* $\Phi_I(E_\bullet) = \Omega_I(E_\bullet) \cap X$.

Proof. We first prove Part (1): From the known description of $T(\Omega_I(E_\bullet))_M$ as the space of maps $\gamma : M \to V/M$ so that $\gamma(M \cap E_b) \subset (E_b + M)/M$ for all b, we see that for γ to also be in $\operatorname{Hom}(M, M^\perp/M)$, the condition is

$$\gamma(M \cap E_b) \subset ((E_b + M) \cap M^\perp)/M \text{ for any } b \in [2n]. \tag{19}$$

By Lemma 35, $E_b \cap M^\perp / E_b \cap M$ is of dimension $|b \geq \tilde{I} \sqcup I| - |b \geq I| = |b \geq \tilde{I}|$ for any $b \in [2n]$. Putting $b = i_a$, the condition given in Equation (19) can be rewritten as

$$\gamma(E(M)_a) \subseteq E(M^\perp/M)_{\lambda_a}, \forall a \in [r].$$

The dimension of this vector space is clearly $|I \geq \tilde{I}|$. This proves Part (1). In particular,

$$\begin{aligned} &\dim\big(\operatorname{Hom}(M, M^\perp/M) \cap T(\Phi_I(E_\bullet))_M\big) \\ &\quad \leq \dim\big(\operatorname{Hom}(M, M^\perp/M) \cap T(\Omega_I(E_\bullet))_M\big) = |I \geq \tilde{I}| = |I > \tilde{I}|, \end{aligned} \tag{20}$$

where the last equality follows since $I \cap \tilde{I} = \emptyset$. We have (by the description of $T(\Omega_I(E_\bullet))_M$ given above)

$$\begin{aligned} &\phi(T(\Phi_I(E_\bullet))_M) \\ &\quad \subseteq \phi(T(\Omega_I(E_\bullet))_M \cap T(X)_M) \\ &\quad \subseteq \{\gamma \in \operatorname{sym}^2 M^* : \gamma(E_b \cap M, E_b^\perp \cap M) = 0,\ \forall b \in [2n]\} \\ &\quad = \{\gamma \in \operatorname{sym}^2 M^* : \gamma(E_b \cap M, E_{2n-b} \cap M) = 0, \forall b \in [2n]\} \\ &\quad = \{\gamma \in \operatorname{sym}^2 M^* : \gamma(E(M)_a, E(M)_{t_a}) = 0, \forall a \in [r]\}. \end{aligned} \tag{21}$$

(To prove the last equality, we have used $\dim(E_{2n-i_a} \cap M) = t_a$. Furthermore, if $E_b \cap M = E_{b+1} \cap M$, then the condition $\gamma(E_b \cap M, E_{2n-b} \cap M) = 0$ implies the condition $\gamma(E_{b+1} \cap M, E_{2n-b-1} \cap M) = 0$.)

Moreover, the last space has dimension $\operatorname{sym}^2(I)$ by the following calculation.

As above, express the last vector space in the more symmetric form

$$V_1 = \{\gamma \in \operatorname{sym}^2 M^* : \gamma(E_b \cap M, E_{2n-b} \cap M) = 0, \forall b \in [2n]\}.$$

Form the "analogous" space

$$V_2 = \{\gamma \in \textstyle\bigwedge^2 M^* : \gamma(E_b \cap M, E_{2n-b} \cap M) = 0, \forall b \in [2n]\}.$$

Clearly,

$$V_1 \oplus V_2 = \{\gamma \in (M \otimes M)^* : \gamma((E_b \cap M) \otimes (E_{2n-b} \cap M)) = 0,\ \forall b \in [2n]\},$$

which, in turn, can be written as

$$S_M(E_\bullet) = \{\gamma \in \mathrm{Hom}(M, M^*) : \gamma(E_b \cap M) \subset \mathrm{ann}(E_{2n-b} \cap M), \forall b \in [2n]\},$$

where $\mathrm{ann}(E_{2n-b} \cap M)$ is the annihilator of $E_{2n-b} \cap M$ in M^*. The last space $S_M(E_\bullet)$ is of dimension $|I > \bar{I}|$. For this note that $\mathrm{ann}(E_{2n-i_a} \cap M)$ is of dimension $r - |I \leq 2n - i_a| = |I > 2n - i_a| = |I > 2n + 1 - i_a| = |i_a > \bar{I}|$, since $I \cap \bar{I} = \emptyset$.

Choose a basis of M compatible with the filtration $\{E_b \cap M\}_{b \in [2n]}$. Then, in this basis, the vector spaces V_1 and V_2 are subspaces of symmetric and skew-symmetric matrices respectively. The difference $\dim V_1 - \dim V_2$ is the number of diagonal terms allowed in V_1, i.e.,

$$\begin{aligned} \dim V_1 - \dim V_2 &= |\{a \in [r] : E_{i_a} \cap M \not\subset E_{2n-i_a} \cap M\}| \\ &= |\{a \in [r] : i_a > 2n - i_a\}| = |I > n|. \end{aligned}$$

Therefore, we conclude

$$\begin{aligned} \dim V_1 + \dim V_2 &= |I > \bar{I}|, \\ \dim V_1 - \dim V_2 &= |I > n|. \end{aligned}$$

Using Equation (26), we obtain $\dim V_1 = \mathrm{sym}^2 I$ and $\dim V_2 = \bigwedge^2 I$. This proves the last assertion in Part (3).

Combining Equations (20) and (21), and using Lemma 33, we get

$$\dim(T(\Phi_I(E_\bullet))_M) \leq \mathrm{sym}^2(I) + |I > \tilde{I}|. \tag{22}$$

But, by Proposition 32, the above inequality is an equality. Hence, all the inclusions and inequalities in Equations (20) and (21) are equalities. This proves Parts (2) and (3).

By the definition, set theoretically $\Phi_I(E_\bullet) = \Omega_I(E_\bullet) \cap X$. Moreover, by Parts (2) and (3), the tangent space $T(\Phi_I(E_\bullet))_M = T(\Omega_I(E_\bullet))_M \cap T(X)_M$. This proves Part (4) and thus completes the proof of the proposition. ☐

Let V be as in the beginning of this section and let $r \leq n$. Fix $M \in \mathrm{IG}(r, V)$ and $I \in \mathfrak{S}(r, 2n)$. Let $U = U_I(M)$ be the set of isotropic flags $E_\bullet$ on V such that $M \in \Phi_I(E_\bullet)$. A flag $E_\bullet \in U$ induces a complete flag $E_\bullet(M)$ on M, and a complete isotropic flag $E_\bullet(M^\perp/M)$ on $M^\perp/M$ (Lemma 34). The following lemma follows by choosing basis elements appropriately.

Lemma 37. *The map $U \to \mathrm{Fl}(M) \times \mathrm{IFl}(M^\perp/M)$ is a surjective fiber bundle with irreducible fibers, where $\mathrm{Fl}(M)$ is the variety of all the complete flags on M and $\mathrm{IFl}(M^\perp/M)$ is the variety of all the isotropic flags on $M^\perp/M$.*

Now, let V_o be a $2r$-dimensional vector space with a nondegenerate symplectic form containing M of dimension r as an isotropic subspace. Let $I^1, \dots, I^s \in \mathfrak{S}(r, 2r)$ be written as $I^j = \{i_1^j < \cdots < i_r^j\}$. Define, for any $j \in [s]$ and $a \in [r]$, $t_a^j = |\bar{I}^j \geq i_a^j|$.

Lemma 38. *The following are equivalent:*

(α) $\prod_{j=1}^s [\bar{\Phi}_{I^j}] \neq 0 \in H^*(\mathrm{IG}(r, 2r))$.

(β) *For some (and hence generic) complete flags $F_\bullet^1, \dots, F_\bullet^s$ on M, the vector space*

$$\{\gamma \in \operatorname{sym}^2 M^* \mid \gamma(F_a^j, F_{t_a^j}^j) = 0,\ a \in [r], j \in [s]\} \tag{23}$$

is of the expected dimension

$$r(r+1)/2 - \sum_{j=1}^s \operatorname{cosym}^2(I^j),$$

where $\operatorname{cosym}^2(I^j) := \frac{r(r+1)}{2} - \operatorname{sym}^2(I^j)$.

Proof. (α) $\Rightarrow$ (β): Choose generic isotropic flags $E_\bullet^1, \dots, E_\bullet^s$ on V_o. Because of the assumption (α), $\bigcap_{j=1}^s \Phi_{I^j}(E_\bullet^j)$ is nonempty and, by simultaneously translating each $E_\bullet^j$ by a single element of $\mathrm{Sp}(2r)$, we can assume that M belongs to this intersection and the intersection is transverse. Let $F_\bullet^1, \dots, F_\bullet^s$ be the induced flags on M.

The vector space (23) is the tangent space to the scheme theoretic intersection $\bigcap_{j=1}^s \Phi_{I^j}(E_\bullet^j)$ at M (by Proposition 36(3)). Again applying Proposition 36(3), the transversality of the intersection implies that (β) holds.

Now, assume (β). Choose (generic) complete flags $F_\bullet^1, \dots, F_\bullet^s$ on M such that (β) is satisfied. Now choose isotropic flags $E_\bullet^1, \dots, E_\bullet^s$ on V_o such that $M \in \bigcap_{j=1}^s \Phi_{I^j}(E_\bullet^j)$ and the induced flags on M are $F_\bullet^1, \dots, F_\bullet^s$ respectively. This is possible by Lemma 37. Using (β) and Proposition 36(3), we see that $\bigcap_{j=1}^s \Phi_{I^j}(E_\bullet^j)$ is transverse at M. Therefore, using standard facts from intersection theory on a homogeneous space, we see that (α) holds. $\square$

Let $\{\bar{\epsilon}_i\}_{1 \leq i \leq n}$ be the basis of $\mathfrak{h}^C$, where $\bar{\epsilon}_i$ is the diagonal matrix

$$\operatorname{diag}(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0),$$

where 1 is placed in the i-th slot and -1 in the $(2n+1-i)$-th slot. Recall from Section 2 that we have identified $\mathfrak{h}^C$ with $\mathfrak{h}^B$ and denote either of them by $\mathfrak{h}$. Let $\{\epsilon_i\}_{1 \leq i \leq n}$ be the dual basis (of $\mathfrak{h}^*$). Let ρ^C (respectively, ρ^B) be half the sum of positive roots of $\mathrm{Sp}(2n)$ (respectively, $\mathrm{SO}(2n+1)$). Then,

$$2\rho^B = (2n-1)\epsilon_1 + (2n-3)\epsilon_2 + \cdots + 3\epsilon_{n-1} + \epsilon_n,$$

and

$$2\rho^C = (2n)\epsilon_1 + (2n-2)\epsilon_2 + \cdots + 4\epsilon_{n-1} + 2\epsilon_n.$$

Thus,

$$2(\rho^C - \rho^B) = \epsilon, \text{ where } \epsilon := \epsilon_1 + \cdots + \epsilon_n.$$

Let $\{x_1^C, \dots, x_n^C\}$ be the basis of $\mathfrak{h}^C$ as in Section 2 and similarly for x_r^B. It is easy to see that

$$x_i^B = x_i^C = \bar{\epsilon}_1 + \cdots + \bar{\epsilon}_i, \text{ if } 1 \le i \le n-1 \text{ and}$$

$$x_n^B = 2x_n^C = \bar{\epsilon}_1 + \cdots + \bar{\epsilon}_n. \tag{24}$$

Lemma 39. *For any $w \in W^C$,*

$$w\epsilon \in \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_n.$$

Moreover,

$$\mu(w) = \#\ \textit{negative signs in the above} = \frac{1}{2}\big(n - \langle w\epsilon, \epsilon\rangle\big), \tag{25}$$

where $\langle\, ,\, \rangle$ is the normalized W^C-invariant form on $\mathfrak{h}^*$ given by $\langle \epsilon_i, \epsilon_j\rangle = \delta_{i,j}$.

In particular, for any $I \in \mathfrak{S}(r, 2n)$,

$$\mu(w_I) = |I > n|. \tag{26}$$

Proof. Since the symmetric group $S_n \subset W^C$ (acting as the permutation group of $\{\epsilon_1, \dots, \epsilon_n\}$) acts trivially on ϵ, the first part follows easily. Applying s_n to $\pm\epsilon_1 \pm \cdots \pm \epsilon_n$, the number of minus signs either increases by 1 or decreases by 1, whereas the application of any s_i to $\pm\epsilon_1 \pm \cdots \pm \epsilon_n$ (for any $i < n$) keeps the number of minus signs unchanged. Thus, the number of negative signs in $w\epsilon \le \mu(w)$, for any $w \in W^C$. Further, the last equality in Equation (25) is trivially satisfied. Now, the longest element $w_0 \in W^C$ makes all the signs in $w_0\epsilon$ negative and $\mu(w_0) = n$. Thus, Equation (25) follows for w_0 and hence for any $w \in W^C$. (Otherwise, taking a reduced decomposition of w and extending it to a reduced decomposition of w_0, we would get a contradiction.)

Equation (26) follows from Equation (25) immediately. □

Now, we are ready to prove Theorem 30.

8.3. Proof of Theorem 30. We first observe that, under the assumption (17), the following two conditions are equivalent.

(γ_1) $\prod_{j=1}^s [\bar{\Phi}_{I^j}] \neq 0 \in H^*(\mathrm{IG}(r, 2n), \odot_0)$.

(γ_2) $\prod_{j=1}^s [\bar{\Phi}_{I^j}] \neq 0 \in H^*(\mathrm{IG}(r, 2n))$ and the condition (β_1) is satisfied.

To see this we use [BK, Theorem 15] and the following lemma together with the fact that $\mathrm{cosym}^2(I^j) = \mathrm{cosym}^2(I^j_o)$ (which follows from Equation (26)).

Lemma 40. *For $I \in \mathfrak{S}(r, 2n)$, $\chi^C_{w_I}(x_r^C) = \mathrm{codim}(\Phi_I) + \mathrm{codim}(\Phi_{I_o})$, where $\chi^C_{w_I} := \rho^C + w_I^{-1}\rho^C - 2\rho^{L_r^C}$ and $\rho^{L_r^C}$ is half the sum of positive roots of the Levi subgroup L_r^C.*

Proof. Consider the exact sequence (18). Let $L_M = \mathrm{GL}(M)\times \mathrm{Sp}(M^\perp/M)$ be the Levi subgroup of the stabilizer P_M of M in $\mathrm{Sp}(2n)$. The connected center $\mathbb{C}^*$ of L_M (which is the center of $\mathrm{GL}(M)$) acts on the tangent space $T(X)_M$ as $t\mapsto t^{-1}$ (multiplication by t^{-1}) on $\mathrm{Hom}(M, M^\perp/M)$ and by $t\mapsto t^{-2}$ on $\mathrm{sym}^2(M)^*$. We conclude the proof by using Proposition 36 and the formula for $\chi^C_{w_I}$ given at the end of page 192 in [BK]. □

Let V be a vector space of dimension $2n$ with a nondegenerate symplectic form and let $M\subset V$ be an isotropic subspace of dimension r. Let $X = \mathrm{IG}(r, 2n)$ and assume the condition (β_1). Let $E_\bullet = (E^1_\bullet, \dots, E^s_\bullet)$ be an s-tuple of isotropic flags on V such that for all $j\in[s], M\in \Phi_{I^j}(E^j_\bullet)$. Consider the exact sequence (as in Lemma 33):

$$0\to \mathrm{Hom}(M, M^\perp/M)\xrightarrow{\xi} T(X)_M \xrightarrow{\phi} \mathrm{sym}^2 M^*\to 0.$$

Let us abbreviate

$$\mathcal{A}^0 = \mathrm{Hom}(M, M^\perp/M),\ \mathcal{B}^0 = T(X)_M,\ \text{and } \mathcal{C}^0 = \mathrm{sym}^2 M^*.$$

For $j = 1,\dots, s$, let

$$P^j := \mathrm{Hom}(M, M^\perp/M)\cap (T(\Phi_{I^j}(E^j_\bullet))_M),$$
$$Q^j := T(\Phi_{I^j}(E^j_\bullet))_M,$$
$$R^j = \phi(T(\Phi_{I^j}(E^j_\bullet))_M).$$

Define

$$\mathcal{A}^1 = \bigoplus_{j=1}^s \mathcal{A}^0/P^j,\ \mathcal{B}^1 = \bigoplus_{j=1}^s \mathcal{B}^0/Q^j,\ \mathcal{C}^1 = \bigoplus_{j=1}^s \mathcal{C}^0/R^j.$$

There are natural differentials

$$\mathcal{A}^0\to\mathcal{A}^1,\ \mathcal{B}^0\to\mathcal{B}^1,\ \mathcal{C}^0\to\mathcal{C}^1,$$

and an exact sequence of two-term complexes

$$0\to \mathcal{A}^*\xrightarrow{\xi}\mathcal{B}^*\xrightarrow{\phi}\mathcal{C}^*\to 0. \tag{27}$$

Our assumption (17) implies that $\chi(\mathcal{B}^*) = 0$, and the assumption (β_1) (in view of Proposition 36(3)) implies that $\chi(\mathcal{C}^*) = 0$. Therefore, from the above exact sequence, $\chi(\mathcal{A}^*) = 0$ as well. Thus, by Proposition 36(1) and (2),

$$2(n-r)r - \sum_{j=1}^s |\mu^j| = 0. \tag{28}$$

Moreover, $h^0(\mathcal{B}^*) = 0$ if and only if $h^0(\mathcal{A}^*) = 0$ and $h^0(\mathcal{C}^*) = 0$ from the above exact sequence (27).

We now prove:

$(\alpha) \Rightarrow (\beta)$: Let $E_\bullet = (E^1_\bullet, \dots, E^s_\bullet)$ be a generic s-tuple of isotropic flags on V such that for all $j \in [s], M \in \Phi_{I^j}(E^j_\bullet)$. This can be achieved by the virtue of (α) and by simultaneously translating each $E^j_\bullet$ by the same element of $\mathrm{Sp}(2n)$. Since $E_\bullet$ is generic, $\Omega = \bigcap_{j=1}^s \Phi_{I^j}(E^j_\bullet)$ is a transverse intersection at M and hence $h^1(\mathcal{B}^*) = 0$. Thus, $h^0(\mathcal{B}^*) = 0$ as well and we get the following:

- $h^0(\mathcal{A}^*) = 0$. Hence, by Proposition 36(1) and (2), Equation (28) and Corollary 17 applied to $V = M^\perp/M$, condition (β_2) holds.
- $h^0(\mathcal{C}^*) = 0$ which implies condition (β_3) (using Lemma 38, Proposition 36(3) and the condition (β_1)).

$(\beta) \Rightarrow (\alpha)$: Assume (β). Find an s-tuple $\mathcal{E}_\bullet = (E^1_\bullet, \dots, E^s_\bullet)$ of isotropic flags on V such that M lies in the intersection $\Omega = \bigcap_{j=1}^s \Phi_{I^j}(E^j_\bullet)$. The parameter space of such $\mathcal{E}_\bullet$ is irreducible (Lemma 37); pick a generic such $\mathcal{E}_\bullet$. It follows that the induced flags on M and $M^\perp/M$ are generic. Therefore, our assumptions $(\beta_1), (\beta_2)$, Equation (28), Corollary 17 applied to $V = M^\perp/M$ and Proposition 36(1) and (2) imply that $h^0(\mathcal{A}^*) = 0$. Moreover, our assumption (β_3), Proposition 36(3) and Lemma 38 imply that $h^0(\mathcal{C}^*) = 0$. Hence, $h^0(\mathcal{B}^*) = 0$ and thus $h^1(\mathcal{B}^*) = 0$. Therefore, Ω is a transverse intersection at M, and hence (α) holds by the equivalence of (γ_1) and (γ_2). This proves Theorem 30. □

9. Horn's problem for odd orthogonal groups

In this section we extend the results from the last section to the odd orthogonal groups. The proofs are similar, so we will be brief. Refer to the Section 2.3 for various notation. Fix $1 \le r \le n$ and let $J^1, \dots, J^s \in \mathfrak{S}'(r, 2n+1)$ be such that

$$\sum_{j=1}^{s} \operatorname{codim}(\Psi_{J^j}) = \dim \mathrm{OG}(r, 2n+1). \tag{29}$$

Let a_j be the largest integer $\le n$ in $J^j \sqcup \bar{J'}^j$ and set $b_j := 2n+2-a_j$. Let β_j be the order preserving bijection of $(J^j \sqcup \bar{J'}^j) \setminus \{a_j, b_j\}$ with $[2r-2]$ and let J^j_o be the subset $\beta_j(J^j \setminus \{a_j, b_j\})$ of $[2r-2]$ (of cardinality $r-1$). For any $a \in [r]$ and $j \in [s]$, let $\lambda^j_a := |i^j_a \ge \tilde{J}^j|$ and $\mu^j_a := 2n+1-2r-\lambda^j_a$, where $J^j = \{i^j_1 < \cdots < i^j_r\}$ and $\tilde{J}^j := [2n+1] \setminus (J^j \sqcup \bar{J'}^j)$.

Theorem 41. *Under the assumption* (29)*, the following conditions* (α) *and* (β) *are equivalent.*

(α) $\prod_{j=1}^{s}[\bar{\Psi}_{J^j}] \neq 0 \in H^*(\mathrm{OG}(r, 2n+1), \odot_0)$.

(β) (β_1) $\dim \mathrm{OG}(r, 2r) = \sum_{j=1}^{s} \mathrm{co} \bigwedge^2(J^j)$, *where* $\mathrm{OG}(r, 2r)$ *is the orthogonal Grassmannian of isotropic r-dimensional subspaces in the $2r$-dimensional space with a symmetric nondegenerate form and*

$$\mathrm{co} \bigwedge^2(J) := r(r-1)/2 - \bigwedge^2(J).$$

(β_2) $\left(V_{\mu^1} \otimes \cdots \otimes V_{\mu^s}\right)^{SL(r)} \neq 0$.

(β_3) $\prod_{j=1}^{s}[\bar{\Phi}_{J_o^j}] \neq 0 \in H^*(\mathrm{IG}(r-1, 2r-2))$.

Similar to Remark 31, we have the following:

Remark 42. Assume that the condition (β_1) is satisfied. Then, the condition (β_2) is equivalent to the following condition.

For any $1 \leq d \leq r$ and subsets $B^1, \ldots, B^s$ of $[r]$ each of cardinality d such that the product $\prod_{j=1}^{s}[\bar{\Omega}_{B^j}] \neq 0 \in H^*(\mathrm{Gr}(d, r))$, the following inequality holds

$$\sum_{j=1}^{s} \sum_{a \in B^j} \mu_a^j \leq d(2n+1-2r).$$

Before we can complete the proof of the theorem, we need an interplay between the following three homogeneous spaces.

9.1. Fundamental triple of classical homogeneous spaces. As is well known, the following three homogenous spaces have essentially equivalent intersection theories.

(1) The isotropic Grassmannian $\mathrm{IG}(r-1, 2r-2)$.
(2) The orthogonal Grassmannian $\mathrm{OG}(r-1, 2r-1)$.
(3) The orthogonal Grassmannians $\mathrm{OG}^{\pm}(r, 2r)$, where $\mathrm{OG}^{\pm}(r, 2r)$ are the two components of the orthogonal Grassmannian $\mathrm{OG}(r, 2r)$.

The relation between (1) and (2) is discussed in the appendix. The spaces (2) and (3) are actually homeomorphic. Let V_o be a $2r$-dimensional vector space with a symmetric nondegenerate bilinear form. Fix a subspace T of V_o of dimension $2r-1$, such that the quadratic form is nondegenerate on T. Then, there are homeomorphisms

$$\varphi^{\pm} : \mathrm{OG}^{\pm}(r, V_o) \xrightarrow{\sim} \mathrm{OG}(r-1, T),$$

which send N to $N \cap T$. Notice that N cannot be a subspace of T (because T does not have isotropic subspaces of dimension r).

Fix one component and denote it by $\mathrm{OG}^+(r,2r)$. Then, as in [BL, §3.5], the Schubert cells in $\mathrm{OG}^+(r,2r)$ are parameterized by the subsets $I \subset [2r]$ of cardinality r satisfying:

- $I \cap \bar{I} = \emptyset$, where $\bar{I} := \{2r+1-a | a \in I\}$.
- The number of elements in I that are less than or equal to r has the same parity as r.

One easily sees that the above set is in bijection with $\mathfrak{S}(r-1, 2r-2)$, the indexing set for the Schubert cells in $\mathrm{IG}(r-1, 2r-2)$. This bijection takes I to $\beta(I \setminus \{r, r+1\})$, where β is the order preserving bijection $[2r] \setminus \{r, r+1\} \xrightarrow{\sim} [2r-2]$. By Theorem 45 and the above homeomorphism φ^+, this bijection preserves non-zeroness of intersection numbers.

9.2. Proof of Theorem 41. Under the assumption (29), the following two conditions are equivalent.

(γ_1) $\prod_{j=1}^{s} [\bar{\Psi}_{J^j}] \neq 0 \in H^*(\mathrm{OG}(r, 2n+1), \odot_0)$.

(γ_2) $\prod_{j=1}^{s} [\bar{\Psi}_{J^j}] \neq 0 \in H^*(\mathrm{OG}(r, 2n+1))$ and the condition (β_1) is satisfied.

A straightforward reworking of the proof of Theorem 30 yields Theorem 41 with condition (β_3) replaced by the following:

(β_3') For generic complete flags $F_\bullet^1, \dots, F_\bullet^s$ on an r-dimensional vector space M, the vector space

$$\{\gamma \in \bigwedge^2 M^* \mid \gamma(F_a^j, F_{t_a^j}^j) = 0,\ a \in [r], j \in [s]\} \tag{30}$$

is of the expected dimension

$$r(r-1)/2 - \sum_{j=1}^{s} \mathrm{co} \bigwedge^2 (J^j),$$

where $J^j := \{i_1^j < \cdots < i_r^j\}$ and $t_a^j := |\bar{J'}^j \geq i_a^j|$.

In the following lemma, we will consider only those isotropic flags $F_\bullet$ on V_o such that $F_r \in \mathrm{OG}^+(r,2r)$. We have the following analogue of Proposition 36(3) for the Grassmannian $\mathrm{OG}^+(r,2r)$.

Lemma 43. *The tangent space at a point P to the shifted Schubert cell in $\mathrm{OG}^+(r,2r)$ parameterized by $J = \{i_1 < \cdots < i_r\}$ and corresponding to an isotropic flag $F_\bullet$ on V_o is given by*

$$\{\gamma \in \bigwedge^2 P^* \mid \gamma(F(P)_a, F(P)_{t_a}) = 0, \forall a \in [r]\}, \tag{31}$$

where $t_a := |\bar{J} \geq i_a|$ and (as earlier) $F(P)_\bullet$ is the filtration on P induced from that of $F_\bullet$ indexed by the dimension.

Let τ_r be the permutation in the symmetric group S_{2r} which transposes r and $r+1$. Then, observe that in the above lemma, the space (31) corresponding to J is the same as that for $s_r(J)$.

Using an obvious analogue of Lemma 38 (with $\mathrm{IG}(r,2r)$ replaced by $\mathrm{OG}^+(r,2r)$) and the relation between structure constants in $\mathrm{OG}^+(r,2r)$ and $\mathrm{IG}(r-1,2r-2)$ as given in Subsection 9.1, we conclude that the conditions (β_3) and (β_3') are the same. This proves Theorem 41. □

10. Appendix. Relation between intersection theory of homogeneous spaces for $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$

For completeness, we include a proof of the following well-known Theorem 45 (cf., e.g., [BS, pages 2674–2675]). As in Section 2, there is a canonical Weyl group equivariant identification between the Cartan subalgebras $\mathfrak{h}^C$ and $\mathfrak{h}^B$ (and also the Weyl groups W^C and W^B) of $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ respectively. We will make these identifications and denote them by $\mathfrak{h}$ and W. Let A_i^C (respectively, A_i^B) be the BGG operators for $\mathrm{Sp}(2n)$ (respectively, $\mathrm{SO}(2n+1)$) acting on $S(\mathfrak{h}^*)=\mathbb{C}[\epsilon_1,\dots,\epsilon_n]$ corresponding to the simple reflections s_i, where $\{\epsilon_1,\dots,\epsilon_n\}$ is the basis of $\mathfrak{h}^*$ as in [Bo, Planche II and III].

Lemma 44.

$$A_i^B = A_i^C, \qquad 1\le i<n,$$
$$\frac{1}{2}A_n^B = A_n^C.$$

For any $w\in W$ with reduced decomposition $w=s_{i_1}\cdots s_{i_p}$, define $A_w^C = A_{i_1}^C\cdots A_{i_p}^C$ and similarly define A_w^B. (They do not depend upon the choice of the reduced decomposition.) Consider the isomorphism

$$\phi : H^*\Big(\mathrm{SO}(2n+1)/B^B,\mathbb{C}\Big) \to H^*\Big(\mathrm{Sp}(2n)/B^C,\mathbb{C}\Big),$$

induced from the Borel isomorphism via the identity map

$$\frac{\mathbb{C}[\epsilon_1,\dots,\epsilon_n]}{I^B} \xrightarrow[\sim]{I} \frac{\mathbb{C}[\epsilon_1,\dots,\epsilon_n]}{I^C},$$

where $I^B=I^C$ is the ideal generated by the positive degree W-invariants in $\mathbb{C}[\epsilon_1,\dots,\epsilon_n]$. Let $\{[\bar\Lambda_w(C)]\}_{w\in W}\subset H^*(\mathrm{Sp}(2n)/B^C)$ be the Schubert basis and similarly for $\{[\bar\Lambda_w(B)]\}_{w\in W}$. Let $p_e^C\in\mathbb{C}[\epsilon_1,\dots,\epsilon_n]$ be a representative of $[\bar\Lambda_e(C)]$ and similarly p_e^B for $[\bar\Lambda_e(B)]$. Then, by [BGG], we can take

$$p_e^B = \frac{\left(\prod_{i=1}^n \epsilon_i\right)\left(\prod_{1\le i<j\le n}(\epsilon_i^2-\epsilon_j^2)\right)}{|W|},$$

and

$$p_e^C = \frac{2^n\left(\prod_{i=1}^n \epsilon_i\right)\left(\prod_{1\le i<j\le n}(\epsilon_i^2-\epsilon_j^2)\right)}{|W|}.$$

Thus,

$$p_e^C = 2^n\, p_e^B.$$

By loc cit., $p_w^C := A^C_{w^{-1}} p_e^C$ represents the class $[\bar{\Lambda}_w(C)]$ and similarly $p_w^B := A^B_{w^{-1}} p_e^B$ represents the class $[\bar{\Lambda}_w(B)]$. Hence,

$$p_w^C = 2^{n-\mu(w)} p_w^B,$$

where (as in Section 8) $\mu(w)$ represents the number of times the simple reflection s_n appears in any reduced decomposition of w. Thus, we have the following:

Theorem 45. *The algebra isomorphism ϕ satisfies:*

$$\phi\big([\bar{\Lambda}_w(B)]\big) = 2^{\mu(w)-n}\, [\bar{\Lambda}_w(C)], \textit{ for any } w \in W.$$

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