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DESCENT OF LINE BUNDLES TO GIT QUOTIENTS OF FLAG VARIETIES BY MAXIMAL TORUS

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Dedicated to Bertram Kostant on his eightieth birthday

Abstract. Let G be a connected, simply connected semisimple complex algebraic group with a maximal torus T and let P be a parabolic subgroup containing T. Let $\mathcal{L}_P(\lambda)$ be a homogeneous ample line bundle on the flag variety $Y = G/P$. We give a necessary and sufficient condition for $\mathcal{L}_P(\lambda)$ to descend to a line bundle on the GIT quotient $Y(\lambda)/\!\!/T$. As a consequence of this result, we get the precise list of P-regular weights λ for which the line bundle $\mathcal{L}_P(\lambda)$ descends to the GIT quotient $Y(\lambda)/\!\!/T$.

1. Introduction

Let G be a connected, simply connected semisimple complex algebraic group with a maximal torus T and let P be a parabolic subgroup containing T. We denote their Lie algebras by the corresponding Gothic characters. The following theorem is one of our main results.

Theorem 3.9. *Let $\mathcal{L}_P(\lambda)$ be a homogeneous ample line bundle on the flag variety $Y = G/P$. Then, the line bundle $\mathcal{L}_P(\lambda)$ descends to a line bundle on the GIT quotient $Y(\lambda)/\!\!/T$ if and only if, for all the semisimple subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$ (in particular,* $\operatorname{rank}\mathfrak{s} = \operatorname{rank}\mathfrak{g}$),

$$\lambda \in \sum_{\alpha\in\triangle^+(\mathfrak{s})} \mathbb{Z}\alpha,$$

where $\triangle^+(\mathfrak{s})$ is the set of positive roots of $\mathfrak{s}$.

Using the above theorem, we explicitly get exactly for which λ the line bundle $\mathcal{L}_P(\lambda)$ descends to $Y(\lambda)/\!\!/T$. This is our second main result.

In the following Q (resp., Λ) is the root (resp., weight) lattice and we follow the indexing convention as in Bourbaki [B].

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Theorem 3.10. *With the notation as in the above theorem, the line bundle $\mathcal{L}_P(\lambda)$ descends to a line bundle on the GIT quotient $Y(\lambda)/\!\!/T$ if and only if λ is of the following form depending upon the type of G:*

(a) *G of type A_ℓ $(\ell \geqslant 1)$: $\lambda \in Q$.*
(b) *G of type B_ℓ $(\ell \geqslant 3)$: $\lambda \in 2Q$.*
(c) *G of type C_ℓ $(\ell \geqslant 2)$: $\lambda \in \mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell = 2\Lambda$.*
(d_1) *G of type D_4: $\lambda \in \{n_1\alpha_1 + 2n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4 \mid n_i \in \mathbb{Z}$ and $n_1 + n_3 + n_4$ is even$\}$.*
(d_2) *G of type D_ℓ $(\ell \geqslant 5)$: $\lambda \in \{2n_1\alpha_1 + 2n_2\alpha_2 + \cdots + 2n_{\ell-2}\alpha_{\ell-2} + n_{\ell-1}\alpha_{\ell-1} + n_\ell\alpha_\ell \mid n_i \in \mathbb{Z}$ and $n_{\ell-1} + n_\ell$ is even$\}$.*
(e) *G of type G_2: $\lambda \in \mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$.*
(f) *G of type F_4: $\lambda \in \mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4$.*
(g) *G of type E_6: $\lambda \in 6\Lambda$.*
(h) *G of type E_7: $\lambda \in 12\Lambda$.*
(i) *G of type E_8: $\lambda \in 60Q$.*

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2. Notation

Let G be a connected, simply connected, semisimple algebraic group over the field $\mathbb{C}$ of complex numbers. We fix a Borel subgroup B of G and a maximal torus T contained in B. Let B^- be the opposite Borel subgroup of G (i.e., the Borel subgroup B^- such that $B^- \cap B = T$). We denote the unipotent radicals of B, B^- by U, U^-, respectively. We denote the Lie algebra of any algebraic group by the corresponding Gothic character. In particular, the Lie algebras of G, B, T are denoted by $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$, respectively. Let $\triangle = \triangle(\mathfrak{g})$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$ and let $\triangle^+ = \triangle^+(\mathfrak{g})$ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$) with $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ the set of simple roots. For $\alpha \in \Delta$, we denote the corresponding root space in $\mathfrak{g}$ by $\mathfrak{g}_\alpha$. Let W be the Weyl group of G with $\{s_1, \ldots, s_\ell\} \subset W$ the set of simple reflections corresponding to the simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$, respectively. Let $Q := \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i \subset \mathfrak{t}^*$ be the root lattice and let

$$\Lambda := \{\lambda \in \mathfrak{t}^* \mid \lambda(\alpha_i^\vee) \in \mathbb{Z} \text{ for all } i\}$$

be the weight lattice, where $\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ are the simple coroots corresponding to the simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$, respectively. Let $\Lambda^+ \subset \Lambda$ be the set of dominant weights, i.e.,

$$\Lambda^+ := \{\lambda \in \mathfrak{t}^* \mid \lambda(\alpha_i^\vee) \in \mathbb{Z}^+ \text{ for all } i\},$$

where $\mathbb{Z}^+$ is the set of nonnegative integers. For $\lambda \in \Lambda^+$, we denote by $V(\lambda)$ the irreducible representation of $\mathfrak{g}$ with highest weight λ.

Let X be the group of characters of T. Then, X can be identified with Λ by taking the derivative of characters. For $\lambda \in \Lambda$, we denote the corresponding character of T by e^λ. Then, e^λ uniquely extends to a character of B.

We also consider (standard) parabolic subgroups $P \supset B$ of G. Define

$$\Pi_P = \{\text{simple roots } \alpha_i \mid -\alpha_i \text{ is a root of } \mathfrak{p}\},$$

$$\Lambda_P = \{\lambda \in \Lambda \mid \lambda(\alpha_i^\vee) = 0 \text{ for } \alpha_i \in \Pi_P\}, \quad \Lambda_P^+ = \Lambda_P \cap \Lambda^+,$$

and

$$\Lambda_P^\circ = \{\lambda \in \Lambda_P \mid \lambda(\alpha_i^\vee) > 0 \text{ for each } \alpha_i \in \Pi \backslash \Pi_P\}.$$

In particular, for $P = B$, $\Lambda_B = \Lambda$ and $\Lambda_B^+ = \Lambda^+$. We recall that e^λ extends to a character of P if and only if $\lambda \in \Lambda_P$. For $\lambda \in \Lambda_P$, let $\mathcal{L}_P(\lambda) = G \times_P \mathbb{C}_{-\lambda}$ be the homogeneous line bundle on the flag variety G/P associated to the principal P-bundle $G \to G/P$ via the one-dimensional representation $\mathbb{C}_{-\lambda}$ of P given by the character $e^{-\lambda}$. Then, $\mathcal{L}_P(\lambda)$ is an ample line bundle if and only if $\lambda \in \Lambda_P^\circ$. For $g \in G$ and $v \in \mathbb{C}_{-\lambda}$, we denote the equivalence class of (g, v) in $G \times_P \mathbb{C}_{-\lambda}$ by $[g, v]$.

3. Proofs

For any set of positive roots $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_n\}$, let $\mathbb{Z}\boldsymbol{\beta} := \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n \subset Q$ and let $\mathfrak{g}(\boldsymbol{\beta})$ be the semisimple subalgebra of $\mathfrak{g}$ with roots

$$\triangle(\mathfrak{g}(\boldsymbol{\beta})) := \triangle(\mathfrak{g}) \cap \mathbb{Z}\boldsymbol{\beta}.$$

We recall the following well-known result.

Lemma 3.1. *For any $\lambda \in \Lambda^+$ and any set of positive roots $\boldsymbol{\beta}$ such that the index of $\mathbb{Z}\boldsymbol{\beta}$ in the root lattice Q is finite, the 0-weight space of the submodule $U(\mathfrak{g}(\boldsymbol{\beta}))\cdot v_\lambda^+ \subset V(\lambda)$ is nonzero if and only if $\lambda \in \mathbb{Z}\boldsymbol{\beta}$, where v_λ^+ is a nonzero highest weight vector of $V(\lambda)$. Observe that since $\mathbb{Z}\boldsymbol{\beta}$ is of finite index in Q, the full Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}(\boldsymbol{\beta})$.*

Let P be a standard parabolic subgroup. For the action of T on G/P via the left multiplication, the isotropy subgroup $I_{gP} \subset T$ at any $gP \in G/P$ is clearly given by

$$I_{gP} = T \cap gPg^{-1}.$$

Lemma 3.2. *For $gP \in G/P$ and any $\mu \in \Lambda_P$, the isotropy subgroup I_{gP} acts trivially on the fiber $\mathcal{L}_P(\mu)|_{gP}$ if and only if*

$$e^\mu|_{P \cap g^{-1}Tg} \equiv 1.$$

Proof. For $t \in I_{gP} = T \cap gPg^{-1}$ and any nonzero vector $v_{-\mu} \in \mathbb{C}_{-\mu}$,

$$\begin{aligned} t[g, v_{-\mu}] &= [tg, v_{-\mu}] \\ &= [gg^{-1}tg, v_{-\mu}] \\ &= [g, (g^{-1}tg)v_{-\mu}], \quad \text{since } g^{-1}tg \in P, \\ &= [g, e^{-\mu}(g^{-1}tg)v_{-\mu}]. \end{aligned}$$

Thus,

$$t[g, v_{-\mu}] = [g, v_{-\mu}] \text{ for } t \in T \cap gPg^{-1} \qquad \text{if and only if} \qquad e^{-\mu}(g^{-1}tg) = 1.$$

This proves the lemma. $\square$

Lemma 3.3. *For any $g = \bar{w}up$, for $\bar{w} \in N(T), u \in U_P^-$ and $p \in P$, we have*

$$P \cap g^{-1}Tg = p^{-1}(T \cap u^{-1}Tu)p, \tag{1}$$

where $N(T)$ is the normalizer of T in G and U_P^- is the unipotent radical of the opposite parabolic P^- (where the Lie algebra of P^- is defined by $\mathfrak{p}^- := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{p})} \mathfrak{g}_{-\alpha}$, $\Delta(\mathfrak{p})$ being the set of roots of $\mathfrak{p}$).

Moreover, in the above, if we just assume that $u \in U^-$, then

$$P \cap g^{-1}Tg \supset p^{-1}(T \cap u^{-1}Tu)p. \tag{2}$$

Observe that $\bigcup_{\bar{w} \in N(T)} \bar{w}U_P^- P = G$. This follows since $Z := \bigcup_{\bar{w} \in N(T)} \bar{w}U_P^- P/P$ is a T-stable open subset of G/P and the complement $(G/P) \setminus Z$ does not contain any T-fixed points of G/P.

Proof. We first prove (1). We have

$$P \cap g^{-1}Tg = P \cap p^{-1}u^{-1}\bar{w}^{-1}T\bar{w}up,$$

i.e.,

$$P \cap g^{-1}Tg = p^{-1}(P \cap u^{-1}Tu)p. \tag{3}$$

Now, for any $t \in T$, if $u^{-1}tu \in P$, then $u^{-1}tu = t$. To prove this, observe that $u^{-1}tut^{-1} \in P \cap U_P^- = (1)$. Thus, $p^{-1}(P \cap u^{-1}Tu)p \subset p^{-1}Tp$ and hence $P \cap g^{-1}Tg \subset p^{-1}Tp \cap g^{-1}Tg$. The reverse inclusion is, of course, obvious. This proves equality (1).

Inclusion (2) of course follows from equality (3). □

For $Y = G/P$ and an ample line bundle $\mathcal{L}_P(\lambda)$ on Y (i.e., $\lambda \in \Lambda_P^\circ$), by $Y^{\text{ss}}(\lambda)$ we mean the set of semistable points in Y with respect to the action of T via the left multiplication on Y and T-linearlized ample line bundle $\mathcal{L}_P(\lambda)$. Then, as is well known, $Y^{\text{ss}}(\lambda) \subset Y$ is a Zariski open subset. Moreover, by Lemma 3.1, for any $n \geqslant 1$, $V(n\lambda)^T \neq 0$ if and only if $n\lambda \in Q$. Since Q is of finite index in Λ, we get from the next Lemma 3.4 that

$$Y^{\text{ss}}(\lambda) \neq \varnothing.$$

Lemma 3.4. *Let $\lambda \in \Lambda_P^\circ$. Then, $gP \in Y^{\text{ss}}(\lambda)$ if and only if $gv_{n\lambda}^+$ has a nonzero component in the zero weight space for some $n \geqslant 1$, where $v_{n\lambda}^+$ is a highest weight vector of $V(n\lambda)$.*

Proof. By definition, the point $gP \in Y$ is semistable if and only if there exists a section $\sigma \in H^0(Y, \mathcal{L}_P(n\lambda))^T$ (for some $n \geqslant 1$), such that $\sigma(gP) \neq 0$. Consider the G-equivariant isomorphism

$$\chi : V(n\lambda)^* \xrightarrow{\sim} H^0(Y, \mathcal{L}_P(n\lambda)),$$

where $\chi(f)(gP) = [g, (g^{-1}f)|_{\mathbb{C}v_{n\lambda}^+}]$. Thus,

$$\chi(f)(gP) \neq 0 \iff f(gv_{n\lambda}^+) \neq 0,$$

and hence gP is semistable if and only if $gv_{n\lambda}^+$ has a nonzero component in the zero weight space. □

Lemma 3.5. *For $gP \in Y^{\mathrm{ss}}(\lambda)$,*

$$e^{n\lambda}|_{P\cap g^{-1}Tg} \equiv 1 \quad \text{for some} \quad n \geqslant 1. \tag{4}$$

In particular,

$$e^{\lambda}|_{(P\cap g^{-1}Tg)^{\circ}} \equiv 1, \tag{5}$$

where $(P\cap g^{-1}Tg)^{\circ}$ is the identity component of $P\cap g^{-1}Tg$.

Proof. By Lemma 3.4, gP is semistable if and only if for some $n \geqslant 1$, $[gv^{+}_{n\lambda}]_0 \neq 0$, where $[gv^{+}_{n\lambda}]_0$ denotes its component in the zero weight space. For $t \in T$,

$$[tgv^{+}_{n\lambda}]_0 = [gv^{+}_{n\lambda}]_0. \tag{6}$$

But, for $t \in T\cap gPg^{-1}$,

$$[tgv^{+}_{n\lambda}]_0 = [gg^{-1}tgv^{+}_{n\lambda}]_0 = e^{n\lambda}(g^{-1}tg)[gv^{+}_{n\lambda}]_0. \tag{7}$$

Combining (6) and (7), we get $e^{n\lambda}(g^{-1}tg) = 1$. This proves (4).

The identity (5) follows immediately from (4) since a connected torus is a divisible group. $\square$

Lemma 3.6. *For any $x \in \mathfrak{u}^-$, let $\boldsymbol{\beta}_x$ be the subset of $\triangle^+$ such that $x = \sum_{\alpha\in\boldsymbol{\beta}_x} x_{-\alpha}$, where $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ is nonzero. Then, for $u = \operatorname{Exp} x$,*

$$T\cap u^{-1}Tu = \bigcap_{\alpha\in\boldsymbol{\beta}_x} \{t \in T \mid e^{\alpha}(t) = 1\}. \tag{8}$$

In particular, for a weight $\mu \in \Lambda$, we have $e^{\mu}|_{T\cap u^{-1}Tu} \equiv 1$ if and only if $\mu \in \mathbb{Z}\boldsymbol{\beta}_x$.

Proof. Take $t \in T$ such that $utu^{-1} \in T$. Then,

$$utu^{-1}t^{-1} \in T\cap U^- = \{1\}.$$

Thus, $utu^{-1} = t$, i.e., $tut^{-1} = u$. But,

$$\begin{aligned} tut^{-1} &= \operatorname{Exp}(\operatorname{Ad} t\cdot x) \\ &= \operatorname{Exp}\Bigl(\sum_{\alpha\in\boldsymbol{\beta}_x} e^{-\alpha}(t)x_{-\alpha}\Bigr) \\ &= \operatorname{Exp}(x). \end{aligned}$$

Since $\operatorname{Exp}|_{\mathfrak{u}^-}$ is a bijection,

$$\sum_{\alpha\in\boldsymbol{\beta}_x} e^{-\alpha}(t)x_{-\alpha} = \sum_{\alpha\in\boldsymbol{\beta}_x} x_{-\alpha}.$$

Thus, $e^{-\alpha}(t) = 1$ for any $\alpha \in \boldsymbol{\beta}_x$. This proves the inclusion

$$T\cap u^{-1}Tu \subset \bigcap_{\alpha\in\boldsymbol{\beta}_x} \{t \in T \mid e^{\alpha}(t) = 1\}.$$

The reverse inclusion follows by reversing the above calculation.

To prove the 'In particular' statement, consider the isomorphism $\xi : T \simeq \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*)$, $\xi(t)\mu = e^{\mu}(t)$ for $t \in T$ and $\mu \in \Lambda$. From this identification and (8) it is easy to see that $\xi(T \cap u^{-1}Tu) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda/\mathbb{Z}\boldsymbol{\beta}_x, \mathbb{C}^*)$. Now, if $\mu \in \Lambda\backslash(\mathbb{Z}\boldsymbol{\beta}_x)$, there exists a homomorphism $f : \Lambda/\mathbb{Z}\boldsymbol{\beta}_x \to \mathbb{C}^*$ such that $f(\mu) \neq 1$. From this we conclude the 'In particular' statement. $\square$

Since $Y^{\mathrm{ss}}(\lambda)$ is nonempty and Zariski open in Y, we can find a semistable point of the form $\mathrm{Exp}(x)P$, where $x \in \mathfrak{u}^-$ and $\boldsymbol{\beta}_x = \triangle^+$.

More generally, take any subset $\boldsymbol{\beta} \subset \triangle^+$ such that $\mathbb{Z}\boldsymbol{\beta}$ is of finite index in Q. Then, for some $n \geqslant 1, n\lambda \in \mathbb{Z}\boldsymbol{\beta}$ and hence, by Lemma 3.1,

$$\big(U(\mathfrak{g}(\boldsymbol{\beta})) \cdot v_{n\lambda}^+\big)^T \neq (0).$$

In particular,

$$\big(G(\boldsymbol{\beta})/P(\boldsymbol{\beta})\big) \cap Y^{\mathrm{ss}}(\lambda) \neq \varnothing,$$

where $G(\boldsymbol{\beta})$ is the connected (semisimple) subgroup of G with Lie algebra $\mathfrak{g}(\boldsymbol{\beta})$ and $P(\boldsymbol{\beta}) := P \cap G(\boldsymbol{\beta})$ is a parabolic subgroup of $G(\boldsymbol{\beta})$. Again, by Zariski density, we can find an element of the form $\mathrm{Exp}(x)P \in Y^{\mathrm{ss}}(\lambda)$ such that $x \in \mathfrak{u}^-$ and $\boldsymbol{\beta}_x$ is the set of all the positive roots of $G(\boldsymbol{\beta})$ (i.e., all the roots of $B \cap G(\boldsymbol{\beta})$).

Lemma 3.7. *For any subset $S \subset \triangle^+$, the quotient group*

$$T_S/T_S^{\circ} \simeq \mathrm{Tor}(\Lambda/\mathbb{Z}S),$$

where $T_S := \bigcap_{\alpha \in S}\{t \in T \mid e^{\alpha}(t) = 1\}$, T_S° denotes its identity component, and Tor *denotes the torsion subgroup.*

Proof. Recall the identification $\xi : T \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*)$ from the proof of Lemma 3.6. Under this identification

$$\xi(T_S) = \mathrm{Hom}_{\mathbb{Z}}\big(\Lambda/\mathbb{Z}S, \mathbb{C}^*\big).$$

Decompose

$$\Lambda/\mathbb{Z}S \simeq \mathrm{Tor}\big(\Lambda/\mathbb{Z}S\big) \oplus F,$$

where F is a free $\mathbb{Z}$-module. Thus,

$$\mathrm{Hom}_{\mathbb{Z}}\big(\Lambda/\mathbb{Z}S, \mathbb{C}^*\big) \simeq \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Tor}\big(\Lambda/\mathbb{Z}S\big), \mathbb{C}^*\big) \times \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{C}^*\big).$$

But $\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{C}^*) \simeq (\mathbb{C}^*)^m$, where $m := \mathrm{rk}\, F$. Thus, $T_S^{\circ} \simeq \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{C}^*)$ and hence

$$T_S/T_S^{\circ} \simeq \mathrm{Tor}(\Lambda/\mathbb{Z}S)^{\vee},$$

where $\mathrm{Tor}(\Lambda/\mathbb{Z}S)^{\vee} := \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Tor}(\Lambda/\mathbb{Z}S), \mathbb{C}^*\big)$. $\square$

Let H be a reductive group, X a projective variety, and $\mathcal{L}$ an ample H-equivariant line bundle on X. Then, recall that the *GIT quotient* $X/\!\!/H$ is by definition the uniform categorical quotient of the (open) set of semistable points X^{ss} by H (see [MFK, Theorem 1.10]).

We recall the following 'descent' lemma of Kempf (see [DN, Theorem 2.3]).

Lemma 3.8. *Let X, H, and $\mathcal{L}$ be as above. Then, an H-equivariant vector bundle $\mathcal{S}$ on X descends to a vector bundle on $X/\!\!/H$ (i.e., there exists a vector bundle $\mathcal{S}'$ on $X/\!\!/H$ such that its pull-back to X^{ss} under the canonical H-equivariant structure is H-equivariantly isomorphic to the restriction of $\mathcal{S}$ to X^{ss}) if and only if for any $x \in X^{\mathrm{ss}}$, the isotropy subgroup I_x acts trivially on the fiber $\mathcal{S}_x$. In fact (though we do not need this), for the 'if' part, it suffices to assume that I_x acts trivially for only those $x \in X^{\mathrm{ss}}$ such that the orbit $H \cdot x$ is closed in X^{ss}.*

Let P be a standard parabolic subgroup and let $\mathcal{L}_P(\lambda)$ be a homogeneous ample line bundle (i.e., $\lambda \in \Lambda_P^\circ$) on $Y = G/P$. Denote the GIT quotient of Y by T with respect to $\mathcal{L}_P(\lambda)$ by $Y(\lambda)/\!\!/T$. The following is one of our main results.

Theorem 3.9. *With the notation as above, the line bundle $\mathcal{L}_P(\lambda)$ descends to a line bundle on $Y(\lambda)/\!\!/T$ if and only if for all the semisimple subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$ (in particular,* $\operatorname{rank} \mathfrak{s} = \operatorname{rank} \mathfrak{g}$*),*

$$\lambda \in \mathbb{Z}\triangle^+(\mathfrak{s}),$$

where $\triangle^+(\mathfrak{s}) := \triangle^+ \cap \triangle(\mathfrak{s})$ is the set of positive roots of $\mathfrak{s}$.

Proof. By Lemma 3.8, $\mathcal{L}_P(\lambda)$ descends to $Y(\lambda)/\!\!/T$ if and only if for all the semistable points $gP = \bar{w}uP \in G/P$ (for $\bar{w} \in N(T)$ and $u \in U_P^-$), the isotropy subgroup I_{gP} acts trivially on the fiber $\mathcal{L}_P(\lambda)_{|gP}$. By Lemmas 3.2 and 3.3, this is equivalent to the requirement that $e^\lambda{}_{|T \cap u^{-1}Tu} \equiv 1$. Further, by Lemma 3.6, this is equivalent to the requirement that $\lambda \in \mathbb{Z}\boldsymbol{\beta}_x$, where $x \in \mathfrak{u}^-$ is the element with $\operatorname{Exp} x = u$.

By the discussion above Lemma 3.7, for any semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$, there exists an element $x \in \mathfrak{u}^-$ such that $\operatorname{Exp}(x)P \in G/P$ is semistable and, moreover, $\boldsymbol{\beta}_x = \triangle^+(\mathfrak{s})$. Thus, we get by Lemmas 3.2, 3.3, and 3.6 that $\lambda \in \mathbb{Z}\triangle^+(\mathfrak{s})$ for any such $\mathfrak{s}$ if the line bundle $\mathcal{L}_P(\lambda)$ descends to $Y(\lambda)/\!\!/T$.

Conversely, take any semistable point $\bar{w}\operatorname{Exp}(x)P \in G/P$ for $\bar{w} \in N(T)$ and $x \in \mathfrak{u}_P^-$. If $\mathbb{Z}\boldsymbol{\beta}_x \subset Q$ is not of finite index in Q, choose simple roots $\boldsymbol{\alpha}_x = \{\alpha_{i_1}, \dots, \alpha_{i_j}\}$ such that $\mathbb{Q}\boldsymbol{\beta}_x \cap \mathbb{Q}\boldsymbol{\alpha}_x = (0)$ and $\mathbb{Q}\boldsymbol{\beta}_x + \mathbb{Q}\boldsymbol{\alpha}_x = \bigoplus_{i=1}^{\ell} \mathbb{Q}\alpha_i$, where $\mathbb{Q}\boldsymbol{\beta}_x := \sum_{\beta \in \boldsymbol{\beta}_x} \mathbb{Q}\beta \subset \mathfrak{t}^*$ and $\mathbb{Q}\boldsymbol{\alpha}_x := \bigoplus_{n=1}^{j} \mathbb{Q}\alpha_{i_n} \subset \mathfrak{t}^*$. With this choice of $\boldsymbol{\alpha}_x$, the torsion submodule

$$\operatorname{Tor}(\Lambda/\mathbb{Z}\boldsymbol{\beta}_x) \hookrightarrow \operatorname{Tor}(\Lambda/(\mathbb{Z}\boldsymbol{\beta}_x + \mathbb{Z}\boldsymbol{\alpha}_x)). \tag{9}$$

To see this observe that we have the following short exact sequence:

$$0 \to \mathbb{Z}\boldsymbol{\alpha}_x \to \Lambda/\mathbb{Z}\boldsymbol{\beta}_x \to \Lambda/(\mathbb{Z}\boldsymbol{\beta}_x + \mathbb{Z}\boldsymbol{\alpha}_x) \to 0.$$

From this the assertion (9) follows easily.

Let $\mathfrak{s}$ be a semisimple subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ such that

$$\mathbb{Z}\triangle^+(\mathfrak{s}) = \mathbb{Z}\boldsymbol{\beta}_x + \mathbb{Z}\boldsymbol{\alpha}_x.$$

Choose an element $y = y_{\mathfrak{s}} \in \mathfrak{u}^-$ such that $\boldsymbol{\beta}_y = \triangle^+(\mathfrak{s})$ and $\operatorname{Exp}(y)P \in Y^{\mathrm{ss}}(\lambda)$. With this choice of y, by Lemma 3.6, we see that $T \cap v^{-1}Tv \subset T \cap u^{-1}Tu$, where

$u := \operatorname{Exp} x$ and $v := \operatorname{Exp} y$. We further show that the inclusion $T \cap v^{-1}Tv \subset T \cap u^{-1}Tu$ induces a surjective map

$$(T \cap v^{-1}Tv)/(T \cap v^{-1}Tv)^\circ \twoheadrightarrow (T \cap u^{-1}Tu)/(T \cap u^{-1}Tu)^\circ. \tag{10}$$

By Lemmas 3.6 and 3.7,

$$(T \cap v^{-1}Tv)/(T \cap v^{-1}Tv)^\circ \simeq \operatorname{Tor}\big(\Lambda/\mathbb{Z}\boldsymbol{\beta}_y\big)^\vee$$

and

$$(T \cap u^{-1}Tu)/(T \cap u^{-1}Tu)^\circ \cong \operatorname{Tor}\big(\Lambda/\mathbb{Z}\boldsymbol{\beta}_x\big)^\vee.$$

Combining the above identifications with (9) and the injectivity of $\mathbb{C}^*$, we get (10).

Since $\bar{w}uP$ is a semistable point, by Lemmas 3.3 and 3.5, $e^\lambda|_{(T\cap u^{-1}Tu)^\circ} \equiv 1$. Moreover, by the assumption and Lemma 3.6, $e^\lambda|_{T\cap v^{-1}Tv} \equiv 1$. Thus, $e^\lambda|_{T\cap u^{-1}Tu} \equiv 1$ by (10). This proves the theorem. □

Using the above theorem, we explicitly get exactly for which λ the bundle $\mathcal{L}_P(\lambda)$ descends to the GIT quotient $Y(\lambda)/\!\!/T$.

In the following, we follow the indexing convention as in Bourbaki [B, Planche I–IX].

Theorem 3.10. *Let G be a connected, simply connected simple algebraic group, $P \subset G$ a standard parabolic subgroup and let $\mathcal{L}_P(\lambda)$ be a homogeneous ample line bundle on the flag variety $Y = G/P$ (i.e., $\lambda \in \Lambda_P^\circ$). Then, the line bundle $\mathcal{L}_P(\lambda)$ descends to a line bundle on the GIT quotient $Y(\lambda)/\!\!/T$ if and only if λ is of the following form depending upon the type of G (in addition to $\lambda \in \Lambda_P^\circ$):*

(a) *G of type A_ℓ $(\ell \geqslant 1)$: $\lambda \in Q$.*
(b) *G of type B_ℓ $(\ell \geqslant 3)$: $\lambda \in 2Q$.*
(c) *G of type C_ℓ $(\ell \geqslant 2)$: $\lambda \in \mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell = 2\Lambda$.*
(d$_1$) *G of type D_4: $\lambda \in \{n_1\alpha_1 + 2n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4 \mid n_i \in \mathbb{Z}$ and $n_1 + n_3 + n_4$ is even$\}$.*
(d$_2$) *G of type D_ℓ $(\ell \geqslant 5)$: $\lambda \in \{2n_1\alpha_1 + 2n_2\alpha_2 + \cdots + 2n_{\ell-2}\alpha_{\ell-2} + n_{\ell-1}\alpha_{\ell-1} + n_\ell\alpha_\ell \mid n_i \in \mathbb{Z}$ and $n_{\ell-1} + n_\ell$ is even$\}$.*
(e) *G of type G_2: $\lambda \in \mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$.*
(f) *G of type F_4: $\lambda \in \mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4$.*
(g) *G of type E_6: $\lambda \in 6\Lambda$.*
(h) *G of type E_7: $\lambda \in 12\Lambda$.*
(i) *G of type E_8: $\lambda \in 60Q$.*

Proof. The theorem follows by using Theorem 3.9 and the classification of the semisimple subalgebras of $\mathfrak{g}$ of maximal rank due to Borel–Siebenthal (see [W, Theorem 8.10.8]). In the following chart, we only list all the proper maximal semisimple subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ containing the (fixed) Cartan subalgebra $\mathfrak{t}$ up to a conjugation under the Weyl group. In the following, θ denotes the highest root and $\mathfrak{s}_i$ denotes the semisimple subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ with simple roots $\{\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \alpha_\ell, -\theta\}$:

(a) A_ℓ $(\ell \geqslant 1)$: none.

(b) B_ℓ ($\ell \geqslant 3$): $\mathfrak{s}_i, 2 \leqslant i \leqslant \ell$.
(c) C_ℓ ($\ell \geqslant 2$): $\mathfrak{s}_i, 1 \leqslant i \leqslant \ell - 1$.
(d) D_ℓ ($\ell \geqslant 4$): $\mathfrak{s}_i, 2 \leqslant i \leqslant \ell - 2$.
(e) G_2: $\mathfrak{s}_i$, $i = 1, 2$.
(f) F_4: $\mathfrak{s}_i, i = 1, 2, 4$.
(g) E_6: $\mathfrak{s}_i, i = 2, 3, 4, 5$.
(h) E_7: $\mathfrak{s}_i, i = 1, 2, 3, 5, 6$.
(i) E_8: $\mathfrak{s}_i, i = 1, 2, 5, 7, 8$.

We denote by $L(\mathfrak{g}; \alpha_1, \dots, \alpha_\ell)$ the intersection of $\mathbb{Z}\triangle^+(\mathfrak{s})$ (inside the root lattice Q) as $\mathfrak{s}$ varies over all possible semisimple subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ containing the (fixed) Cartan subalgebra $\mathfrak{t}$.

In the following, we determine $L(\mathfrak{g}; \alpha_1, \dots, \alpha_\ell)$ for each simple $\mathfrak{g}$.

(a) A_ℓ ($\ell \geqslant 1$): In this case $L(\mathsf{A}_\ell; \alpha_1, \dots, \alpha_\ell) = Q$.
(c) C_ℓ ($\ell \geqslant 2$): By the above chart,

$$\begin{aligned} L(\mathsf{C}_\ell; \alpha_1, \dots, \alpha_\ell) = [&(\mathbb{Z}\theta + L(\mathsf{C}_{\ell-1}; \alpha_2, \dots, \alpha_\ell)) \\ &\cap (L(\mathsf{C}_2; \alpha_1, -\theta) + L(\mathsf{C}_{\ell-2}; \alpha_3, \dots, \alpha_\ell)) \cap \cdots \\ &\cap (L(\mathsf{C}_{\ell-2}; \alpha_{\ell-3}, \dots, \alpha_1, -\theta) + L(\mathsf{C}_2; \alpha_{\ell-1}, \alpha_\ell)) \\ &\cap (L(\mathsf{C}_{\ell-1}; \alpha_{\ell-2}, \dots, \alpha_1, -\theta) + \mathbb{Z}\alpha_\ell)]_W, \end{aligned}$$

where $[M]_W$ denotes $\bigcap_{w \in W} wM$. By induction, for $j < \ell$,

$$L(\mathsf{C}_j; \alpha_{\ell-j+1}, \dots, \alpha_\ell) = \mathbb{Z}2\alpha_{\ell-j+1} + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell$$

and

$$\begin{aligned} L(\mathsf{C}_j; \alpha_{j-1}, \dots, \alpha_1, -\theta) &= \mathbb{Z}2\alpha_{j-1} + \cdots + \mathbb{Z}2\alpha_1 + \mathbb{Z}\theta \\ &= \mathbb{Z}2\alpha_{j-1} + \cdots + \mathbb{Z}2\alpha_1 + \mathbb{Z}(2\alpha_j + \cdots + 2\alpha_{\ell-1} + \alpha_\ell), \end{aligned}$$

since

$$\theta(\mathsf{C}_\ell; \alpha_1, \dots, \alpha_\ell) = 2\alpha_1 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell.$$

Thus,

$$L(\mathsf{C}_\ell; \alpha_1, \dots, \alpha_\ell) = [\mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell]_W.$$

But, $\mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell$ is W-stable and hence

$$L(\mathsf{C}_\ell; \alpha_1, \dots, \alpha_\ell) = \mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell.$$

This proves part (c) of the theorem.

(d) D_ℓ ($\ell \geqslant 4$): We first consider the case of D_4. In this case,

$$\begin{aligned} L(\mathsf{D}_4; \alpha_1, \dots, \alpha_4) &= [L(\mathsf{A}_1; \alpha_1) + L(\mathsf{A}_1; -\theta) + L(\mathsf{A}_1; \alpha_3) + L(\mathsf{A}_1; \alpha_4)]_W \\ &= [\mathbb{Z}\alpha_1 + \mathbb{Z}\theta + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4]_W \\ &= [\mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4]_W. \end{aligned}$$

Now, the sublattice $L' := \{n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4 : n_i \in \mathbb{Z}, n_2$ is even and $\sum_{i=1}^4 n_i$ is even$\} \subset \mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4$ is W-stable (as is easy to see). Moreover, clearly the index of L' in $\mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4$ is 2 and $\mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4$ is not W-stable (as can be seen by applying the second simple reflection s_2). Thus $[\mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4]_W = L'$, proving the theorem in this case.

We now come to the case of general D_ℓ. By the above chart and the A_ℓ case,

$$\begin{aligned} L(\mathsf{D}_\ell; \alpha_1, \dots, \alpha_\ell) = & [(L(\mathsf{D}_2; \alpha_1, -\theta) + L(\mathsf{D}_{\ell-2}; \alpha_3, \dots, \alpha_\ell)) \cap \cdots \\ & \cap (L(\mathsf{D}_{\ell-2}; \alpha_{\ell-3}, \dots, \alpha_1, -\theta) + L(\mathsf{D}_2; \alpha_{\ell-1}, \alpha_\ell))]_W, \end{aligned}$$

where D_k for $k = 2, 3$ is interpreted as A_k. Set

$$L'(\alpha_{\ell-3}, \alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell) = \left\{ \sum_{i=\ell-3}^{\ell} n_i\alpha_i \mid n_i \in \mathbb{Z}, \sum n_i \text{ is even and } n_{\ell-2} \text{ is even} \right\},$$

and if $\ell - k > 4$, set

$$L'(\alpha_{k+1}, \dots, \alpha_\ell) = \left\{ \sum_{i=k+1}^{\ell} n_i\alpha_i \mid n_i \in \mathbb{Z}, \sum n_i \text{ is even and } n_{k+1}, \dots, n_{\ell-2} \text{ are even} \right\},$$

and if $\ell - k < 4$, set

$$L'(\alpha_{k+1}, \dots, \alpha_\ell) = \mathbb{Z}\alpha_{k+1} + \cdots + \mathbb{Z}\alpha_\ell.$$

By induction and the A_i $(i = 2, 3)$ case, for $2 \leqslant \ell - k < \ell$,

$$L(\mathsf{D}_{\ell-k}; \alpha_{k+1}, \dots, \alpha_\ell) = L'(\alpha_{k+1}, \dots, \alpha_\ell).$$

Thus,

$$\begin{aligned} L(\mathsf{D}_\ell; \alpha_1, \dots, \alpha_\ell) = & \big[(\mathbb{Z}\alpha_1 + \mathbb{Z}\theta + L'(\alpha_3, \dots, \alpha_\ell)) \\ & \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\theta + L'(\alpha_4, \dots, \alpha_\ell)) \\ & \cap (L'(\alpha_3, \alpha_2, \alpha_1, -\theta) + L'(\alpha_5, \dots, \alpha_\ell)) \cap \cdots \\ & \cap (L'(\alpha_{\ell-5}, \dots, \alpha_1, -\theta) + L'(\alpha_{\ell-3}, \alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell)) \\ & \cap (L'(\alpha_{\ell-4}, \dots, \alpha_1, -\theta) + \mathbb{Z}\alpha_{\ell-2} + \mathbb{Z}\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell) \\ & \cap (L'(\alpha_{\ell-3}, \dots, \alpha_1, -\theta) + \mathbb{Z}\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell)]_W \\ = & [L'(\alpha_1, \dots, \alpha_\ell)]_W. \end{aligned}$$

It is easy to see that $L'(\alpha_1, \dots, \alpha_\ell)$ is W-invariant and hence

$$[L'(\alpha_1, \dots, \alpha_\ell)]_W = L'(\alpha_1, \dots, \alpha_\ell).$$

This proves the theorem in the case of D_ℓ.

(b) B_ℓ ($\ell \geqslant 3$): By the above chart, we get that

$$\begin{aligned} L(\mathsf{B}_\ell;\alpha_1,\dots,\alpha_\ell) = [&(\mathbb{Z}\alpha_1 + \mathbb{Z}\theta + L(\mathsf{B}_{\ell-2};\alpha_3,\dots,\alpha_\ell)) \\ &\cap (L(\mathsf{A}_3;\alpha_1,\alpha_2,-\theta) + L(\mathsf{B}_{\ell-3};\alpha_4,\dots,\alpha_\ell)) \\ &\cap (L(\mathsf{D}_4;\alpha_3,\alpha_2,\alpha_1,-\theta) + L(\mathsf{B}_{\ell-4};\alpha_5,\dots,\alpha_\ell)) \cap \cdots \\ &\cap (L(\mathsf{D}_{\ell-3};\alpha_{\ell-4},\dots,\alpha_1,-\theta) + L(\mathsf{B}_3;\alpha_{\ell-2},\alpha_{\ell-1},\alpha_\ell)) \\ &\cap (L(\mathsf{D}_{\ell-2};\alpha_{\ell-3},\dots,\alpha_1,-\theta) + L(\mathsf{C}_2;\alpha_\ell,\alpha_{\ell-1})) \\ &\cap (L(\mathsf{D}_{\ell-1};\alpha_{\ell-2},\dots,\alpha_1,-\theta) + \mathbb{Z}\alpha_\ell) \\ &\cap L(\mathsf{D}_\ell;\alpha_{\ell-1},\dots,\alpha_1,-\theta)]_W. \end{aligned}$$

By using the result for D_p and A_3 and also, by induction, for $\mathsf{B}_j (j < \ell)$, we get that (with L' as in the proof of the D_ℓ case)

$$\begin{aligned} L(\mathsf{B}_\ell;\alpha_1,\dots,\alpha_\ell) &= [(\mathbb{Z}\alpha_1 + \mathbb{Z}\theta + \mathbb{Z}2\alpha_3 + \cdots + \mathbb{Z}2\alpha_\ell) \\ &\quad \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\theta + \mathbb{Z}2\alpha_4 + \cdots + \mathbb{Z}2\alpha_\ell) \\ &\quad \cap (L'(\alpha_3,\alpha_2,\alpha_1,-\theta) + \mathbb{Z}2\alpha_5 + \cdots + \mathbb{Z}2\alpha_\ell) \cap \cdots \\ &\quad \cap (L'(\alpha_{\ell-4},\dots,\alpha_1,-\theta) + \mathbb{Z}2\alpha_{\ell-2} + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}2\alpha_\ell) \\ &\quad \cap (L'(\alpha_{\ell-3},\dots,\alpha_1 - \theta) + \mathbb{Z}2\alpha_\ell + \mathbb{Z}\alpha_{\ell-1}) \\ &\quad \cap (L'(\alpha_{\ell-2},\dots,\alpha_1,-\theta) + \mathbb{Z}\alpha_\ell) \ \cap L'(\alpha_{\ell-1},\dots,\alpha_1,-\theta)]_W \\ &= [\mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_\ell]_W \\ &= 2[Q]_W \\ &= 2Q, \quad \text{since } Q \text{ is } W\text{-stable.} \end{aligned}$$

This proves the theorem for the case of B_ℓ.

(e) G_2: By the chart and the theorem for the case of A_2,

$$\begin{aligned} L(\mathsf{G}_2;\alpha_1,\alpha_2) &= [(L(\mathsf{A}_2;\alpha_2,-\theta)) \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\theta)]_W \\ &= [(\mathbb{Z}\alpha_2 + \mathbb{Z}\theta) \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\theta)]_W \\ &= [(\mathbb{Z}\alpha_2 + \mathbb{Z}3\alpha_1) \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2)]_W \\ &= [\mathbb{Z}3\alpha_1 + \mathbb{Z}2\alpha_2]_W \\ &= \mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2, \end{aligned}$$

since $\mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$ is W-stable, whereas $\mathbb{Z}3\alpha_1 + \mathbb{Z}2\alpha_2$ is not W-stable and $\mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$ is of index 2 inside $\mathbb{Z}3\alpha_1 + \mathbb{Z}2\alpha_2$.

(f) F_4: By the chart,

$$\begin{aligned} L(\mathsf{F}_4;\alpha_1,\alpha_2,\alpha_3,\alpha_4) = [&(\mathbb{Z}\theta + L(\mathsf{C}_3;\alpha_4,\alpha_3,\alpha_2)) \\ &\cap (L(\mathsf{A}_2;-\theta,\alpha_1) + L(\mathsf{A}_2;\alpha_3,\alpha_4)) \\ &\cap L(\mathsf{B}_4;-\theta,\alpha_1,\alpha_2,\alpha_3)]_W. \end{aligned}$$

By the theorem for $\mathsf{A}_2, \mathsf{C}_3$, and B_4, we get

$$\begin{aligned} L(\mathsf{F}_4; \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= [(\mathbb{Z}\theta + \mathbb{Z}2\alpha_4 + \mathbb{Z}2\alpha_3 + \mathbb{Z}\alpha_2) \\ &\quad \cap (\mathbb{Z}\theta + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4) \\ &\quad \cap (\mathbb{Z}2\theta + \mathbb{Z}2\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}2\alpha_3)]_W \\ &= [(\mathbb{Z}2\alpha_1 + \mathbb{Z}2\alpha_4 + \mathbb{Z}2\alpha_3 + \mathbb{Z}\alpha_2) \\ &\quad \cap (\mathbb{Z}3\alpha_2 + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4) \\ &\quad \cap (\mathbb{Z}4\alpha_4 + \mathbb{Z}2\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}2\alpha_3)]_W \\ &= [M]_W, \text{ where } M := \mathbb{Z}2\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}2\alpha_3 + \mathbb{Z}4\alpha_4. \end{aligned}$$

Now,

$$\mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4 \subset [M]_W,$$

since $\mathbb{Z}6\alpha_1+\mathbb{Z}6\alpha_2+\mathbb{Z}12\alpha_3+\mathbb{Z}12\alpha_4$ is W-stable (as can be easily seen). Conversely, take $\mu \in [M]_W$. Then, $w\mu \in M$ for any $w \in W$. Since

$$s_i w\mu = w\mu - (w\mu)(\alpha_i^\vee)\alpha_i;$$

for any $\mu \in [M]_W$ and any $w \in W$,

$$w\mu(\alpha_2^\vee) \in 6\mathbb{Z} \text{ and } w\mu(\alpha_4^\vee) \in 4\mathbb{Z}.$$

Since $\alpha_1^\vee \in W.\alpha_2^\vee$ (see [H, §10.4, Lemma C]), we get

$$\mu(\alpha_1^\vee) \in 6\mathbb{Z}, \mu(\alpha_2^\vee) \in 6\mathbb{Z}, \text{ and } \mu(\alpha_4^\vee) \in 4\mathbb{Z}.$$

Take $\mu = n_1 2\alpha_1 + n_2 6\alpha_2 + n_3 2\alpha_3 + n_4 4\alpha_4 \in [M]_W$. Then, from the above, we get

$$4n_1 - 6n_2 \in 6\mathbb{Z}, \quad -2n_1 + 12n_2 - 2n_3 \in 6\mathbb{Z}, \quad \text{and} \quad -2n_3 + 8n_4 \in 4\mathbb{Z}.$$

This gives that $n_1 \in 3\mathbb{Z}$ and $n_3 \in 6\mathbb{Z}$. Thus, $\mu \in \mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}4\alpha_4$. Considering $s_3(\mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}4\alpha_4)$, we see that

$$[\mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}4\alpha_4]_W \subset \mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4.$$

Thus, we conclude that

$$[M]_W = \mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4.$$

This completes the proof of the theorem in the case of F_4.

(g) E_6: By the chart and the case of A_ℓ,

$$\begin{aligned}
L(\mathsf{E}_6;\alpha_1,\dots,\alpha_6) &= [(\mathbb{Z}\theta + L(\mathsf{A}_5;\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6))\\
&\quad \cap (\mathbb{Z}\alpha_1 + L(\mathsf{A}_5;-\theta,\alpha_2,\alpha_4,\alpha_5,\alpha_6))\\
&\quad \cap (L(\mathsf{A}_2;\alpha_1,\alpha_3) + L(\mathsf{A}_2;-\theta,\alpha_2) + L(\mathsf{A}_2;\alpha_5,\alpha_6))\\
&\quad \cap (L(\mathsf{A}_5;\alpha_1,\alpha_3,\alpha_4,\alpha_2,-\theta) + \mathbb{Z}\alpha_6)]_W\\
&= [(\mathbb{Z}\theta + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4 + \mathbb{Z}\alpha_5 + \mathbb{Z}\alpha_6)\\
&\quad \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\theta + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_4 + \mathbb{Z}\alpha_5 + \mathbb{Z}\alpha_6)\\
&\quad \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_3 + \mathbb{Z}\theta + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_5 + \mathbb{Z}\alpha_6)\\
&\quad \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4 + \mathbb{Z}\alpha_2 + \mathbb{Z}\theta + \mathbb{Z}\alpha_6)]_W\\
&= [(\mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4 + \mathbb{Z}\alpha_5 + \mathbb{Z}\alpha_6)\\
&\quad \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}2\alpha_3 + \mathbb{Z}\alpha_4 + \mathbb{Z}\alpha_5 + \mathbb{Z}\alpha_6)\\
&\quad \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}3\alpha_4 + \mathbb{Z}\alpha_5 + \mathbb{Z}\alpha_6)\\
&\quad \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4 + \mathbb{Z}2\alpha_5 + \mathbb{Z}\alpha_6)]_W\\
&= [M_6]_W,
\end{aligned}$$

where $M_6 := \mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}2\alpha_3 + \mathbb{Z}3\alpha_4 + \mathbb{Z}2\alpha_5 + \mathbb{Z}\alpha_6$.

Clearly, $6\Lambda \subset M_6$ and since Λ is W-stable,

$$6\Lambda \subset [M_6]_W. \tag{11}$$

Conversely, take $\mu \in [M_6]_W$. Then, for any $w \in W, w\mu \in M_6$. Since $s_i(w\mu) = w\mu - (w\mu)(\alpha_i^\vee)\alpha_i$; for any $\mu \in [M_6]_W$ and any $w \in W$, $w\mu(\alpha_2^\vee) \in 2\mathbb{Z}$, and $w\mu(\alpha_4^\vee) \in 3\mathbb{Z}$. Since E_6 is simplylaced, the Weyl group W acts transitively on the coroots [H, §10.4, Lemma C]. Thus, $\mu(\alpha_i^\vee) \in 2\mathbb{Z}\cap 3\mathbb{Z} = 6\mathbb{Z}$ for all the simple coroots $\alpha_i^\vee$. This proves that $\mu \in 6\Lambda$, i.e.,

$$[M_6]_W \subset 6\Lambda. \tag{12}$$

Comparing (11) and (12) we get $[M_6]_W = 6\Lambda$. This proves the theorem in the case of E_6.

(h) E_7: By the chart and the result for the cases of $\mathsf{A}_\ell, \mathsf{D}_\ell$, we get

$$\begin{aligned}
L(\mathsf{E}_7;\alpha_1,\dots,\alpha_7) &= [(\mathbb{Z}\theta + L(\mathsf{D}_6;\alpha_7,\alpha_6,\dots,\alpha_2))\\
&\quad \cap\, L(\mathsf{A}_7;-\theta,\alpha_1,\alpha_3,\alpha_4,\dots,\alpha_7)\\
&\quad \cap\, (L(\mathsf{A}_2;-\theta,\alpha_1) + L(\mathsf{A}_5;\alpha_2,\alpha_4,\alpha_5,\alpha_6,\alpha_7))\\
&\quad \cap\, (L(\mathsf{A}_5;-\theta,\alpha_1,\alpha_3,\alpha_4,\alpha_2) + L(\mathsf{A}_2;\alpha_6,\alpha_7))\\
&\quad \cap\, (L(\mathsf{D}_6;-\theta,\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_2) + \mathbb{Z}\alpha_\ell)]_W\\
&= [(\mathbb{Z}\theta + L'(\alpha_7,\alpha_6,\dots,\alpha_2))\\
&\quad \cap\, (\mathbb{Z}\theta + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4 + \cdots + \mathbb{Z}\alpha_7)\\
&\quad \bigcap\, (\mathbb{Z}\theta + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_4 + \cdots + \mathbb{Z}\alpha_7)\\
&\quad \cap\, (\mathbb{Z}\theta + \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_4 + \mathbb{Z}\alpha_6 + \mathbb{Z}\alpha_7)\\
&\quad \cap\, (L'(-\theta,\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_2) + \mathbb{Z}\alpha_7)]_W\\
&= [M_7]_W,
\end{aligned}$$

where $M_7 := \mathbb{Z}4\alpha_1 + \mathbb{Z}2\alpha_2 + \mathbb{Z}6\alpha_3 + \mathbb{Z}2\alpha_4 + \mathbb{Z}6\alpha_5 + \mathbb{Z}4\alpha_6 + \mathbb{Z}2\alpha_7$.

Clearly, $12\Lambda \subset M_7$ and since Λ is W-stable, $12\Lambda \subset [M_7]_W$.

For any $\mu \in [M_7]_W$, by considering $(w\mu)(\alpha_1^\vee)$ and $(w\mu)(\alpha_3^\vee)$ as in the proof of the theorem for the case of E_6, we get that $\mu(\alpha_i^\vee) \in 4\mathbb{Z} \cap 6\mathbb{Z} = 12\mathbb{Z}$ for all the simple coroots $\alpha_i^\vee$. Thus,

$$[M_7]_W \subset 12\Lambda \text{ and hence } [M_7]_W = 12\Lambda.$$

This takes care of the case of E_7.

Finally, we come to the following:

(i) E_8: By the chart and the theorem for $\mathsf{E}_6, \mathsf{E}_7, \mathsf{D}_8$, and A_ℓ, we get (denoting the weight lattice of E_i by $\Lambda(\mathsf{E}_i)$).

$$\begin{aligned} L(\mathsf{E}_8; \alpha_1, \dots, \alpha_8) &= [L(\mathsf{D}_8; -\theta, \alpha_8, \alpha_7, \dots, \alpha_2) \\ &\quad \cap L(\mathsf{A}_8; \alpha_1, \alpha_3, \alpha_4, \dots, \alpha_8, -\theta) \\ &\quad \cap (L(\mathsf{A}_4; \alpha_1, \alpha_3, \alpha_4, \alpha_2) + L(\mathsf{A}_4; \alpha_6, \alpha_7, \alpha_8, -\theta)) \\ &\quad \cap (L(\mathsf{E}_6; \alpha_1, \dots, \alpha_6) + L(\mathsf{A}_2; \alpha_8, -\theta)) \\ &\quad \cap (L(\mathsf{E}_7; \alpha_1, \dots, \alpha_7) + \mathbb{Z}\theta)]_W \\ &= [L'(\theta, \alpha_8, \alpha_7, \dots, \alpha_2) \\ &\quad \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4 + \cdots + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \\ &\quad \cap (\mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_4 + \mathbb{Z}\alpha_6 + \mathbb{Z}\alpha_7 + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \\ &\quad \cap (6\Lambda(\mathsf{E}_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \cap (12\Lambda(\mathsf{E}_7) + \mathbb{Z}\theta)]_W \\ &= [M_8' \cap (6\Lambda(\mathsf{E}_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \cap (12\Lambda(\mathsf{E}_7) + \mathbb{Z}\theta)]_W, \end{aligned}$$

where

$$\begin{aligned} M_8' := \{\mathbb{Z}4\alpha_1 + n3\alpha_2 + m\alpha_3 + \mathbb{Z}2\alpha_4 + \mathbb{Z}10\alpha_5 + \mathbb{Z}2\alpha_6 + \mathbb{Z}2\alpha_7 + \mathbb{Z}2\alpha_8 \\ \mid n, m \in \mathbb{Z} \text{ and } n+m \text{ is even}\}. \end{aligned}$$

But the coefficient of α_3 in any element of $6\Lambda(\mathsf{E}_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta$ is even. Thus,

$$\begin{aligned} &[M_8' \cap (6\Lambda(\mathsf{E}_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \cap (12\Lambda(\mathsf{E}_7) + \mathbb{Z}\theta)]_W \\ &\qquad = [M_8 \cap (6\Lambda(\mathsf{E}_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \cap (12\Lambda(\mathsf{E}_7) + \mathbb{Z}\theta)]_W, \end{aligned}$$

where

$$M_8 := \mathbb{Z}4\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}2\alpha_3 + \mathbb{Z}2\alpha_4 + \mathbb{Z}10\alpha_5 + \mathbb{Z}2\alpha_6 + \mathbb{Z}2\alpha_7 + \mathbb{Z}2\alpha_8.$$

For any $\mu \in [M_8]_W$, by considering $(w\mu)(\alpha_1^\vee), (w\mu)(\alpha_2^\vee)$, and $(w\mu)(\alpha_5^\vee)$, as in the proof of the theorem for the case of E_6, we get

$$[M_8]_W \subset 60\Lambda(\mathsf{E}_8). \tag{13}$$

Conversely, since $\Lambda(\mathsf{E}_8) = Q(\mathsf{E}_8)$, we get that

$$60\Lambda(\mathsf{E}_8) \subset M_8 \cap (6\Lambda(\mathsf{E}_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \cap (12\Lambda(\mathsf{E}_7) + \mathbb{Z}\theta), \tag{14}$$

and hence

$$60\Lambda(\mathsf{E}_8) \subset [M_8 \cap (6\Lambda(\mathsf{E}_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \cap (12\Lambda(\mathsf{E}_7) + \mathbb{Z}\theta)]_W.$$

To prove (14), it suffices to show that

$$60\alpha_7 \in 6\Lambda(\mathsf{E}_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta \tag{15}$$

and

$$60\alpha_8 \in 12\Lambda(\mathsf{E}_7) + \mathbb{Z}\theta. \tag{16}$$

To prove (15), observe that

$$\begin{aligned} 60\alpha_7 &= 20\theta - 40\alpha_8 - 20(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6) \\ &= 20\theta - 40\alpha_8 - 60\omega_6(\mathsf{E}_6), \end{aligned}$$

where $\omega_6(\mathsf{E}_6)$ is the sixth fundamental weight of E_6. Similarly, to prove (16), observe that

$$\begin{aligned} 60\alpha_8 &= 30\theta - 30(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7) \\ &= 30\theta - 60\omega_7(\mathsf{E}_7). \end{aligned}$$

This proves (15) and (16) and thus (14). Combining (13) and (14), we get that

$$[M_8 \cap (6\Lambda(\mathsf{E}_6) + \mathbb{Z}\alpha_8 + \mathbb{Z}\theta) \cap (12\Lambda(\mathsf{E}_7) + \mathbb{Z}\theta)]_W = 60\Lambda(\mathsf{E}_8) = 60Q.$$

This proves the theorem for E_8 and hence the theorem is completely proved. □

Remark 3.11. Theorem 3.10(a) was obtained earlier by Howard [Ho].

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