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Explicit determination of the Picard group of moduli spaces of semistable G-bundles on curves

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Introduction

Let G be a connected, simply-connected, simple affine algebraic group and $\mathcal{C}_g$ be a smooth irreducible projective curve of any genus $g \geq 1$ over $\mathbb{C}$. Denote by $\mathfrak{M}_{\mathcal{C}_g}(G)$ the moduli space of semistable principal G-bundles on $\mathcal{C}_g$. Let $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ be the Picard group of $\mathfrak{M}_{\mathcal{C}_g}(G)$ and let X be the infinite Grassmannian of the affine Kac-Moody group associated to G. It is known that $\text{Pic}(X) \simeq \mathbb{Z}$ and is generated by a homogenous line bundle $\mathfrak{L}_{\chi_0}$. Also, as proved by Kumar-Narasimhan [KN], there exists a canonical injective group homomorphism

$$\beta : \text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G)) \hookrightarrow \text{Pic}(X),$$

which takes $\Theta_V(\mathcal{C}_g, G) \mapsto \mathfrak{L}_{\chi_0}^{m_V}$ for any finite dimensional representation V of G, where $\Theta_V(\mathcal{C}_g, G)$ is the theta bundle associated to the G-module V and m_V is its Dynkin index (cf. Theorem 2.2). As an immediate corollary, they obtained that

$$\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G)) \simeq \mathbb{Z},$$

generalizing the corresponding result for $G = SL(n)$ proved by Drezet-Narasimhan [DN]. However, the precise image of β was not known for non-classical G excluding G_2. (For classical G and G_2, see [KN], [LS], [BLS].) The main aim of this paper is to determine the image of β for an arbitrary G. It is shown that the image of β is generated by $\mathfrak{L}_{\chi_0}^{m_G}$, where m_G is the least common multiple of the coefficients of the coroot $\theta^\vee$ written in terms of the simple coroots, θ being the highest root of G (cf. Theorem 2.4, see also Proposition 2.3 and the subsequent discussion where m_G is explicitly given for each G). As a consequence, we obtain that the theta bundles $\Theta_V(\mathcal{C}_g, G)$, where V runs over all the finite dimensional representations of G, generate $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ (cf. Theorem 1.3). In fact, it is shown that there is a fundamental weight ω_d such that the theta bundle $\Theta_{V(\omega_d)}(\mathcal{C}_g, G)$ corresponding to the irreducible highest weight G-module $V(\omega_d)$ with highest weight

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ω_d generates $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ (cf. Theorem 2.4). All these fundamental weights ω_d are explicitly determined in Proposition 2.3.

It may be mentioned that Picard group of the moduli *stack* of G-bundles is studied in [LS], [BLS], [T_2].

We now briefly outline the idea of the proofs. Recall that, by a celebrated result of Narasimhan-Seshadri, the underlying real analytic space $M_g(G)$ of $\mathfrak{M}_{\mathcal{C}_g}(G)$ admits a description as the space of representations of the fundamental group $\pi_1(\mathcal{C}_g)$ into a fixed compact form of G up to conjugation. In particular, $M_g(G)$ depends only upon g and G (and not on the specific choice of the projective curve $\mathcal{C}_g$). Moreover, this description gives rise to a standard embedding $i_g : M_g(G) \hookrightarrow M_{g+1}(G)$.

Let V be any finite dimensional representation of G. We first show that the first Chern class of the theta bundle $\Theta_V(\mathcal{C}_g, G)$ does not depend upon the choice of the smooth projective curve $\mathcal{C}_g$, as long as g is fixed (cf. Proposition 1.6).

We next show that the first Chern class of $\Theta_V(\mathcal{C}_{g+1}, G)$ restricts to the first Chern class of $\Theta_V(\mathcal{C}_g, G)$ under the embedding i_g (cf. Proposition 1.8). This result is proved by first reducing the case of general G to $SL(n)$ and then reducing the case of $SL(n)$ to $SL(2)$. The corresponding result for $SL(2)$ is obtained by showing that the inclusion $M_g(SL(2)) \hookrightarrow M_{g+1}(SL(2))$ induces isomorpism in cohomology $H^2(M_{g+1}(SL(2)), \mathbb{Z}) \simeq H^2(M_g(SL(2)), \mathbb{Z})$ (cf. Proposition 1.7). The last result for H^2 with rational coefficients is fairly well known (and follows easily by observing that the symplectic form on $M_{g+1}(G)$ restricts to the symplectic form on $M_g(G)$) but the result with integral coefficients is more delicate and is proved in Section 4. The proof involves the calculation of the determinant bundle of the Poincaré bundle on $\mathcal{C}_g \times \mathcal{J}_{\mathcal{C}_g}$, $\mathcal{J}_{\mathcal{C}_g}$ being the Jacobian of $\mathcal{C}_g$ which consists of the isomorphism classes of degree 0 line bundles on $\mathcal{C}_g$.

By virtue of the above mentioned two propositions (Propositions 1.6 and 1.8), to prove our main result determining $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ stated in the first paragraph for any $g \geq 1$, it suffices to consider the case of genus $g = 1$.

In the genus $g = 1$ case, $\mathfrak{M}_{\mathcal{C}_1}(G)$ admits a description as the weighted projective space $\mathbb{P}(1, a_1^\vee, a_2^\vee, \dots, a_k^\vee)$, where $a_i^\vee$ are the coefficients of the coroot $\theta^\vee$ written in terms of the simple coroots and k is the rank of G (cf. Theorems 3.1 and 3.3). The ample generator of the Picard group of $\mathbb{P}(1, a_1^\vee, a_2^\vee, \dots, a_k^\vee)$ is known to be $\mathcal{O}_{\mathbb{P}(1,a_1^\vee,a_2^\vee,\dots,a_k^\vee)}(m_G)$ (cf. Theorem 3.4). In section 3, we show that $\Theta_{V(\omega_d)}(\mathcal{C}_1, G)$ is, in fact, $\mathcal{O}_{\mathbb{P}(1,a_1^\vee,a_2^\vee,\dots,a_k^\vee)}(m_G)$, and hence it is the ample generator of $\text{Pic}(\mathfrak{M}_{\mathcal{C}_1}(G))$. The proof makes use of the Verlinde formula determining the dimension of the space of global sections $H^0(\mathfrak{M}_{\mathcal{C}_g}(G), \mathfrak{L})$ (cf. Theorem 3.5).

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1. Statement of the main theorem and its proof

For a topological space X, $H^i(X)$ denotes the singular cohomology of X with integral coefficients, unless otherwise explicitly stated.

Let G be a connected, simply-connected, simple affine algebraic group over $\mathbb{C}$. This will be our tacit assumption on G throughout the paper. Let $\mathcal{C}_g$ be a smooth irreducible projective curve (over $\mathbb{C}$) of genus g, which we assume to be ≥ 1. Let $\mathfrak{M}_{\mathcal{C}_g} = \mathfrak{M}_{\mathcal{C}_g}(G)$ be the moduli space of semistable principal G-bundles on $\mathcal{C}_g$.

We begin by recalling the following result due to Kumar-Narasimhan [KN, Theorem 2.4]. (In loc cit. the genus g is assumed to be ≥ 2. For the genus $g = 1$ case, the result follows from Theorems 3.1, 3.3 and 3.4.)

Theorem 1.1. *With the notation as above,*

$$\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}) \simeq \mathbb{Z},$$

where $\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g})$ *is the group of isomorphism classes of algebraic line bundles on* $\mathfrak{M}_{\mathcal{C}_g}$.

In particular, any nontrivial line bundle on $\mathfrak{M}_{\mathcal{C}_g}$ *is ample or its inverse is ample.*

Definition 1.2. *Let $\mathcal{F}$ be a family of vector bundles on $\mathcal{C}_g$ parametrized by a variety X, i.e., $\mathcal{F}$ is a vector bundle over $\mathcal{C}_g \times X$. Then, the 'determinant of the cohomology' gives rise to the determinant bundle* $\mathrm{Det}(\mathcal{F})$ *of the family $\mathcal{F}$, which is a line bundle over the base X. By definition, the fiber of* $\mathrm{Det}(\mathcal{F})$ *over any $x \in X$ is given by the expression:*

$$\mathrm{Det}(\mathcal{F})|_x = \wedge^{top}(H^0(\mathcal{C}_g, \mathcal{F}_x))^* \otimes \wedge^{top}(H^1(\mathcal{C}_g, \mathcal{F}_x)),$$

where $\mathcal{F}_x$ is the restriction of $\mathcal{F}$ to $\mathcal{C}_g \times x$ (cf., e.g., [L, Chap. 6, §1], [KM]).

Let $\mathcal{R}(G)$ denote the set of isomorphism classes of all the finite dimensional algebraic representations of G. For any V in $\mathcal{R}(G)$, we have the Θ-bundle $\Theta_V(\mathcal{C}_g) = \Theta_V(\mathcal{C}_g, G)$ on $\mathfrak{M}_{\mathcal{C}_g}$, which is an algebraic line bundle whose fibre at any principal G-bundle $E \in \mathfrak{M}_{\mathcal{C}_g}$ is given by the expression

$$\Theta_V(\mathcal{C}_g)|_E = \wedge^{top}(H^0(\mathcal{C}_g, E_V))^* \otimes \wedge^{top}(H^1(\mathcal{C}_g, E_V)),$$

where E_V is the associated vector bundle $E \times_G V$ on $\mathcal{C}_g$. Observe that the moduli space $\mathfrak{M}_{\mathcal{C}_g}$ does not parametrize a universal family of G-bundles, however, the theta bundle $\Theta_V(\mathcal{C}_g)$ (which is essentially the determinant bundle if there were a universal family parametrized by $\mathfrak{M}_{\mathcal{C}_g}$) still exists (cf. [$K_1$, §3.7]).

Now, we can state the main result of this paper.

Theorem 1.3.

$$\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}) = < \Theta_V(\mathcal{C}_g), V \in \mathcal{R}(G) >,$$

where the notation $< \ >$ *denotes the group generated by the elements in the bracket.*

Lemma 1.4.

$$c : \mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}) \simeq H^2(\mathfrak{M}_{\mathcal{C}_g}, \mathbb{Z}),$$

where c maps any line bundle $\mathfrak{L}$ *to its first Chern class* $c_1(\mathfrak{L})$.

In particular,

$$H^2(\mathfrak{M}_{\mathcal{C}_g}, \mathbb{Z}) \simeq \mathbb{Z}.$$

The first Chern class of the ample generator of $\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g})$ is called the *positive generator* of $H^2(\mathfrak{M}_{\mathcal{C}_g}, \mathbb{Z})$.

Proof. Consider the following exact sequence of abelian groups:

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{f} \mathbb{C}^* \to 0,$$

where $f(x) = e^{2\pi i x}$. This gives rise to the following exact sequence of sheaves on $\mathfrak{M}_{\mathcal{C}_g}$ endowed with the analytic topology:

$$0 \to \bar{\mathbb{Z}} \to \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}} \to \bar{\mathcal{O}}^*_{\mathfrak{M}_{\mathcal{C}_g}} \to 0,$$

where $\bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}$ is the sheaf of holomorphic functions on $\mathfrak{M}_{\mathcal{C}_g}$, $\bar{\mathcal{O}}^*_{\mathfrak{M}_{\mathcal{C}_g}}$ is the sheaf of invertible elements of $\bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}$ and $\bar{\mathbb{Z}}$ is the constant sheaf corresponding to the abelian group $\mathbb{Z}$.

The above sequence, of course, induces the following long exact sequence in cohomology:

$$\cdots \to H^1(\mathfrak{M}_{\mathcal{C}_g}, \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}) \to H^1(\mathfrak{M}_{\mathcal{C}_g}, \bar{\mathcal{O}}^*_{\mathfrak{M}_{\mathcal{C}_g}}) \xrightarrow{\bar{c}} H^2(\mathfrak{M}_{\mathcal{C}_g}, \mathbb{Z})$$
$$\to H^2(\mathfrak{M}_{\mathcal{C}_g}, \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}) \to \cdots .$$

First of all,

$$\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}) \simeq H^1(\mathfrak{M}_{\mathcal{C}_g}, \mathcal{O}^*_{\mathfrak{M}_{\mathcal{C}_g}}), \tag{1}$$

where $\mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}$ is the sheaf of algebraic functions on $\mathfrak{M}_{\mathcal{C}_g}$ and $\mathcal{O}^*_{\mathfrak{M}_{\mathcal{C}_g}}$ is the subsheaf of invertible elements of $\mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}$.

Moreover, by GAGA, $\mathfrak{M}_{\mathcal{C}_g}$ being a projective variety,

$$H^1(\mathfrak{M}_{\mathcal{C}_g}, \mathcal{O}^*_{\mathfrak{M}_{\mathcal{C}_g}}) \simeq H^1(\mathfrak{M}_{\mathcal{C}_g}, \bar{\mathcal{O}}^*_{\mathfrak{M}_{\mathcal{C}_g}}), \tag{2}$$

and also, for any $p \geq 0$,

$$H^p(\mathfrak{M}_{\mathcal{C}_g}, \mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}) \simeq H^p(\mathfrak{M}_{\mathcal{C}_g}, \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}). \tag{3}$$

By Kumar-Narasimhan [KN, Theorem 2.8], $H^i(\mathfrak{M}_{\mathcal{C}_g}, \mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}) = 0$ for $i > 0$. Hence, under the identification (1), by (2)–(3) and the above long exact cohomology sequence,

$$\operatorname{Pic}(\mathfrak{M}_{\mathcal{C}_g}) \overset{\sim c}{\to} H^2(\mathfrak{M}_{\mathcal{C}_g}, \mathbb{Z}),$$

where c is the map $\bar{c}$ under the above identifications. Moreover, as is well known, c is the first Chern class map. □

Let us fix a maximal compact subgroup K of G. Denote the Riemann surface with g handles, considered only as a topological manifold, by C_g. Thus, the underlying topological manifold of $\mathcal{C}_g$ is C_g. Define $M_g(G) := \varphi^{-1}(1)/\mathrm{Ad}K$, where $\varphi : K^{2g} \to K$ is the commutator map $\varphi(k_1, k_2, \ldots, k_{2g}) = [k_1, k_2][k_3, k_4] \cdots [k_{2g-1}, k_{2g}]$ and $\varphi^{-1}(1)/\mathrm{Ad}K$ refers to the quotient of $\varphi^{-1}(1)$ by K under the diagonal adjoint action of K on K^{2g}.

Now, we recall the following fundamental result due to Narasimhan-Seshadri [NS] for vector bundles and extended for an arbitrary G by Ramanathan [R_1, R_2].

Consider the standard generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ of $\pi_1(\mathcal{C}_g)$ (cf. [N, §14]). Then, we have the presentation:

$$\pi_1(\mathcal{C}_g) = F[a_1, \ldots, a_g, b_1, \ldots, b_g] / < [a_1, b_1] \cdots [a_g, b_g] >,$$

where $F[a_1, \ldots, a_g, b_1, \ldots, b_g]$ denotes the free group generated by $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ and $< [a_1, b_1] \cdots [a_g, b_g] >$ denotes the normal subgroup generated by the single element $[a_1, b_1] \cdots [a_g, b_g]$.

Theorem 1.5. *Having chosen the standard generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ of $\pi_1(\mathcal{C}_g)$, there exists a canonical isomorphism of real analytic spaces:*

$$\theta_{\mathcal{C}_g}(G) : M_g(G) \simeq \mathfrak{M}_{\mathcal{C}_g}(G).$$

In the sequel, we will often make this identification. Define a class $\alpha \in H^2(Mg(G), \mathbb{Z})$ to be *positive* if it is a positive multiple of the positive generator of $H^2(\mathfrak{M}_{\mathcal{C}_g}(G), \mathbb{Z})$ under the identification $\theta_{\mathcal{C}_g}(G)^*$. Then, for any fixed g, the positivity of α does not depend upon the choice of the algebraic curve $\mathcal{C}_g$. To prove this, follow the argument as in the proof of Proposition 1.8 to reduce it to the case of $G = SL(2)$. In this case it follow from the identity (11) of the proof of Lemma 4.1.

Proposition 1.6. *For any $V \in \mathcal{R}(G)$, $c(\Theta_V(\mathcal{C}_g, G))$, under the above identification $\theta_{\mathcal{C}_g}(G)$, does not depend on the choice of the projective variety structure $\mathcal{C}_g$ on the Riemann surface C_g for any fixed g.*

Proof. Let $\rho : G \to SL(V)$ be the given representation. By taking a K-invariant Hermitian form on V we get $\rho(K) \subset SU(n)$, where $n = \dim V$. For any principal G-bundle E on $\mathcal{C}_g$, let $E_{SL(V)}$ be the principal $SL(V)$-bundle over $\mathcal{C}_g$

obtained by the extension of the structure group via ρ. Then, if E is semistable, so is $E_{SL(V)}$, giving rise to a variety morphism $\hat{\rho} : \mathfrak{M}_{\mathcal{C}_g}(G) \to \mathfrak{M}_{\mathcal{C}_g}(SL(V))$ (cf. [RR, Theorem 3.18]). Hence, we get the commutative diagram:

$$\begin{array}{ccc} \mathfrak{M}_{\mathcal{C}_g}(G) & \xrightarrow{\hat{\rho}} & \mathfrak{M}_{\mathcal{C}_g}(SL(V)) \\ \uparrow & & \uparrow \\ M_g(G) & \xrightarrow{\bar{\rho}} & M_g(SL(V)), \end{array} \tag{D$_1$}$$

where $\bar{\rho}$ is induced from the commutative diagram:

$$\begin{array}{ccc} K^{2g} & \xrightarrow{\varphi} & K \\ \Big\downarrow \rho^{\times 2g} & & \Big\downarrow \rho \\ SU(n)^{2g} & \xrightarrow[\varphi]{} & SU(n). \end{array}$$

The diagram (D$_1$) induces the following commutative diagram in cohomology:

$$\begin{array}{ccc} H^2(\mathfrak{M}_{\mathcal{C}_g}(SL(V)), \mathbb{Z}) & \xrightarrow{\hat{\rho}^*} & H^2(\mathfrak{M}_{\mathcal{C}_g}(G), \mathbb{Z}) \\ \Big\downarrow \Vert & & \Big\downarrow \Vert \\ H^2(M_g(SL(V)), \mathbb{Z}) & \xrightarrow[\bar{\rho}^*]{} & H^2(M_g(G), \mathbb{Z}). \end{array} \tag{D$_2$}$$

By the construction of the Θ-bundle, $\hat{\rho}^*(\Theta_V(\mathcal{C}_g, SL(V))) = \Theta_V(\mathcal{C}_g, G)$, where $\hat{\rho}^*$ also denotes the pullback of line bundles and V is thought of as the standard representation of $SL(V)$.

Thus, using the functoriality of the Chern class, we get

$$\hat{\rho}^*(c(\Theta_V(\mathcal{C}_g, SL(V)))) = c(\Theta_V(\mathcal{C}_g, G)). \tag{1}$$

By Drezet-Narasimhan [DN], $c(\Theta_V(\mathcal{C}_g, SL(V)))$ is the unique positive generator of $H^2(\mathfrak{M}_{\mathcal{C}_g}(SL(V)), \mathbb{Z})$ and thus is independent of the choice of $\mathcal{C}_g$ under the identification $\theta_{\mathcal{C}_g}(SL(V))^*$. Consequently, by (1) and the above commutative diagram (D$_2$), $c(\Theta_V(\mathcal{C}_g, G))$ is independent of the choice of $\mathcal{C}_g$. □

From now on we will denote the cohomology class $c(\Theta_V(\mathcal{C}_g, G))$ in $H^2(M_g(G), \mathbb{Z})$, under the identification $\theta_{\mathcal{C}_g}(G)^*$, by $c(\Theta_V(g, G))$.

Consider the embedding

$$i_g = i_g(G) : M_g(G) \hookrightarrow M_{g+1}(G)$$

induced by the inclusion of $K^{2g} \to K^{2g+2}$ via $(k_1, \dots, k_{2g}) \mapsto (k_1, \dots, k_{2g}, 1, 1)$.

By virtue of the map i_g, we will identify $M_g(G)$ as a subspace of $M_{g+1}(G)$. In particular, we get the following induced sequence of maps in the second cohomology.

$$H^2(M_1(G), \mathbb{Z}) \xleftarrow{i_1^*} H^2(M_2(G), \mathbb{Z}) \xleftarrow{i_2^*} H^2(M_3(G), \mathbb{Z}) \xleftarrow{i_3^*} \cdots.$$

Proposition 1.7. *For $G = SL(2)$, the maps $i_g^* : H^2(M_{g+1}(G), \mathbb{Z}) \to H^2(M_g(G), \mathbb{Z})$ take the positive generator of $H^2(M_{g+1}(SL(2)), \mathbb{Z})$ to the positive generator of $H^2(M_g(SL(2)), \mathbb{Z})$.*

In particular, i_g^* are isomorphisms for any $g \geq 1$.

We shall prove this proposition in Section 4.

Proposition 1.8. *For any $V \in \mathcal{R}(G)$ and any $g \geq 1$, $i_g^*(c(\Theta_V(g+1, G))) = c(\Theta_V(g, G))$.*

Proof. We first claim that it suffices to prove the above proposition for $G = SL(n)$ and the standard n-dimensional representation V of $SL(n)$.

Let $\rho : G \to SL(V)$ be the given representation. Consider the following commutative diagram:

$$\begin{array}{ccc} M_g(G) & \overset{i_g}{\hookrightarrow} & M_{g+1}(G) \\ \bar{\rho} \downarrow & & \downarrow \bar{\rho} \\ M_g(SL(V)) & \overset{i_g}{\hookrightarrow} & M_{g+1}(SL(V)), \end{array}$$

where $\bar{\rho}$ is the map defined in the proof of Proposition 1.6. It induces the commutative diagram:

$$\begin{array}{ccc} H^2(M_g(G), \mathbb{Z}) & \overset{i_g^*}{\leftarrow} & H^2(M_{g+1}(G), \mathbb{Z}) \\ \bar{\rho}^* \uparrow & & \bar{\rho}^* \uparrow \\ H^2(M_g(SL(V)), \mathbb{Z}) & \overset{i_g^*}{\leftarrow} & H^2(M_{g+1}(SL(V)), \mathbb{Z}). \end{array}$$

Therefore, using the commutativity of the above diagram and equation (1) of Proposition 1.6, supposing that $i_g^*(c(\Theta_V(g+1, SL(V)))) = c(\Theta_V(g, SL(V)))$, we get $i_g^*(c(\Theta_V(g+1, G))) = c(\Theta_V(g, G))$. Hence, Proposition 1.8 is established for any G provided we assume its validity for $G = SL(V)$ and its standard representation in V.

We further reduce the proposition from $SL(n)$ to $SL(2)$. As in the proof of Proposition 1.6, consider the mappings

$$\bar{\rho} : M_g(SL(2)) \to M_g(SL(n)), \text{ and}$$

$$\hat{\rho} : \mathfrak{M}_{\mathcal{C}_g}(SL(2)) \to \mathfrak{M}_{\mathcal{C}_g}(SL(n))$$

induced by the inclusions

$$SU(2) \to SU(n) \text{ and } SL(2) \to SL(n),$$

given by $m \mapsto diag(m, 1, \ldots, 1)$.

The maps $\bar{\rho}$ and $\hat{\rho}$ induce the commutative diagram:

$$\begin{array}{ccc} H^2(M_g(SL(n)),\mathbb{Z}) & \xrightarrow{\bar{\rho}^*} & H^2(M_g(SL(2)),\mathbb{Z}) \\ \uparrow \| & & \uparrow \| \\ H^2(\mathfrak{M}_{\mathcal{C}_g}(SL(n)),\mathbb{Z}) & \xrightarrow{\hat{\rho}^*} & H^2(\mathfrak{M}_{\mathcal{C}_g}(SL(2)),\mathbb{Z}). \end{array}$$

By the construction of the Θ-bundle, $\hat{\rho}^*\big(\Theta_V(\mathcal{C}_g, SL(n))\big) = \Theta_{V_2}(\mathcal{C}_g, SL(2))$, where V_2 is the standard 2-dimensional representation of $SL(2)$.

Thus, using the functoriality of the Chern class, we get

$$\hat{\rho}^*(c(\Theta_V(\mathcal{C}_g, SL(n)))) = c(\Theta_{V_2}(\mathcal{C}_g, SL(2))). \tag{1}$$

Using one more time the result of Drezet-Narasimhan that $c(\Theta_V(\mathcal{C}_g, SL(n)))$ is the unique positive generator of $H^2(\mathfrak{M}_{\mathcal{C}_g}(SL(n)))$ for any n (cf. Proof of Proposition 1.6), we see that $\hat{\rho}^*$ is surjective and hence an isomorphism by Lemma 1.4.

Consider the following commutative diagram:

$$\begin{array}{ccc} H^2(M_g(SL(n)),\mathbb{Z}) & \xleftarrow{i_g^*} & H^2(M_{g+1}(SL(n)),\mathbb{Z}) \\ \bar{\rho}^* \downarrow & & \bar{\rho}^* \downarrow \\ H^2(M_g(SL(2)),\mathbb{Z}) & \xleftarrow{i_g^*} & H^2(M_{g+1}(SL(2)),\mathbb{Z}). \end{array}$$

Suppose that the proposition is true for $G = SL(2)$ and the standard representation V_2, i.e.,

$$i_g^*(c(\Theta_{V_2}(g+1, SL(2)))) = c(\Theta_{V_2}(g, SL(2))). \tag{2}$$

Then, using the commutativity of the above diagram and (1) together with the fact that $\bar{\rho}^*$ is an isomorphism, we get that

$$i_g^*(c(\Theta_V(g+1, SL(n)))) = c(\Theta_V(g, SL(n))).$$

Finally, (2) follows from the result of Drezet-Narasimhan cited above and Proposition 1.7. Hence the proposition is established for any G (once we prove Proposition 1.7). □

Proposition 1.9. *For $g = 1$, Theorem 1.3 is true.*

The proof of this proposition will be given in Section 3.

Proof of Theorem 1.3. Denote the subgroup $< \Theta_V(\mathcal{C}_g, G), V \in \mathcal{R}(G) >$ of $\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ by $\mathrm{Pic}^\Theta(\mathfrak{M}_{\mathcal{C}_g}(G))$.

Set $H^2_\Theta(M_g(G)) := c(\mathrm{Pic}^\Theta(\mathfrak{M}_{\mathcal{C}_g}(G)))$. By virtue of Proposition 1.6, this is well defined, i.e., $H^2_\Theta(M_g(G))$ does not depend upon the choice of the projective variety structure $\mathcal{C}_g$ on C_g. Moreover, by Proposition 1.8, $i_g^*(H^2_\Theta(M_{g+1}(G))) = H^2_\Theta(M_g(G))$.

Thus, we get the following commutative diagram, where the upward arrows are inclusions and the maps in the bottom horizontal sequence are induced from the maps i_g^*.

$$\begin{array}{ccccccc} H^2(M_1(G)) & \overset{i_1^*}{\leftarrow} & H^2(M_2(G)) & \overset{i_2^*}{\leftarrow} & H^2(M_3(G)) & \overset{i_3^*}{\leftarrow} & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ H^2_\Theta(M_1(G)) & \leftarrow & H^2_\Theta(M_2(G)) & \leftarrow & H^2_\Theta(M_3(G)) & \leftarrow & \cdots . \end{array}$$

By Proposition 1.9 and Lemma 1.4, $H^2(M_1(G)) = H^2_\Theta(M_1(G))$. Then, i_1^* is surjective and hence an isomorphism (by using Lemma 1.4 again). Thus, by the commutativity of the above diagram, the inclusion $H^2_\Theta(M_2(G)) \hookrightarrow H^2(M_2(G))$ is an isomorphism. Arguing the same way, we get that $H^2(M_g(G)) = H^2_\Theta(M_g(G))$ for all g. This completes the proof of the theorem by virtue of the isomorphism c of Lemma 1.4. □

2. Comparison of the Picard groups of $\mathfrak{M}_{\mathcal{C}_g}$ and the infinite Grassmannian

As earlier, let G be a connected, simply-connected, simple affine algebraic group over $\mathbb{C}$. We fix a Borel subgroup B of G and a maximal torus $T \subset B$. Let $\mathfrak{h}$ (resp. $\mathfrak{b}$) be the Lie algebra of T (resp. B). Let $\Delta_+ \subset \mathfrak{h}^*$ be the set of positive roots (i.e., the roots of $\mathfrak{b}$ with respect to $\mathfrak{h}$) and let $\{\omega_i\}_{1\le i\le k} \subset \mathfrak{h}^*$ be the set of fundamental weights, where k is the rank of G. As earlier, $\mathcal{R}(G)$ denotes the set of isomorphism classes of all the finite dimensional algebraic representations of G. This is a semigroup under the direct sum of two representations. Let $R(G)$ denote the associated Grothendieck group. Then, $R(G)$ is a ring, where the product is induced from the tensor product of two representations. Then, the fundamental representations $\{V(\omega_i)\}_{1\le i\le k}$ generate the representation ring $R(G)$ as a ring [A].

Let X be the infinite Grassmannian associated to the affine Kac-Moody group $\mathcal{G}$ corresponding to G, i.e., $X := \mathcal{G}/\mathcal{P}$, where $\mathcal{P}$ is the standard maximal parabolic subgroup of $\mathcal{G}$ (cf. [K_2, §13.2.12]; in loc cit., X is denoted by $\mathcal{Y} = \mathcal{X}^Y$). It is known that $\mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}$ and is generated by the homogenous line bundle $\mathfrak{L}_{\chi_0}$ (cf. [K_2, Proposition 13.2.19]).

We recall the following definition from [D,§2].

Definition 2.1. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two (finite dimensional) complex simple Lie algebras and $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. There exists a unique number $m_\varphi \in \mathbb{C}$, called the* Dynkin index *of the homomorphism φ, satisfying*

$$\langle \varphi(x), \varphi(y)\rangle = m_\varphi \langle x, y\rangle, \text{ for all } x, y \in \mathfrak{g}_1,$$

where $\langle , \rangle$ is the Killing form on $\mathfrak{g}_1$ (and $\mathfrak{g}_2$) normalized so that $\langle \theta, \theta\rangle = 2$ for the highest root θ.

For a Lie algebra $\mathfrak{g}_1$ as above and a finite dimensional representation V of $\mathfrak{g}_1$, by the Dynkin index m_V of V, we mean the Dynkin index of the Lie algebra homomorphism $\rho : \mathfrak{g}_1 \to sl(V)$, where $sl(V)$ is the Lie algebra of trace 0 endomorphisms of V.

Then, for any two finite dimensional representations V and W of $\mathfrak{g}_1$, we have, by [D, Chap. 1, §2] or [KN, Lemma 4.5],

$$m_{V\otimes W} = m_V \dim W + m_W \dim V. \tag{1}$$

We recall the following main result of Kumar-Narasimhan [KN, Theorem 2.4]. In loc cit., it is proved under the assumption $g \geq 2$. But it remains true for $g = 1$ by virtue of Theorem 1.1 and the following identity (1) which is proved in [KNR] for any $g \geq 0$.

Theorem 2.2. *There exists a 'natural' injective group homomorphism*

$$\beta : \operatorname{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G)) \hookrightarrow \operatorname{Pic}(X).$$

Moreover, by [KNR, Theorem 5.4] (see also [Fa]), for any $V \in \mathcal{R}(G)$,

$$\beta(\Theta_V(\mathcal{C}_g, G)) = \mathfrak{L}_{\chi_0}^{\otimes m_V}, \tag{1}$$

where V is thought of as a module for $\mathfrak{g}$ under differentiation and m_V is its Dynkin index.

We also recall the following result from [D, Table 5], [KN, Proposition 4.7], or [LS, §2]. We follow the indexing convention as in [B, Planche I-IX].

Proposition 2.3. *For any simple Lie algebra $\mathfrak{g}$, there exists a (not unique in general) fundamental weight ω_d such that $m_{V(\omega_d)}$ divides each of $\{m_{V(\omega_i)}\}_{1\leq i\leq k}$. Thus, by (1) of Definition 2.1, $m_{V(\omega_d)}$ divides m_V for any $V \in \mathcal{R}(G)$.*

The following table gives the list of all such ω_d's and the corresponding Dynkin index $m_{V(\omega_d)}$.

Type of G	ω_d	$m_{V(\omega_d)}$
A_k $(k \geq 1)$	ω_1, ω_k	1
C_k $(k \geq 2)$	ω_1	1
B_k $(k \geq 3)$	ω_1	2
D_k $(k \geq 4)$	ω_1	2
G_2	ω_1	2
F_4	ω_4	6
E_6	ω_1, ω_6	6
E_7	ω_7	12
E_8	ω_8	60.

For B_3, ω_3 also satisfies $m_{V(\omega_3)} = 2$; for D_4, ω_3 and ω_4 both have $m_{V(\omega_3)} = m_{V(\omega_4)} = 2$.

Let θ be the highest root of G. Observe that, for any G, $m_{V(\omega_d)}$ is the least common multiple of the coefficients of the coroot $\theta^\vee$ written in terms of the simple coroots. We shall denote $m_{V(\omega_d)}$ by m_G.

Combining the above result with Theorem 1.3, we get the following.

Theorem 2.4. *For any $\mathcal{C}_g$ with $g \geq 1$ and G as in Section 1, the Picard group* $\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ *is freely generated by the Θ-bundle $\Theta_{V(\omega_d)}(\mathcal{C}_g, G)$, where ω_d is any fundamental weight as in the above proposition.*

In particular,

$$Im(\beta) \text{ is freely generated by } \mathfrak{L}_{\chi_0}^{\otimes m_G}. \tag{1}$$

Proof. By Theorem 1.3,

$$\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G)) =< \Theta_V(\mathcal{C}_g, G), V \in \mathcal{R}(G) > .$$

Thus, by Theorem 2.2 and Proposition 2.3,

$$\mathrm{Im}(\beta) = < \mathfrak{L}_{\chi_0}^{\otimes m_V}, V \in \mathcal{R}(G) > = < \mathfrak{L}_{\chi_0}^{\otimes m_G} > .$$

This proves (1).

Since β is injective, by the above description of $\mathrm{Im}(\beta)$, $\Theta_{V(\omega_d)}(\mathcal{C}_g, G)$ freely generates $\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$, proving the theorem. □

Following the same argument as in [So, §4], using Theorem 2.4 and Proposition 2.3, we get the following corollary for genus $g \geq 2$. For genus $g = 1$, use Theorems 3.1 and 3.3 together with [BR, Theorem 7.1.d]. This corollary is due to [BLS], [So].

Corollary 2.5. *Let G be any group and $\mathcal{C}_g$ be any curve as in Section 1. Then, the moduli space $\mathfrak{M}_{\mathcal{C}_g}(G)$ is locally factorial if and only if G is of type A_k ($k \geq 1$) or C_k ($k \geq 2$).*

3. Proof of Proposition 1.9

Let G be as in the beginning of Section 1. In this section, we identify $\mathfrak{M}_{\mathcal{C}1}(G)$ with a weighted projective space and show that the generator of $\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_1}(G))$ is $\Theta_{V(\omega_d)}(\mathcal{C}_1, G)$ as claimed.

We recall the following theorem due independently to Laszlo [La, Theorem 4.16] and Friedman-Morgan-Witten [FMW, §2].

Theorem 3.1. *Let $\mathcal{C}_1$ be a smooth, irreducible projective curve of genus* 1. *Then, there is a natural variety isomorphim between the moduli space $\mathfrak{M}_{\mathcal{C}_1}(G)$ and $(\mathcal{C}_1 \otimes_{\mathbb{Z}} Q^\vee)/W$, where $Q^\vee$ is the coroot lattice of G and W is its Weyl group acting canonically on $Q^\vee$ (and acting trivially on $\mathcal{C}_1$).*

Definition 3.2. *Let $N = (n_0, \dots, n_k)$ be a $k+1$-tuple of positive integers. Consider the polynomial ring $\mathbb{C}[z_0, \dots, z_k]$ graded by $\deg z_i = n_i$. The scheme Proj$(\mathbb{C}[z_0, \dots, z_k])$ is said to be the weighted projective space of type N and we denote it by $\mathbb{P}(N)$.*

Consider the standard (nonweighted) projective space $\mathbb{P}^k :=$ Proj $(\mathbb{C}[w_0, \dots, w_k])$, where each $\deg w_i = 1$. Then, the graded algebra homomorphism $\mathbb{C}[z_0, \dots, z_k] \to \mathbb{C}[w_0, \dots, w_k]$, $z_i \mapsto w_i^{n_i}$, induces a morphism $\delta : \mathbb{P}^k \to \mathbb{P}(N)$.

The following theorem is due to Looijenga [Lo]. His proof had a gap; a complete proof of a more general result is outlined by Bernshtein-Shvartsman [BS].

Theorem 3.3. *Let $\mathcal{C}_1$ be an elliptic curve. Then, the variety $(\mathcal{C}_1 \otimes_{\mathbb{Z}} Q^\vee)/W$ is the weighted projective space of type $(1, a_1^\vee, a_2^\vee, \dots, a_k^\vee)$, where $a_i^\vee$ are the coefficients of the coroot $\theta^\vee$ written in terms of the simple coroots $\{\alpha_i^\vee\}$ (and, as earlier, k is the rank of G).*

The following table lists the weighted projective space isomorphic to $\mathfrak{M}_{\mathcal{C}_1}(G)$ corresponding to any G. In this table the entries beyond 1 are precisely the numbers $(a_1^\vee, a_2^\vee, \dots, a_k^\vee)$ following the convention as in Bourbaki [B, Planche I-IX].

Type of G	Type of the weighted projective space
$A_k (k \geq 1)$, C_k $(k \geq 2)$	$(1, 1, 1, \dots, 1)$
B_k $(k \geq 3)$	$(1, 1, 2, \dots, 2, 1)$
D_k $(k \geq 4)$	$(1, 1, 2, \dots, 2, 1, 1)$
G_2	$(1, 1, 2)$
F_4	$(1, 2, 3, 2, 1)$
E_6	$(1, 1, 2, 2, 3, 2, 1)$
E_7	$(1, 2, 2, 3, 4, 3, 2, 1)$
E_8	$(1, 2, 3, 4, 6, 5, 4, 3, 2)$.

We recall the following result from the theory of weighted projective spaces (see, e.g., Beltrametti-Robbiano [BR, Lemma 3B.2.c and Theorem 7.1.c]).

Theorem 3.4. *Let $N = (n_0, \dots, n_k)$ and assume $\gcd\{n_0, \dots, n_k\} = 1$. Then, we have the following.*
(a) $\mathrm{Pic}(\mathbb{P}(N)) \simeq \mathbb{Z}$. *In fact, the morphism δ of Definition 3.2 induces an injective map* $\delta^* : \mathrm{Pic}(\mathbb{P}(N)) \to \mathrm{Pic}(\mathbb{P}^k)$.

Moreover, the ample generator of $\mathrm{Pic}(\mathbb{P}(N))$ *maps to $\mathcal{O}_{\mathbb{P}^k}(s)$ under δ^*, where s is the least common multiple of $\{n_0, \dots, n_k\}$. We denote this ample generator by $\mathcal{O}_{\mathbb{P}(N)}(s)$.*
(b) For any $d \geq 0$,

$$H^0(\mathbb{P}(N), \mathcal{O}_{\mathbb{P}(N)}(s)^{\otimes d}) = \mathbb{C}[z_0, \dots, z_k]_{ds},$$

where $\mathbb{C}[z_0, \dots, z_k]_{ds}$ denotes the subspace of $\mathbb{C}[z_0, \dots, z_k]$ consisting of homogeneous elements of degree ds.

Using Theorems 3.1, 3.3 and 3.4 and the fact that the least common multiple of the numbers $\{1, a_1^\vee, a_2^\vee, \ldots, a_k^\vee\}$ for each G is the Dynkin index $m_G = m_{V(\omega_d)}$, we have

$$\Theta_{V(\omega_d)}(\mathcal{C}_1, G) = \mathcal{O}_{\mathbb{P}(1,a_1^\vee,a_2^\vee,\ldots,a_k^\vee)}(m_G)^{\otimes p} \qquad (*)$$

for some positive integer p. The value of m_G is given in Proposition 2.3 for any G.

We recall the following basic result, the first part of which is due independently to Beauville-Laszlo [BL], Faltings [Fa] and Kumar-Narasimhan-Ramanathan [KNR]. The second part of the theorem as in (1) is the celebrated Verlinde formula for the dimension of the space of conformal blocks essentially due to Tsuchiya-Ueno-Yamada [TUY] (together with works [Fa, Appendix] and [T_1]).

Theorem 3.5. *For any ample line bundle $\mathfrak{L} \in \mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ and $\ell \geq 0$, there is an isomorphism (canonical up to scalar multiples):*

$$H^0(\mathfrak{M}_{\mathcal{C}_g}(G), \mathfrak{L}^{\otimes \ell}) \simeq L(\mathcal{C}_g, \ell m_{\mathfrak{L}}),$$

where $L(\mathcal{C}_g, \ell)$ is the space of conformal blocks corresponding to the one marked point on $\mathcal{C}_g$ and trivial representation attached to it with central charge ℓ (cf., e.g., [TUY] for the definition of conformal blocks) and $m_{\mathfrak{L}}$ is the positive integer such that $\beta(\mathfrak{L}) = \mathfrak{L}_{\chi_0}^{\otimes m_{\mathfrak{L}}}$, β being the map as in Theorem 2.2.

Moreover, the dimension $F_g(\ell)$ of the space $L(\mathcal{C}_g, \ell)$ is given by the following Verlinde formula:

$$F_g(\ell) = t_\ell^{g-1} \sum_{\mu \in P_\ell} \prod_{\alpha \in \Delta_+} \left| 2 \sin\left(\frac{\pi}{\ell + h} < \alpha, \mu + \rho >\right)\right|^{2-2g}, \qquad (1)$$

where

$< \, , \, > :=$ *Killing form on $\mathfrak{h}^*$ normalized so that $< \theta, \theta >= 2$ for the highest root θ,*
$\Delta_+ :=$ *the set of positive roots,*
$P_\ell := \{$*dominant integral weights* $\mu | < \mu, \theta > \leq \ell\}$,
$\rho :=$ *half sum of positive roots,*
$h :=< \rho, \theta > +1$, *the dual Coxeter number,*
$t_\ell := (\ell + h)^{\mathrm{rank}\, G} (\# P / Q_{lg})$,

and P is the weight lattice and Q_{lg} is the sublatttice of the root lattice Q generated by the long roots.

In fact, we only need to use the above theorem for the case of genus $g = 1$. For $g = 1$, the Verlinde formula (1) clearly reduces to the identity:

$$F_1(\ell) = \# P_\ell.$$

Of course,

$$P_\ell = \{(n_1, \dots, n_k) \in (\mathbb{Z}_+)^k : \sum_{i=1}^{k} n_i a_i^\vee \leq \ell\}.$$

Proof of Proposition 1.9. Using the specialization of Theorem 3.5 to $g = 1$, we see that

$$\dim H^0(\mathfrak{M}_{\mathcal{C}_1}(G), \Theta_{V(\omega_d)}(\mathcal{C}_1, G)) = \# P_{m_G}.$$

On the other hand, by Theorems 3.1, 3.3 and 3.4(b),

$$\dim H^0(\mathfrak{M}_{\mathcal{C}_1}(G), \mathcal{O}_{\mathbb{P}(1,a_1^\vee,a_2^\vee,\dots,a_k^\vee)}(m_G)^{\otimes p}) = \dim(\mathbb{C}[z_0, \dots, z_k]_{pm_G}) = \# P_{pm_G}.$$

Hence, in the equation $(*)$ following Theorem 3.4, $p = 1$ and $\Theta_{V(\omega_d)}(\mathcal{C}_1, G)$ is the (ample) generator of $\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_1}(G))$. This proves Proposition 1.9. □

4. Proof of Proposition 1.7

In this section, we take $G = SL(2)$ and abbreviate $\mathfrak{M}_{\mathcal{C}_g}(SL(2))$ by $\mathfrak{M}_{\mathcal{C}_g}$ etc. Let $\mathfrak{M}^{\mathrm{red}}_{\mathcal{C}_g}$ be the closed subvariety of the moduli space $\mathfrak{M}_{\mathcal{C}_g}$ consisting of decomposable bundles on $\mathcal{C}_g$ (which are semistable of rank-2 with trivial determinant). Let $\mathfrak{J}_{\mathcal{C}_g}$ be the Jacobian of $\mathcal{C}_g$. Recall that the underlying set of the variety $\mathfrak{J}_{\mathcal{C}_g}$ consists of all the isomorphism classes of line bundles on $\mathcal{C}_g$ with trivial first Chern class. Then, there is a surjective morphism $\xi = \xi_{\mathcal{C}_g} : \mathfrak{J}_{\mathcal{C}_g} \to \mathfrak{M}^{\mathrm{red}}_{\mathcal{C}_g} \subset \mathfrak{M}_{\mathcal{C}_g}$, taking $\mathfrak{L} \mapsto \mathfrak{L} \oplus \mathfrak{L}^{-1}$. Moreover, $\xi^{-1}(\xi(\mathfrak{L})) = \{\mathfrak{L}, \mathfrak{L}^{-1}\}$. The Jacobian $\mathfrak{J}_{\mathcal{C}_g}$ admits the involution τ taking $\mathfrak{L} \mapsto \mathfrak{L}^{-1}$.

Let T be a maximal torus of the maximal compact subgroup $SU(2)$ of $SL(2)$, which we take to be the diagonal subgroup of $SU(2)$. Similar to the identification $\theta_{\mathcal{C}_g}$ as in Theorem 1.5, setting $J_g := T^{2g}$, there is an isomorphism of real analytic spaces $\bar{\theta}_{\mathcal{C}_g} : J_g \to \mathfrak{J}_{\mathcal{C}_g}$ making the following diagram commutative:

$$\begin{array}{ccc} J_g & \xrightarrow{\bar{\theta}_{\mathcal{C}_g}} & \mathfrak{J}_{\mathcal{C}_g} \\ f_g \downarrow & & \downarrow \xi_{\mathcal{C}_g} \\ M_g & \xrightarrow{\theta_{\mathcal{C}_g}} & \mathfrak{M}_{\mathcal{C}_g}, \end{array} \qquad \text{(E)}$$

where $f_g : J_g \to M_g$ is induced from the standard inclusion $T^{2g} \subset SU(2)^{2g}$. We will explicitly describe the isomorphism $\bar{\theta}_{\mathcal{C}_g}$ in the proof of the following lemma.

Recall the definition of the map $i_g : M_g \to M_{g+1}$ from Section 1 and let $r_g : J_g \to J_{g+1}$ be the map $(t_1, \dots, t_{2g}) \mapsto (t_1, \dots, t_{2g}, 1, 1)$. Then, we have the following commutative diagram:

$$\begin{array}{ccc} J_g & \stackrel{f_g}{\rightarrow} & M_g \\ r_g \downarrow & & \downarrow i_g \\ J_{g+1} & \stackrel{f_{g+1}}{\rightarrow} & M_{g+1}. \end{array} \tag{F}$$

Let x_{g+1} denote the positive generator of $H^2(M_{g+1}, \mathbb{Z})$. Then, by Lemma 1.4 and Theorem 1.5,

$$i_g^*(x_{g+1}) = d_g x_g,$$

for some integer d_g. We will prove that $d_g = 1$, which will of course prove Proposition 1.7. Set $y_g := f_g^*(x_g)$; $f_g^* : H^2(M_g, \mathbb{Z}) \to H^2(J_g, \mathbb{Z})$ being the map in cohomology induced from f_g.

Lemma 4.1. $y_g \neq 0$ *and* $r_g^*(y_{g+1}) = y_g$ *as elements of* $H^2(J_g, \mathbb{Z})$.

Proof. There exists a unique universal line bundle $\mathcal{P}$, called the *Poincaré bundle* on $\mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g}$ such that, for each $\mathfrak{L} \in \mathfrak{J}_{\mathcal{C}_g}$, $\mathcal{P}$ restricts to the line bundle $\mathfrak{L}$ on $\mathcal{C}_g \times \mathfrak{L}$, and $\mathcal{P}$ restricted to $x_o \times \mathfrak{J}_{\mathcal{C}_g}$ is trivial for a fixed base point $x_o \in \mathcal{C}_g$ (cf. [ACGH, Chap. IV, §2]).

Let $\mathcal{F}$ be the rank-2 vector bundle $\mathcal{P} \oplus \hat{\tau}^*(\mathcal{P})$ over the base space $\mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g}$, and think of $\mathcal{F}$ as a family of rank-2 bundles on $\mathcal{C}_g$ parametrized by $\mathfrak{J}_{\mathcal{C}_g}$, where $\hat{\tau} : \mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g} \to \mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g}$ is the involution $I \times \tau$.

By Drezet-Narasimhan [DN], we have $x_g = c_1(\Theta_{V_2}(\mathcal{C}_g, SL(2)))$ for the standard representation V_2 of $SL(2)$. Using the functoriality of Chern class,

$$\xi_{\mathcal{C}_g}^*(x_g) = c_1(\operatorname{Det} \mathcal{F}), \tag{1}$$

where $\operatorname{Det} \mathcal{F}$ denotes the determinant line bundle over $\mathfrak{J}_{\mathcal{C}_g}$ associated to the family $\mathcal{F}$ (cf. Definition 1.2). Recall that the fiber of $\operatorname{Det} \mathcal{F}$ at any $\mathfrak{L} \in \mathfrak{J}_{\mathcal{C}_g}$ is given by the expression

$$\begin{aligned} \operatorname{Det} \mathcal{F}_{|\mathfrak{L}} &= \wedge^{top}\big(H^0(\mathcal{C}_g, \mathfrak{L} \oplus \mathfrak{L}^{-1})^*\big) \otimes \wedge^{top}\big(H^1(\mathcal{C}_g, \mathfrak{L} \oplus \mathfrak{L}^{-1})\big) \\ &= \wedge^{top}\big(H^0(\mathcal{C}_g, \mathfrak{L})^* \oplus H^0(\mathcal{C}_g, \mathfrak{L}^{-1})^*\big) \\ &\quad \otimes \wedge^{top}\big(H^1(\mathcal{C}_g, \mathfrak{L}) \oplus H^1(\mathcal{C}_g, \mathfrak{L}^{-1})\big) \\ &= \wedge^{top}\big(H^0(\mathcal{C}_g, \mathfrak{L})^*\big) \otimes \wedge^{top}\big(H^0(\mathcal{C}_g, \mathfrak{L}^{-1})^*\big) \otimes \wedge^{top}\big(H^1(\mathcal{C}_g, \mathfrak{L})\big) \\ &\quad \otimes \wedge^{top}\,(H^1(\mathcal{C}_g, \mathfrak{L}^{-1})) \\ &= \big(\operatorname{Det} \mathcal{P}\big)_{|\mathfrak{L}} \otimes \big(\tau^*(\operatorname{Det} \mathcal{P})\big)_{|\mathfrak{L}}. \end{aligned} \tag{2}$$

Applying the Grothendieck-Riemann-Roch theorem (cf. [F, Example 15.2.8]) for the projection $\mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g} \stackrel{\pi}{\rightarrow} \mathfrak{J}_{\mathcal{C}_g}$ gives

$$\operatorname{ch}(R\pi_* \mathcal{P}) = \pi_*(\operatorname{ch} \mathcal{P} \cdot \operatorname{Td} T_\pi), \tag{3}$$

where ch is the Chern character and Td T_π denotes the Todd genus of the relative tangent bundle of $\mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g}$ along the fibers of π. By the definition of Det $\mathcal{P}$ and $R\pi_*\mathcal{P}$,

$$c_1(\operatorname{Det}\mathcal{P}) = -\operatorname{ch}(R\pi_*\mathcal{P})_{[2]}, \tag{4}$$

where, for a cohomology class y, $y_{[n]}$ denotes the component of y in H^n. Since $\mathcal{P}$ restricted to $x_o \times \mathfrak{J}_{\mathcal{C}_g}$ is trivial and for any $\mathfrak{L} \in \mathfrak{J}_{\mathcal{C}_g}$, $\mathcal{P}$ restricts to the line bundle $\mathfrak{L}$ on $\mathcal{C}_g \times \mathfrak{L}$ (with the trivial Chern class), we get

$$c_1(\mathcal{P}) \in H^1(\mathcal{C}_g) \otimes H^1(\mathfrak{J}_{\mathcal{C}_g}). \tag{5}$$

Thus, using (3)–(4),

$$\begin{aligned} -c_1(\operatorname{Det}\mathcal{P}) &= \pi_*\left((\operatorname{ch}\,\mathcal{P}\cdot \operatorname{Td} T_\pi)_{[4]}\right) \\ &= \pi_*\left(\frac{c_1(\mathcal{P})^2}{2} + \frac{c_1(\mathcal{P})\cdot c_1(T_\pi)}{2}\right) \\ &= \pi_*\left(c_1(\mathcal{P})^2\right)/2. \end{aligned} \tag{6}$$

The last equality follows from (5), since the cup product $c_1(\mathcal{P})\cdot c_1(T_\pi)$ vanishes, $c_1(T_\pi)$ being in $H^2(\mathcal{C}_g) \otimes H^0(\mathfrak{J}_{\mathcal{C}_g})$.

Recall the presentation of $\pi_1(\mathcal{C}_g)$ given just above Theorem 1.5. Then, $H_1(\mathcal{C}_g,\mathbb{Z}) = \oplus_{i=1}^g \mathbb{Z}a_i \oplus \oplus_{i=1}^g \mathbb{Z}b_i$. Moreover, the $\mathbb{Z}$-module dual basis $\{a_i^*, b_i^*\}_{i=1}^g$ of $H^1(\mathcal{C}_g,\mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(H_1(\mathcal{C}_g,\mathbb{Z}),\mathbb{Z})$ satisfies $a_i^*\cdot a_j^* = 0 = b_i^*\cdot b_j^*$, $a_i^*\cdot b_j^* = \delta_{ij}[\mathcal{C}_g]$, where $[\mathcal{C}_g]$ denotes the positive generator of $H^2(\mathcal{C}_g,\mathbb{Z})$.

Having fixed a base point x_o in $\mathcal{C}_g$, define the algebraic map

$$\psi\ :\ \mathcal{C}_g \to \mathfrak{J}_{\mathcal{C}_g},\ \ x \mapsto \mathcal{O}(x - x_o).$$

Of course, $\mathfrak{J}_{\mathcal{C}_g}$ is canonically identified as $H^1(\mathcal{C}_g,\mathcal{O}_{\mathcal{C}_g})/H^1(\mathcal{C}_g,\mathbb{Z})$. Thus, as a real analytic space, we can identify

$$\begin{aligned} \mathfrak{J}_{\mathcal{C}_g} &\simeq H^1(\mathcal{C}_g,\mathbb{R})/H^1(\mathcal{C}_g,\mathbb{Z}) \simeq H^1(\mathcal{C}_g,\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \\ &\simeq \operatorname{Hom}_{\mathbb{Z}}\big(H_1(\mathcal{C}_g,\mathbb{Z}),\mathbb{R}/\mathbb{Z}\big) = J_g \end{aligned} \tag{7}$$

obtained from the $\mathbb{R}$-vector space isomorphism

$$H^1(\mathcal{C}_g,\mathbb{R}) \simeq H^1(\mathcal{C}_g,\mathcal{O}_{\mathcal{C}_g}),$$

induced from the inclusion $\mathbb{R} \subset \mathcal{O}_{\mathcal{C}_g}$, where the last equality in (7) follows by using the basis $\{a_1, b_1, \dots, a_g, b_g\}$ of $H_1(\mathcal{C}_g,\mathbb{Z})$. The induced map, under the identification (7),

$$\psi_* : H_1(\mathcal{C}_g,\mathbb{Z}) \to H_1(\mathfrak{J}_{\mathcal{C}_g},\mathbb{Z}) \simeq H^1(\mathcal{C}_g,\mathbb{Z})$$

is the Poincaré duality isomorphism. To see this, identify

$$\operatorname{Hom}_{\mathbb{Z}}\big(H_1(\mathcal{C}_g,\mathbb{Z}),\mathbb{R}/\mathbb{Z}\big) \simeq \operatorname{Hom}_{\mathbb{Z}}\big(H^1(\mathcal{C}_g,\mathbb{Z}),\mathbb{R}/\mathbb{Z}\big) \tag{8}$$

using the Poincaré duality isomorphim: $H_1(\mathcal{C}_g, \mathbb{Z}) \simeq H^1(\mathcal{C}_g, \mathbb{Z})$. Then, under the identifications (7)–(8), the map

$$\psi : \mathcal{C}_g \to \mathrm{Hom}_{\mathbb{Z}}\big(H^1(\mathcal{C}_g, \mathbb{Z}), \mathbb{R}/\mathbb{Z}\big)$$

can be described as

$$\psi(x)([\omega]) = e^{2\pi i \int_{x_o}^{x} \omega},$$

for any closed 1-form ω on $\mathcal{C}_g$ representing the cohomology class $[\omega] \in H^1(\mathcal{C}_g, \mathbb{Z})$ (cf. [M, Theorem 2.5]), where $\int_{x_o}^{x} \omega$ denotes the integral of ω along any path in $\mathcal{C}_g$ from x_o to x.

Since

$$\psi_* : H_1(\mathcal{C}_g, \mathbb{Z}) \to H_1(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z}) \simeq H^1(\mathcal{C}_g, \mathbb{Z})$$

is the Poincaré duality isomorphism, it is easy to see that the cohomology map induced form ψ:

$$\psi^* : H^1(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z}) \simeq H_1(\mathcal{C}_g, \mathbb{Z}) \to H^1(\mathcal{C}_g, \mathbb{Z})$$

is given by

$$\psi^*(a_i) = b_i^*, \;\; \psi^*(b_i) = -a_i^* \;\; \text{for all } 1 \le i \le g. \tag{9}$$

In particular, ψ^* is an isomorphism. Moreover, the isomorphism does not depend on the choice of x_o.

Consider the map

$$\mathcal{C}_g \times \mathcal{C}_g \overset{I \times \psi}{\to} \mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g}.$$

Let $\mathcal{P}' := (I \times \psi)^*(\mathcal{P})$. Then, $\mathcal{P}'$ is the unique line bundle over $\mathcal{C}_g \times \mathcal{C}_g$ satisfying the following properties:

$$\mathcal{P}'|_{\mathcal{C}_g \times x} = \mathcal{O}(x - x_o) \;\; \text{and} \;\; \mathcal{P}'|_{x_o \times \mathcal{C}_g} \text{ is trivial.}$$

Consider the following line bundle over $\mathcal{C}_g \times \mathcal{C}_g$:

$$\mathcal{O}_{\mathcal{C}_g \times \mathcal{C}_g}(\triangle) \otimes (\mathcal{O}(-x_o) \boxtimes 1) \otimes (1 \boxtimes \mathcal{O}(-x_o)),$$

where $\triangle$ denotes the diagonal in $\mathcal{C}_g \times \mathcal{C}_g$. One sees that this bundle also satisfies the restriction properties mentioned above and hence it must be isomorphic with $\mathcal{P}'$. Consequently,

$$c_1(\mathcal{P}') = c_1(\mathcal{O}_{\mathcal{C}_g \times \mathcal{C}_g}(\triangle)) + c_1(\mathcal{O}(-x_o) \boxtimes 1) + c_1(1 \boxtimes \mathcal{O}(-x_o)).$$

Using the definition of $\mathcal{P}'$ and the functoriality of the Chern classes,

$$c_1(\mathcal{P}') = c_1((I \times \psi)^*(\mathcal{P})) = (I \times \psi)^* c_1(\mathcal{P}). \tag{10}$$

By (5), $c_1(\mathcal{P}) \in H^1(\mathcal{C}_g) \otimes H^1(\mathfrak{J}_{\mathcal{C}_g})$, and hence $c_1(\mathcal{P}') \in H^1(\mathcal{C}_g) \otimes H^1(\mathcal{C}_g)$. Moreover,

$$c_1(\mathcal{O}(-x_o) \boxtimes 1) + c_1(1 \boxtimes \mathcal{O}(-x_o)) \in H^2(\mathcal{C}_g) \otimes H^0(\mathcal{C}_g) \oplus H^0(\mathcal{C}_g) \otimes H^2(\mathcal{C}_g).$$

Thus, $c_1(\mathcal{P}')$ is the component of $c_1(\mathcal{O}_{\mathcal{C}_g \times \mathcal{C}_g}(\triangle))$ in $H^1(\mathcal{C}_g) \otimes H^1(\mathcal{C}_g)$. Hence, by Milnor-Stasheff [MS, Theorem 11.11],

$$c_1(\mathcal{P}') = -\sum_{i=1}^{g} a_i^* \otimes b_i^* + \sum_{i=1}^{g} b_i^* \otimes a_i^*.$$

Therefore, by (10),

$$c_1(\mathcal{P}) = -\sum_{i=1}^{g} a_i^* \otimes \psi^{*-1}(b_i^*) + \sum_{i=1}^{g} b_i^* \otimes \psi^{*-1}(a_i^*),$$

and thus, by (6),

$$\begin{aligned} c_1(\operatorname{Det} \mathcal{P}) &= -\frac{1}{2}\pi_*(c_1(\mathcal{P})^2) \\ &= -\frac{1}{2}\pi_*\Big(\big(-\sum_{i=1}^{g} a_i^* \otimes \psi^{*-1}(b_i^*) + \sum_{i=1}^{g} b_i^* \otimes \psi^{*-1}(a_i^*)\big)^2\Big) \\ &= -\frac{1}{2}\pi_*\left(\sum_{i=1}^{g} a_i^* \cdot b_i^* \otimes \psi^{*-1}(b_i^*) \cdot \psi^{*-1}(a_i^*)\right. \\ &\quad \left. + \sum_{i=1}^{g} b_i^* \cdot a_i^* \otimes \psi^{*-1}(a_i^*) \cdot \psi^{*-1}(b_i^*)\right) \\ &= -\sum_{i=1}^{g} \psi^{*-1}(b_i^*) \cdot \psi^{*-1}(a_i^*) \quad \in H^2(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z}). \end{aligned}$$

Now, the involution τ of $\mathfrak{J}_{\mathcal{C}_g}$ induces the map $-I$ on $H^1(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z})$ (since, under the identification $\bar{\theta}_{\mathcal{C}_g} : J_g \to \mathfrak{J}_{\mathcal{C}_g}$, τ corresponds to the map $x \mapsto x^{-1}$ for $x \in J_g$). Therefore,

$$\tau^*(c_1(\operatorname{Det} \mathcal{P})) = c_1(\operatorname{Det} \mathcal{P}).$$

Hence, by the identities (1)–(2),

$$\begin{aligned} \xi_{\mathcal{C}_g}^*(x_g) &= c_1(\operatorname{Det} \mathcal{F}) \\ &= 2c_1(\operatorname{Det} \mathcal{P}) \\ &= 2\sum_{i=1}^{g} \psi^{*-1}(a_i^*) \cdot \psi^{*-1}(b_i^*), \end{aligned} \tag{11}$$

which is clearly a nonvanishing class in $H^2(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z})$. Moreover, for any $g \geq 2$, under the last equality of (7), the map $r_{g-1} : J_{g-1} \to J_g$ corresponds to the map $H_1(\mathcal{C}_g, \mathbb{Z}) \to H_1(\mathcal{C}_{g-1}, \mathbb{Z})$, $a_i \mapsto a_i, b_i \mapsto b_i$ for $1 \leq i \leq g-1$, $a_g \mapsto 0, b_g \mapsto 0$. Thus, by (9) and (11), $\xi^*_{\mathcal{C}_g}(x_g)$ restricts, via r^*_{g-1}, to the class $\xi^*_{\mathcal{C}_{g-1}}(x_{g-1})$ for any $g \geq 2$. But, by the commutative diagram (E), $\xi^*_{\mathcal{C}_g}(x_g) = y_g$. This proves Lemma 4.1. □

Proof of Proposition 1.7. By the above Lemma 4.1 and the commutative diagram (F), we see that

$$f_g^*(d_g x_g) = f_g^* i_g^*(x_{g+1}) = r_g^*(f_{g+1}^*(x_{g+1})), \text{ i.e., } d_g y_g = y_g.$$

Since the cohomology of J_g is torsion free and y_g is a nonvanishing class, we get $d_g = 1$. This concludes the proof of Proposition 1.7. □

References

[A] Adams, F.: Lectures on Lie Groups. W. A. Benjamin, Inc., 1969

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.: Geometry of Algebraic Curves. Springer-Verlag, 1985

[BL] Beauville, A., Laszlo, Y.: Conformal blocks and generalized theta functions. Commun. Math. Phys. **164**, 385–419 (1994)

[BLS] Beauville, A., Laszlo, Y., Sorger, C.: The Picard group of the moduli of G–bundles on a curve. Compositio Math. **112**, 183–216 (1998)

[BR] Beltrametti, M., Robbiano, L.: Introduction to the theory of weighted projective spaces. Expo. Math. **4**, 111–162 (1986)

[BS] Bernshtein, I.N., Shvartsman, O.V.: Chevalley's theorem for complex crystallographic Coxeter groups. Funct. Anal. Appl. **12**, 308–310 (1978)

[B] Bourbaki, N.: Groupes et Algèbres de Lie, Chap. 4–6. Masson, Paris, 1981

[DN] Drezet, J.–M., Narasimhan, M.S.: Groupe de Picard des variétés de modules de fibrés semi–stables sur les courbes algébriques. Invent. Math. **97**, 53–94 (1989)

[D] Dynkin, E.B.: Semisimple subalgebras of semisimple Lie algebras. Am. Math. Soc. Transl. (Ser. II) **6**, 111–244 (1957)

[Fa] Faltings, G.: A proof for the Verlinde formula. J. Alg. Geom. **3**, 347–374 (1994)

[FMW] Friedman, R., Morgan, J.W., Witten, E.: Principal *G*-bundles over elliptic curves. Mathematical Research Letters **5**, 97–118 (1998)

[F] Fulton, W.: Intersection Theory. Springer, Berlin Heidelberg, 1998

[KM] Knudsen, F., Mumford, D.: The projectivity of the moduli space of stable curves I: preliminaries on "det" and "div". Math. Scand. **39**, 19–55 (1976)

[K_1] Kumar, S.: Infinite Grassmannians and moduli spaces of *G*–bundles. In: Vector Bundles on Curves - New Directions, Lecture Notes in Mathematics **1649**, pp. 1–49, Springer, Berlin Heidelberg, 1997

[K_2] Kumar, S.: Kac-Moody Groups, their Flag Varieties and Representation Theory. Progress in Mathematics vol. **204**, Birkhäuser, 2002

[KN] Kumar, S., Narasimhan, M.S.: Picard group of the moduli spaces of G–bundles. Math. Annalen **308**, 155–173 (1997)

[KNR] Kumar, S., Narasimhan, M.S., Ramanathan, A.: Infinite Grassmannians and moduli spaces of *G*–bundles. Math. Annalen **300**, 41–75 (1994)

[L] Lang, S.: Introduction to Arakelov Theory. Springer, Berlin Heidelberg, 1988
[La] Laszlo, Y.: About G-bundles over elliptic curves. Ann. Inst. Fourier, Grenoble **48**, 413–424 (1998)
[LS] Laszlo, Y., Sorger, C.: The line bundles on the moduli of parabolic G–bundles over curves and their sections. Ann. Scient. Éc. Norm. Sup. **30**, 499–525 (1997)
[Lo] Looijenga, E.: Root systems and elliptic curves. Invent. Math. **38**, 17–32 (1976)
[M] Milne, J.S.: Chap. VII–Jacobian varieties. In: Arithmetic Geometry, Cornell, G., et al., (eds.), Springer, Berlin Heidelberg 1986, 167–212
[MS] Milnor, J.W., Stasheff, J.D.: Characteristic Classes. Annals of Mathematics Studies vol. **76**, Princeton University Press, 1974
[NS] Narasimhan, M.S., Seshadri, C.S.: Stable and unitary vector bundles on a compact Riemann surface. Ann. of Math. **82**, 540–567 (1965)
[N] Narasimhan, R.: Compact Riemann Surfaces. Birkhäuser Verlag, 1992
[RR] Ramanan, S., Ramanathan, A.: Some remarks on the instability flag. Tôhoku Math. J. **36**, 269–291 (1984)
[R_1] Ramanathan, A.: Stable principal bundles on a compact Riemann surface- Construction of moduli space. Thesis, University of Bombay, 1976
[R_2] Ramanathan, A.: Stable principal bundles on a compact Riemann surface. Math. Ann. **213**, 129–152 (1975)
[So] Sorger, C.: On moduli of G-bundles on a curve for exceptional G. Ann. Scient. Éc. Norm. Sup. **32**, 127–133 (1999)
[S] Szenes, A.: The combinatorics of the Verlinde formulas In: Vector Bundles in Algebraic Geometry, Hitchin, N.J., et al., (eds.), Cambridge University Press, 1995
[T_1] Teleman, C.: Lie algebra cohomology and the fusion rules. Commun. Math. Phys. **173**, 265–311 (1995)
[T_2] Teleman, C.: Borel-Weil-Bott theory on the moduli stack of G-bundles over a curve. Invent. Math. **134**, 1–57 (1998)
[TUY] Tsuchiya, A., Ueno, K., Yamada, Y.: Conformal field theory on universal family of stable curves with gauge symmetries. Adv. Stud. Pure Math. **19**, 459–565 (1989)