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Induction functor in noncommutative equivariant cohomology and Dirac cohomology

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Dedicated to Bertram Kostant on his seventy fifth birthday

Abstract

The aim of this paper is to put some recent results of Huang and Pandžić (conjectured by Vogan) and Kostant on Dirac cohomology in a broader perspective. This is achieved by introducing an induction functor in the noncommutative equivariant cohomology. In this context, the results of Huang–Pandžić and Kostant are interpreted as special cases (corresponding to the manifold being a point) of more general results on noncommutative equivariant cohomology introduced by Alekseev and Meinrenken.

Introduction

Let G be a (not necessarily connected) real Lie group and let R be a closed subgroup with their complexified Lie algebras $\mathfrak{g}$ and $\mathfrak{r}$, respectively. We assume that there exists a G-invariant nondegenerate symmetric bilinear form $B_{\mathfrak{g}}$ on $\mathfrak{g}$ such that $B_{\mathfrak{g}|_{\mathfrak{r}}}$ is again nondegenerate. We will impose this restriction on G and R throughout the paper. Let $\mathfrak{p}$ be the orthocomplement $\mathfrak{r}^{\perp}$ of $\mathfrak{r}$ in $\mathfrak{g}$. Then, $B_{\mathfrak{g}|_{\mathfrak{r}}}$ being nondegenerate, we have $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p}$ and,

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moreover, $B_{\mathfrak{g}|_\mathfrak{p}}$ is again nondegenerate. Further, $\mathfrak{p}$ is R-stable under the adjoint action. For example, any compact Lie group G and a closed subgroup R satisfies the above restriction.

Let M be a smooth R-manifold. Then, the deRham complex $\Omega(M)$ of M is canonically a $\mathbb{Z}_+$-graded (and hence $\mathbb{Z}/(2)$-graded) R-differential algebra. We will only consider $\mathbb{Z}/(2)$-graded spaces, algebras etc., so, in the sequel, *by graded we will mean* $\mathbb{Z}/(2)$-*graded*. We define a certain induction functor in noncommutative equivariant cohomology which associates to the R-differential algebra $\Omega(M)$ a differential algebra $\mathrm{Ind}_{G/R}(\Omega(M))$. By definition,

$$\mathrm{Ind}_{G/R}\big(\Omega(M)\big) = \big(\mathcal{W}(\mathfrak{g}) \otimes \Omega(M)\big)_R,$$

where $\mathcal{W}(\mathfrak{g}) := U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$ is the noncommutative Weil algebra (cf. Section 1), $U(\mathfrak{g})$ is the enveloping algebra, $\mathrm{Cl}(\mathfrak{g})$ is the Clifford algebra of $\mathfrak{g}$ with respect to the form $B_\mathfrak{g}$, and the subscript R refers to the subspace of 'R-basic' elements (cf. Section 1). The differential algebra structure on $\mathrm{Ind}_{G/R}(\Omega(M))$ is the restriction of the tensor product differential algebra structure on $\mathcal{W}(\mathfrak{g}) \otimes \Omega(M)$. We prove that the differential algebra $\mathrm{Ind}_{G/R}(\Omega(M))$ is canonically isomorphic (as a differential algebra) with the noncommutative G-equivariant Cartan model $(U(\mathfrak{g}) \overset{\bullet}{\otimes} \Omega(I_G(M)))^G$ of the G manifold $I_G(M) := G \times^R M$ (cf. Theorem 2.2). From this isomorphism, we obtain (as an immediate corollary) that the cohomology $H(\mathrm{Ind}_{G/R}(\Omega(M)))$ is canonically isomorphic with the noncommutative G-equivariant cohomology $\mathcal{H}_G(I_G(M))$ as graded algebras.

We use the above isomorphism to construct a functorial graded linear cochain map $\Phi_M : \mathrm{Ind}_{G/R}(\Omega(M)) \to (U(\mathfrak{r}) \overset{\bullet}{\otimes} \Omega(M))^R$, where the latter is the noncommutative R-equivariant Cartan model of the R-manifold M. Further, we show that Φ_M induces an algebra isomorphism in cohomology, even though, in general, Φ_M by itself is *not* an algebra homomorphism. As a corollary, we obtain a functorial graded algebra isomorphism $\mathcal{H}_G(I_G(M)) \simeq \mathcal{H}_R(M)$.

We now specialize the above results to the case when M is the one-point manifold M^o to obtain some important recent results of Huang–Pandžić and Kostant on Dirac cohomology [Ko2,HP]. In more detail, taking $M = M^o$,

$$\mathrm{Ind}_{G/R}\big(\Omega(M^o)\big) \cong \big(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})\big)^R.$$

We show that the differential d on $\mathrm{Ind}_{G/R}(\Omega(M^o))$ corresponds, under the above isomorphism, with the differential $\mathrm{ad}\,\mathcal{D}^\mathfrak{p}$ on the right-hand side introduced by Kostant, where $\mathcal{D}^\mathfrak{p} \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^R$ is his remarkable cubic Dirac operator $\sum p_\ell \otimes q_\ell + 1 \otimes \gamma_\mathfrak{p}$, where $\{p_\ell\}_\ell$ is any basis of $\mathfrak{p}$ and $\{q_\ell\}_\ell$ is the dual basis with respect to $B_{\mathfrak{g}|_\mathfrak{p}}$ and $\gamma_\mathfrak{p}$ is the Cartan element in $\bigwedge^3(\mathfrak{p})$ under the standard identification $\bigwedge(\mathfrak{p}) \simeq \mathrm{Cl}(\mathfrak{p})$. Recall that in the case when R is a maximal compact subgroup of reductive G, then $\gamma_\mathfrak{p} = 0$ and the operator $\mathcal{D}^\mathfrak{p}$ reduces to the Dirac operator considered by Vogan in defining his Dirac cohomology. Thus, our theorem in the case $M = M^o$ gives that

$$H\big(\big(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})\big)^R, \mathrm{ad}\,\mathcal{D}^\mathfrak{p}\big) \simeq \mathcal{H}_R(M^o) \simeq U(\mathfrak{r})^R, \tag{$*$}$$

which was proved by Huang and Pandžić in the case when R is a maximal compact subgroup of a connected reductive G and by Kostant in the general connected reductive case, i.e., when G and R are connected and reductive (and of course $B_{\mathfrak{g}|_{\mathfrak{r}}}$ is nondegenerate). In fact, from (∗) one obtains the decomposition

$$\mathrm{Ker}(\mathrm{ad}\,\mathcal{D}^{\mathfrak{p}}) = \xi\big(Z(R)\big) \oplus \mathrm{Image}(\mathrm{ad}\,\mathcal{D}^{\mathfrak{p}}),$$

where the homomorphism $\xi : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ is induced from the adjoint actions of $\mathfrak{r}$ on $\mathfrak{g}$ and $\mathfrak{p}$ and $Z(R)$ is the subalgebra of R-invariants $U(\mathfrak{r})^R$. Also, the isomorphism (∗) gives rise to an algebra homomorphism $\eta_R : Z(G) \to Z(R)$. We show that, from the general properties of the Duflo isomorphism, η_R is the unique homomorphism making the following diagram commutative:

$$\begin{array}{ccc} Z(G) & \xrightarrow{\eta_R} & Z(R) \\ {\scriptstyle H_{\mathfrak{g}}}\downarrow & & \downarrow{\scriptstyle H_{\mathfrak{g}}} \\ S(\mathfrak{g})^G & \longrightarrow & S(\mathfrak{r})^R, \end{array}$$

where $H_{\mathfrak{g}}$ is the Harish-Chandra isomorphism and the bottom horizontal map is induced by the orthogonal projection $\mathfrak{g} \to \mathfrak{r}$.

1. Review of noncommutative equivariant cohomology after Alekseev and Meinrenken

Unless otherwise explicitly stated, by vector spaces we mean complex vector spaces and by linear maps complex linear maps.

Let G be a (not necessarily connected) real Lie group with complexified Lie algebra $\mathfrak{g}$. We assume that $\mathfrak{g}$ has a nondegenerate symmetric G-invariant bilinear form $B_{\mathfrak{g}}$ on $\mathfrak{g}$, often denoted as $\langle\cdot\,,\cdot\rangle$. Define the $\mathbb{Z}$-graded super-Lie algebra $\hat{\mathfrak{g}}^*$ as follows. As a vector space,

$$\hat{\mathfrak{g}}^{-1} = \hat{\mathfrak{g}}^0 = \mathfrak{g}, \qquad \hat{\mathfrak{g}}^1 = \mathbb{C}, \qquad \hat{\mathfrak{g}}^n = 0 \quad \text{if } n \neq -1, 0, 1.$$

For $a \in \mathfrak{g}$, the corresponding element in $\hat{\mathfrak{g}}^{-1}$ (respectively $\hat{\mathfrak{g}}^0$) will be denoted by i_a (respectively L_a) and they represent 'contraction' and 'Lie derivation', respectively. We denote the generator of $\hat{\mathfrak{g}}^1$ by d. The bracket relations in $\hat{\mathfrak{g}}^*$ are defined by (for $a, b \in \mathfrak{g}$):

$$[L_a, i_b] = i_{[a,b]}, \qquad [L_a, L_b] = L_{[a,b]}, \qquad [i_a, d] = L_a.$$

By a *super-space* we mean a $\mathbb{Z}/(2)$-graded space. Any $\mathbb{Z}$-graded space of course has a canonical $\mathbb{Z}/(2)$-grading by even and odd components.

Recall that a *G-differential space* is a super-space B which is a Fréchet space, together with a graded smooth action of G on B and a super-Lie algebra homomorphism $\gamma : \hat{\mathfrak{g}} \to \mathrm{End}_{\mathrm{Cont}}\, B$, where $\mathrm{End}_{\mathrm{Cont}}\, B$ denotes the super-Lie algebra consisting of continuous linear endomorphisms of B. Moreover, we assume that the action of G commutes with d (via γ), L_a is the derivative of the G-action and

$$g i_a g^{-1} = i_{g\cdot a} \quad \text{for all } g \in G,\ a \in \mathfrak{g}.$$

The *horizontal subspace* B_{hor} is the space annihilated by $\hat{\mathfrak{g}}^{-1}$, the *invariant subspace* B^G is the subspace invariant under G and the space B_G of *basic elements* is the intersection $B_{\text{hor}} \cap B^G$.

A *G-differential algebra* is a super-algebra B together with the structure of a G-differential space on B such that θ takes values in the super-derivations of B and the action of G on B is via algebra automorphisms.

For a smooth G-manifold M, the deRham complex $\Omega(M)$ with the Fréchet topology provides the most important class of examples of G-differential algebras.

A *homomorphism* between G-differential spaces (respectively algebras) (B_1, θ_1) and (B_2, θ_2) is a continuous homomorphism of super-spaces (respectively super-algebras) $\phi : B_1 \to B_2$ such that

$$\phi(g \cdot b) = g \cdot \phi(b), \quad \phi\big(\theta_1(x)b\big) = \theta_2(x)\phi(b), \quad \text{for } g \in G, x \in \hat{\mathfrak{g}} \text{ and } b \in B_1.$$

There is the (classical) *Weil algebra* $W(\mathfrak{g}) := S(\mathfrak{g}^*) \otimes \bigwedge(\mathfrak{g}^*)$ with the tensor product algebra structure. This is a G-differential algebra under the $\mathbb{Z}_+$-grading

$$W(\mathfrak{g})^n = \bigoplus_{k \geqslant 0} S^k(\mathfrak{g}^*) \otimes \bigwedge^{n-2k}(\mathfrak{g}^*).$$

The action of G is via the coadjoint action. The operators L_a come from the coadjoint action of $\mathfrak{g}$ on $S(\mathfrak{g}^*)$ and $\bigwedge(\mathfrak{g}^*)$. The contraction operator i_a on $W(\mathfrak{g})$ is defined as $I_{S(\mathfrak{g}^*)} \otimes i'_a$, i'_a being the standard contraction operator on $\bigwedge(\mathfrak{g}^*)$. The differential d on $W(\mathfrak{g})$ is the unique super-derivation satisfying (for any $f \in \mathfrak{g}^*$)

$$d(1 \otimes e_f) = 1 \otimes d_\wedge e_f + s_f \otimes 1,$$

where e_f (respectively s_f) is the element f considered as an element of $\mathfrak{g}^* \subset \bigwedge(\mathfrak{g}^*)$ (respectively $\mathfrak{g}^* \subset S(\mathfrak{g}^*)$) and $d_\wedge : \bigwedge(\mathfrak{g}^*) \to \bigwedge(\mathfrak{g}^*)$ is the standard Koszul differential.

We now recall the definition of the *noncommutative Weil algebra* $\mathcal{W}(\mathfrak{g})$ from [AM], which is a G-differential algebra. Recall that the Clifford algebra $\text{Cl}(\mathfrak{g})$ of $\mathfrak{g}$ with respect to the bilinear form $B_{\mathfrak{g}}$ is the quotient of the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$ by the two-sided ideal generated by $2x \otimes x - \langle x, x\rangle,\ x \in \mathfrak{g}$. As a super-algebra $\mathcal{W}(\mathfrak{g})$ is defined as the tensor product of algebras

$$\mathcal{W}(\mathfrak{g}) := U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{g}),$$

where the $\mathbb{Z}/(2)$-grading on $\mathcal{W}(\mathfrak{g})$ comes from the standard grading on the Clifford algebra $\text{Cl}(\mathfrak{g})$, and $U(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$ placed in the even degree part. Both of $U(\mathfrak{g})$ and $\text{Cl}(\mathfrak{g})$ are G-modules under the adjoint action and so is their tensor product. For $a \in \mathfrak{g}$, let L_a be the adjoint action on $\mathcal{W}(\mathfrak{g})$.

Recall that there is a vector space isomorphism given by the symbol map

$$\sigma : \mathrm{Cl}(\mathfrak{g}) \to \bigwedge(\mathfrak{g}),$$

where σ^{-1} is induced from the standard projection map $T(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$ under the identification of $\bigwedge(\mathfrak{g})$ with the skew-symmetric tensors in $T(\mathfrak{g})$. *From now on we will identify* $\mathrm{Cl}(\mathfrak{g})$ *with* $\bigwedge(\mathfrak{g})$ (*via the symbol map*) *as a vector space.* Under this identification, we will denote the product in $\bigwedge(\mathfrak{g})$ by $\odot$, i.e.,

$$x \odot y = \sigma\big(\sigma^{-1}(x) \cdot \sigma^{-1}(y)\big), \quad \text{for } x, y \in \bigwedge(\mathfrak{g}).$$

The exterior product in $\bigwedge(\mathfrak{g})$ will be denoted by $x \wedge y$. Recall that $\bigwedge(\mathfrak{g})$ admits the contraction operator $\bar{i}_a$ (for $a \in \mathfrak{g}$) which is a super-derivation induced from the operator

$$\bar{i}_a b = \langle a, b\rangle, \quad \text{for } b \in \mathfrak{g}.$$

Then, by [AM, Lemma 3.1], the product $\odot$ in $\bigwedge(\mathfrak{g})$ is explicitly given by (for $\omega, \eta \in \bigwedge(\mathfrak{g})$):

$$\omega \odot \eta = \mu\left(\mathrm{Exp}\left(-\frac{1}{2}\sum_k i^1_{a_k} i^2_{b_k}\right)(\omega \otimes \eta)\right),$$

where $\{a_k\}_k$ is any basis of $\mathfrak{g}$ and $\{b_k\}_k$ is the dual basis $\langle a_k, b_\ell\rangle = \delta_{k,\ell}$, $\mu : \bigwedge(\mathfrak{g}) \otimes \bigwedge(\mathfrak{g}) \to \bigwedge(\mathfrak{g})$ is the standard wedge product, $i^1_{a_k}(\omega \otimes \eta) := (\bar{i}_{a_k}\omega) \otimes \eta$, and $i^2_{a_k}(\omega \otimes \eta) = \omega \otimes \bar{i}_{a_k}\eta$.

Define the operator i_a, $a \in \mathfrak{g}$, on $\mathcal{W}(\mathfrak{g})$ by

$$i_a = I_{U(\mathfrak{g})} \otimes \bar{i}_a.$$

For $a \in \mathfrak{g}$, let u_a (respectively c_a) be the corresponding element in $\mathfrak{g} \subset U(\mathfrak{g})$ (respectively $\mathfrak{g} \subset \mathrm{Cl}(\mathfrak{g})$). We also think of u_a (respectively c_a) as the element $u_a \otimes 1$ (respectively $1 \otimes c_a$) of $\mathcal{W}(\mathfrak{g})$.

Finally, we define the differential $d : \mathcal{W}(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g})$ as the commutator

$$dx = \mathrm{ad}\,\mathcal{D}(x),$$

where $\mathcal{D} \in \mathcal{W}(\mathfrak{g})$ is defined by

$$\mathcal{D} = \sum_k u_{a_k} c_{b_k} - 1 \otimes \gamma,$$

$\gamma = \gamma_{\mathfrak{g}} \in \bigwedge^3(\mathfrak{g})^G$ is the G-invariant element (so called the *Cartan element*) defined by

$$\gamma(a, b, c) = \big\langle a, [b, c]\big\rangle, \quad \text{for } a, b, c \in \mathfrak{g},$$

under the identification $\bigwedge(\mathfrak{g}) \simeq \bigwedge(\mathfrak{g}^*)$ induced from the form $\langle \cdot\,,\cdot \rangle$, and $\operatorname{ad}\mathcal{D}$ is the super-adjoint action defined by $\operatorname{ad}\mathcal{D}(x) = \mathcal{D}x - x\mathcal{D}$, for $x \in \mathcal{W}(\mathfrak{g})^{\text{even}}$, and $\operatorname{ad}\mathcal{D}(x) = \mathcal{D}x + x\mathcal{D}$, for $x \in \mathcal{W}(\mathfrak{g})^{\text{odd}}$.

Then, $\mathcal{W}(\mathfrak{g})$ with the above operators i_a, L_a, $d = \operatorname{ad}\mathcal{D}$ and the adjoint action of G is a G-differential algebra called the *noncommutative Weil algebra*.

By [AM, Proposition 3.7 and Eq. (3)], d is given by the formula

$$d(x \otimes \omega) = -\operatorname{ad} u_{a_k}(x) \otimes c_{b_k} \wedge \omega - \left(\frac{u_{a_k}x + xu_{a_k}}{2}\right) \otimes \bar{i}_{b_k}\omega + x \otimes d_\wedge \omega + \frac{1}{4} x \otimes \bar{i}_\gamma \omega,$$
$$\text{for } x \in U(\mathfrak{g}),\ \omega \in \bigwedge(\mathfrak{g}),$$

where $d_\wedge$ is the Koszul differential on $\bigwedge(\mathfrak{g})$ of degree $+1$ under the identification $\bigwedge(\mathfrak{g}) \simeq \bigwedge(\mathfrak{g}^*)$.

Recall [C] that the *G-equivariant cohomology $H_G(B)$* of a G-differential algebra B is, by definition, the cohomology of the basic subalgebra $(W(\mathfrak{g}) \otimes B)_G$ of the tensor product G-differential algebra $W(\mathfrak{g}) \otimes B$ under the tensor product differential $d(x \otimes y) = dx \otimes y + (-1)^{\deg x} x \otimes dy$, for $x \in W(\mathfrak{g})$ and $y \in B$.

Similarly, following [AM], the *noncommutative G-equivariant cohomology $\mathcal{H}_G(B)$* is the cohomology of the basic subalgebra $(\mathcal{W}(\mathfrak{g}) \otimes B)_G$ under the tensor product differential. Then, clearly, $\mathcal{H}_G(B)$ is a super-algebra.

Proposition 1.1. *For any G-differential algebra B, the projection map $\theta : W(\mathfrak{g}) \otimes B \to S(\mathfrak{g}^*) \otimes B$, induced from the standard augmentation map $\bigwedge(\mathfrak{g}^*) \to \mathbb{C}$, induces an algebra isomorphism* (*again denoted by*)

$$\theta : \big(W(\mathfrak{g}) \otimes B\big)_G \xrightarrow{\sim} \big(S(\mathfrak{g}^*) \otimes B\big)^G.$$

Under the above isomorphism, the differential d corresponds to the differential d_G on $(S(\mathfrak{g}^*) \otimes B)^G$ given as follows:

$$d_G = I_{S(\mathfrak{g}^*)} \otimes d - \sum_k s_{a_k^*} \otimes i_{a_k},$$

where $\{a_k\}_k$ is any basis of $\mathfrak{g}$ and $\{a_k^*\}_k$ is the dual basis of $\mathfrak{g}^*$ and $s_{a_k^*}$ denotes the operator acting on $S(\mathfrak{g}^*)$ via the multiplication by a_k^*.

Similarly, we have the following proposition from [AM, Section 4.2].

Proposition 1.2. *For any G-differential algebra B, the projection map*

$$\Theta : \mathcal{W}(\mathfrak{g}) \otimes B \to U(\mathfrak{g}) \otimes B,$$

induced from the standard augmentation map $\bigwedge(\mathfrak{g}) \to \mathbb{C}$, *induces a vector space (but not in general algebra) isomorphism*

$$\Theta : \big(\mathcal{W}(\mathfrak{g}) \otimes B\big)_G \xrightarrow{\sim} \big(U(\mathfrak{g}) \otimes B\big)^G.$$

To distinguish, let $(U(\mathfrak{g}) \overset{\bullet}{\otimes} B)^G$ *denote the vector space* $(U(\mathfrak{g}) \otimes B)^G$ *with the new product* $\odot$ *making* Θ *an algebra isomorphism.*

Under the above isomorphism, the differential d *corresponds to the differential*

$$d_G = I_{U(\mathfrak{g})} \otimes d - \frac{1}{2} \sum_k \big(u^L_{a_k} + u^R_{a_k}\big) \otimes i_{b_k} + \frac{1}{4} I_{U(\mathfrak{g})} \otimes i_\gamma, \tag{1}$$

where $\gamma \in \bigwedge^3(\mathfrak{g})^G$ *is defined earlier,* $u^L_{a_k}$ *(respectively* $u^R_{a_k}$*) denotes the left (respectively right) multiplication in* $U(\mathfrak{g})$ *by* u_{a_k} *and* $\{a_k\}_k, \{b_k\}_k$ *are dual bases of* $\mathfrak{g}$.

By [AM, Proposition 4.3], the multiplication $\odot$ in $(U(\mathfrak{g}) \otimes B)^G$ is given explicitly as the restriction of the multiplication (again denoted by) $\odot$ in $U(\mathfrak{g}) \otimes B$ defined as follows: For $x, y \in U(\mathfrak{g})$, $b_1, b_2 \in B$,

$$(x \otimes b_1) \odot (y \otimes b_2) = xy \otimes \mu\left(\operatorname{Exp}\left(-\frac{1}{2} \sum_k i^1_{a_k} i^2_{b_k}\right)(b_1 \otimes b_2)\right), \tag{2}$$

where $i^1_{a_k}$ and $i^2_{b_k}$ are the contraction operators on $B \otimes B$ with respect to the first and second factors, respectively, and $\mu : B \otimes B \to B$ is the multiplication map.

As in [AM], there exists a G-module isomorphism (depending only on $\mathfrak{g}$), called the *quantization map*,

$$\mathcal{Q} = \mathcal{Q}_{\mathfrak{g}} : W(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g})$$

which intertwines all the operators L_a, i_a and d. $\mathcal{Q}|_{S(\mathfrak{g}^*)}$ is the composite of the isomorphisms

$$S(\mathfrak{g}^*) \to S(\mathfrak{g}) \xrightarrow{D_{\mathfrak{g}}} U(\mathfrak{g}),$$

where the first map is the algebra isomorphism induced from the isomorphism $\mathfrak{g}^* \to \mathfrak{g}$ (coming from $\langle \cdot, \cdot \rangle$) and $D_{\mathfrak{g}}$ is the Duflo isomorphism [D]. (Recall that $D_{\mathfrak{g}}$ is only a linear isomorphism from $S(\mathfrak{g})$ to $U(\mathfrak{g})$ but it is an algebra isomorphism restricted to $S(\mathfrak{g})^{\mathfrak{g}}$ onto the center $U(\mathfrak{g})^{\mathfrak{g}}$. Moreover, it maps isomorphically $S(\mathfrak{g})^G$ onto $U(\mathfrak{g})^G$.) Also, recall that, for $a \in S^p(\mathfrak{g})$, $D_{\mathfrak{g}}(a) = \Sigma(a)$ (mod $U(\mathfrak{g})^{p-1}$), where $U(\mathfrak{g})^p$ is the standard filtration of $U(\mathfrak{g})$ and $\Sigma : S(\mathfrak{g}) \to U(\mathfrak{g})$ is the standard symmetrization map. Also, $\mathcal{Q}|_{\bigwedge(\mathfrak{g}^*)}$ is the isomorphism (induced from $\langle \cdot, \cdot \rangle$)

$$\bigwedge(\mathfrak{g}^*) \xrightarrow{\sim} \bigwedge(\mathfrak{g}).$$

Of course, as earlier, we have identified $\mathrm{Cl}(\mathfrak{g})$ with $\bigwedge(\mathfrak{g})$ via the symbol map σ.

However, $\mathcal{Q} \neq \mathcal{Q}|_{S(\mathfrak{g}^*)} \otimes \mathcal{Q}|_{\bigwedge(\mathfrak{g}^*)}$, in general.

Theorem 1.3 [AM, Theorem 7.1]. *For any G-differential algebra B, the cochain map $\mathcal{Q} \otimes I_B : W(\mathfrak{g}) \otimes B \to \mathcal{W}(\mathfrak{g}) \otimes B$ induces an algebra isomorphism in cohomology*

$$\mathcal{Q}^B : H_G(B) \xrightarrow{\sim} \mathcal{H}_G(B).$$

Definition 1.4. Let $\widehat{\mathcal{Q}}^B = \widehat{\mathcal{Q}}^B_G : (S(\mathfrak{g}^*) \otimes B)^G \to (U(\mathfrak{g}) \overset{\bullet}{\otimes} B)^G$ be the unique map making the following diagram commutative:

$$\begin{array}{ccc} \big(W(\mathfrak{g}) \otimes B\big)_G & \xrightarrow[\sim]{\mathcal{Q}\otimes I_B} & \big(\mathcal{W}(\mathfrak{g}) \otimes B\big)_G \\ \big\downarrow \theta & & \big\downarrow \Theta \\ \big(S(\mathfrak{g}^*) \otimes B\big)^G & \xrightarrow[\sim]{\widehat{\mathcal{Q}}^B} & \big(U(\mathfrak{g}) \overset{\bullet}{\otimes} B\big)^G. \end{array}$$

Then, clearly $\widehat{\mathcal{Q}}^B$ is a cochain isomorphism. In general, $\widehat{\mathcal{Q}}^B$ is not an algebra homomorphism.

2. An induction functor in noncommutative equivariant cohomology

Let G be a real (not necessarily connected) Lie group with complexified Lie algebra $\mathfrak{g}$ and let R be a closed subgroup of G with complexified Lie algebra $\mathfrak{r}$. Assume that $\mathfrak{g}$ admits a G-invariant nondegenerate symmetric bilinear form $B_{\mathfrak{g}} = \langle \cdot\,, \cdot \rangle$ such that $B_{\mathfrak{g}}|_{\mathfrak{r}}$ is nondegenerate. We call such a pair of (G, R) a *quadratic pair*. Thus, we have the decomposition

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p}, \quad \text{where } \mathfrak{p} := \mathfrak{r}^{\perp}.$$

By the G-invariance of $B_{\mathfrak{g}}$, $\mathfrak{p}$ is R-stable under the adjoint action. Moreover, $B_{\mathfrak{g}}|_{\mathfrak{p}}$ also is nondegenerate.

The following definition is influenced by the corresponding definition in the (commutative) equivariant cohomology given in [KV, Definition 32].

Definition 2.1 (Induction functor). For an R-differential complex B, define the complex

$$\mathrm{Ind}_{G/R}(B) = \big(\mathcal{W}(\mathfrak{g}) \otimes B\big)_R$$

equipped with the standard tensor product differential

$$d(x \otimes y) = d_{\mathcal{W}} x \otimes y + (-1)^{\deg x} x \otimes d_B y, \quad x \in \mathcal{W}(\mathfrak{g}),\ y \in B,$$

where $d_{\mathcal{W}}$ is the differential in $\mathcal{W}(\mathfrak{g})$ and d_B is the differential in B.

Since $\mathcal{W}(\mathfrak{g})$ is a G (in particular R) differential complex and B is a R-differential complex, it is easy to see that d indeed keeps the R-basic subspace of $\mathcal{W}(\mathfrak{g}) \otimes B$ stable.

If B is a R-differential algebra, then $\mathrm{Ind}_{G/R}(B)$ is a differential algebra under the tensor product super-algebra structure on $\mathcal{W}(\mathfrak{g}) \otimes B$.

Let M be a smooth real R-manifold and let $\Omega(M)$ be the complexified deRham complex of M. Consider the fiber product G-manifold $I_G(M) := G \times^R M$, where G acts on $I_G(M)$ via the left multiplication on the first factor.

Theorem 2.2. *There exists a* $\mathbb{Z}/(2)$*-graded algebra isomorphism*

$$\psi_M : \big(U(\mathfrak{g}) \dot{\otimes} \Omega\big(I_G(M)\big)\big)^G \xrightarrow{\sim} \mathrm{Ind}_{G/R}\big(\Omega(M)\big)$$

commuting with the differentials, where $(U(\mathfrak{g}) \dot{\otimes} \Omega(I_G(M)))^G$ *is equipped with the Cartan differential* d_G *(cf. Proposition* 1.2*).*

Moreover, ψ_M *is functorial in the sense that for any* R*-equivariant smooth map* $f : M \to N$*, the following diagram is commutative*:

$$\begin{array}{ccc} \big(U(\mathfrak{g}) \dot{\otimes} \Omega\big(I_G(N)\big)\big)^G & \xrightarrow{\psi_N} & \mathrm{Ind}_{G/R}\big(\Omega(N)\big) \\ {\scriptstyle I_{U(\mathfrak{g})}\otimes f_G^*}\downarrow & & \downarrow{\scriptstyle I_{\mathcal{W}(\mathfrak{g})}\otimes f^*} \\ \big(U(\mathfrak{g}) \dot{\otimes} \Omega\big(I_G(M)\big)\big)^G & \xrightarrow{\psi_M} & \mathrm{Ind}_{G/R}\big(\Omega(M)\big), \end{array}$$

where $f^* : \Omega(N) \to \Omega(M)$ *and* $f_G^* : \Omega(I_G(N)) \to \Omega(I_G(M))$ *are the induced maps from* f.

Proof. Under the projection map $p : G \times M \to I_G(M)$, we can identify

$$\Omega\big(I_G(M)\big) \subset \Omega(G \times M).$$

For $\omega \in \Omega(I_G(M))$, by $\omega(1)$ we mean the evaluation of ω at $1 \times M$. Thus, $\omega(1) \in \bigwedge(\mathfrak{g}^*) \otimes \Omega(M)$. Under the identification $\bigwedge(\mathfrak{g}^*) \simeq \bigwedge(\mathfrak{g})$ (induced from the bilinear form $B_{\mathfrak{g}}$), we can (and will) think of $\omega(1) \in \bigwedge(\mathfrak{g}) \otimes \Omega(M)$. Thus, we get the map

$$U(\mathfrak{g}) \otimes \Omega\big(I_G(M)\big) \to \mathcal{W}(\mathfrak{g}) \otimes \Omega(M),$$

$x \otimes \omega \mapsto x \otimes \omega(1)$. Let ψ_M be its restriction to $(U(\mathfrak{g}) \otimes \Omega(I_G(M)))^G$. We need to show that $\psi_M((U(\mathfrak{g}) \otimes \Omega(I_G(M)))^G) \subset \mathrm{Ind}_{G/R}(\Omega(M))$.

It is easy to see that (cf. [KV, p. 147])

$$\Omega\big(I_G(M)\big) = C^\infty\Big(G, \Big(\bigwedge(\mathfrak{g}^*) \otimes \Omega(M)\Big)_{\mathrm{hor} R}\Big)^R,$$

where $C^\infty(G,V)$ denotes the space of C^∞-functions on G with values in V; the R-invariants are taken with respect to the action of R on G via right multiplication, the given action of R on M and the adjoint action on $\bigwedge(\mathfrak{g}^*)$; the contraction i_a ($a \in \mathfrak{r}$) acting on $\bigwedge(\mathfrak{g}^*) \otimes \Omega(M)$ is the standard tensor product contraction. Thus,

$$\big(U(\mathfrak{g}) \otimes \Omega\big(I_G(M)\big)\big)^G = \Big(C^\infty\Big(G, \Big(U(\mathfrak{g}) \otimes \bigwedge(\mathfrak{g}^*) \otimes \Omega(M)\Big)_{\mathrm{hor}R}\Big)^R\Big)^G, \tag{1}$$

where R acts trivially on $U(\mathfrak{g})$; and the G-invariants are taken with respect to the left action of G on G, the adjoint action on $U(\mathfrak{g})$ and the trivial action on $\bigwedge(\mathfrak{g}^*) \otimes \Omega(M)$.

Take $\tilde{\alpha} \in (C^\infty(G, (U(\mathfrak{g}) \otimes \bigwedge(\mathfrak{g}^*) \otimes \Omega(M))_{\mathrm{hor}R})^R)^G$. Then,

$$\tilde{\alpha}\big(gk^{-1}\big) = k\tilde{\alpha}(g), \quad \text{for } g \in G, k \in R.$$

Writing $\tilde{\alpha}(1) = \sum x_i \otimes \omega_i$, $x_i \in U(\mathfrak{g})$, $\omega_i \in \bigwedge(\mathfrak{g}^*) \otimes \Omega(M)$, since $\tilde{\alpha}$ is G-invariant,

$$\sum_i \big(\mathrm{Ad}\big(gk^{-1}\big)x_i\big) \otimes \omega_i = k \cdot \sum_i \big(\mathrm{Ad}(g)x_i\big) \otimes \omega_i = \sum_i \big(\mathrm{Ad}(g)x_i\big) \otimes k\omega_i.$$

Taking $g = k$ in the above identity, we get

$$\sum_i x_i \otimes \omega_i = \sum_i \big(\mathrm{Ad}(k), x_i\big) \otimes k\omega_i.$$

Thus, $\psi_M(\tilde{\alpha}) \in \mathrm{Ind}_{G/R}(\Omega(M))$.

We next show that ψ_M is surjective onto $\mathrm{Ind}_{G/R}(\Omega(M))$. Take $\alpha = \sum_i x_i \otimes \omega_i \in (U(\mathfrak{g}) \otimes (\bigwedge(\mathfrak{g}^*) \otimes \Omega(M)))_R$ and define $\tilde{\alpha} \in (U(\mathfrak{g}) \otimes \Omega(I_G(M)))^G$, under the identification (1), by

$$\tilde{\alpha}(g) = \sum_i \big(\mathrm{Ad}(g)x_i\big) \otimes \omega_i, \quad g \in G.$$

Clearly, $\tilde{\alpha}$ is G-invariant. Further,

$$\tilde{\alpha} \in C^\infty\Big(G, \Big(U(\mathfrak{g}) \otimes \bigwedge(\mathfrak{g}^*) \otimes \Omega(M)\Big)_{\mathrm{hor}R}\Big)^R.$$

To show this, it suffices to prove that for all $g \in G$ and $k \in R$,

$$\sum_i \big(\mathrm{Ad}\big(gk^{-1}\big)x_i\big) \otimes \omega_i = \sum \mathrm{Ad}(g)x_i \otimes k\omega_i. \tag{2}$$

But, α being R-invariant,

$$\sum_i x_i \otimes \omega_i = \sum \big(\mathrm{Ad}(k)x_i\big) \otimes k\omega_i, \quad \text{for all } k \in R. \tag{3}$$

Applying gk^{-1} to (3) we get (2).

The injectivity of ψ_M is clear from the G-invariance of any element in the domain of ψ_M. Thus, ψ_M is a linear isomorphism. We next show that ψ_M is a cochain map.

View $\bigwedge(\mathfrak{g}^*)$ as the space of left-invariant forms on G. For $x \in U(\mathfrak{g})$, $f \in C^\infty(G)$, $\omega_1 \in \bigwedge(\mathfrak{g}^*)$ and $\omega_2 \in \Omega(M)$, by (1.2.1) ($\bar{d}$ being the deRham differentials on G and also on M),

$$d_G(x \otimes f\omega_1 \otimes \omega_2) = x \otimes \bar{d}f \wedge \omega_1 \otimes \omega_2 + x \otimes f d_\wedge \omega_1 \otimes \omega_2 + (-1)^{\deg \omega_1} x \otimes f\omega_1 \otimes \bar{d}\omega_2 \\ - \frac{1}{2}\sum_k \big(u_{a_k}x + xu_{a_k}\big) \otimes f\big(\bar{i}_{b_k}\omega_1\big) \otimes \omega_2 + \frac{1}{4} x \otimes f(\bar{i}_\gamma \omega_1) \otimes \omega_2.$$

(In fact, to be precise, in the above we should have taken $\sum_j x^j \otimes f^j \omega_1^j \otimes \omega_2^j \in (U(\mathfrak{g}) \otimes \Omega(I_G(M)))^G$ instead of just a single term $x \otimes f\omega_1 \otimes \omega_2$. But, for notational convenience, we take a single term.)

Moreover, for any $x \otimes f\omega_1 \otimes \omega_2 \in (U(\mathfrak{g}) \otimes \Omega(I_G(M)))^G$, we get (for any $a_k \in \mathfrak{g}$)

$$(u_{a_k}x - xu_{a_k}) \otimes f\omega_1 \otimes \omega_2 = -x \otimes a_k(f)\omega_1 \otimes \omega_2. \tag{4}$$

Thus,

$$\begin{aligned} &\psi_M d_G(x \otimes f\omega_1 \otimes \omega_2) \\ &\quad = x \otimes (\bar{d}f)(1) \wedge \omega_1 \otimes \omega_2 + x \otimes f(1) d_\wedge \omega_1 \otimes \omega_2 + (-1)^{\deg \omega_1} x \otimes f(1)\omega_1 \otimes \bar{d}\omega_2 \\ &\qquad - \frac{1}{2}\sum_k (u_{a_k}x + xu_{a_k}) \otimes f(1)\big(\bar{i}_{b_k}\omega_1\big) \otimes \omega_2 + \frac{1}{4} x \otimes f(1)(\bar{i}_\gamma \omega_1) \otimes \omega_2 \\ &\quad = -\sum_k (u_{a_k}x - xu_{a_k}) \otimes f(1)a_k^* \wedge \omega_1 \otimes \omega_2 + x \otimes f(1) d_\wedge \omega_1 \otimes \omega_2 \\ &\qquad + (-1)^{\deg \omega_1} x \otimes f(1)\omega_1 \otimes \bar{d}\omega_2 - \frac{1}{2}\sum_k (u_{a_k}x + xu_{a_k}) \otimes f(1)\big(\bar{i}_{b_k}\omega_1\big) \otimes \omega_2 \\ &\qquad + \frac{1}{4} x \otimes f(1)(\bar{i}_\gamma \omega_1) \otimes \omega_2, \quad \text{by (4)}, \end{aligned} \tag{5}$$

where $\{a_k^*\}$ is the basis of $\mathfrak{g}^*$ dual to the basis $\{a_k\}$ of $\mathfrak{g}$.

On the other hand, by the expression of $d_{\mathcal{W}}$ given in Section 1,

$$\begin{aligned} &d\big(x \otimes f(1)\omega_1 \otimes \omega_2\big) \\ &\quad = f(1) d_{\mathcal{W}}(x \otimes \omega_1) \otimes \omega_2 + (-1)^{\deg \omega_1} f(1) x \otimes \omega_1 \otimes \bar{d}\omega_2 \\ &\quad = f(1)\left(-\operatorname{ad} u_{a_k}(x) \otimes c_{b_k} \wedge \omega_1 - \left(\frac{u_{a_k}x + xu_{a_k}}{2}\right) \otimes \bar{i}_{b_k}\omega_1 + x \otimes d_\wedge \omega_1 \right. \\ &\qquad \left. + \frac{1}{4} x \otimes \bar{i}_\gamma \omega_1\right) \otimes \omega_2 + (-1)^{\deg \omega_1} f(1) x \otimes \omega_1 \otimes \bar{d}\omega_2. \end{aligned} \tag{6}$$

Comparing (5) and (6) we get that ψ_M commutes with the differentials.

Finally, we show that ψ_M is an algebra homomorphism. Take two elements $u = x \otimes \sum_i f_i\omega'_i \otimes \omega''_i$ and $v = y \otimes \sum_j g_j\eta'_j \otimes \eta''_j$ in $U(\mathfrak{g}) \dot{\otimes} \Omega(I_G(M))$, where $f_i, g_j \in C^\infty(G \times M)$, $x, y \in U(\mathfrak{g})$, $\omega'_i, \eta'_j \in \bigwedge(\mathfrak{g}^*)$ and $\omega''_i, \eta''_j \in \Omega(M)$. Then, by (1.2.2),

$$\begin{aligned}
&\psi_M(u \odot v)\\
&\quad= \psi_M \sum_{i,j} (-1)^{\deg \eta'_j \deg \omega''_i} \left(xy \otimes f_i g_j \mu\left(\operatorname{Exp}\left(-\frac{1}{2}\sum_k i^1_{a_k} i^2_{b_k}\right)\left(\omega'_i \otimes \eta'_j\right)\right)\right) \otimes \omega''_i \eta''_j\\
&\quad= \sum_{i,j} (-1)^{\deg \eta'_j \deg \omega''_i} xy \otimes \mu\left(\operatorname{Exp}\left(-\frac{1}{2}\sum_k i^1_{a_k} i^2_{b_k}\right)\left(\omega'_i \otimes \eta'_j\right)\right)\\
&\qquad\quad \otimes f_i(1,-) g_j(1,-) \omega''_i \eta''_j\\
&\quad= \sum_{i,j} (-1)^{\deg \eta'_j \deg \omega''_i} xy \otimes \left(\omega'_i \odot \eta'_j\right) \otimes f_i(1,-) g_j(1,-) \omega''_i \eta''_j\\
&\quad= \psi_M(u) \cdot \psi_M(v).
\end{aligned}$$

This completes the proof of the theorem. □

Corollary 2.3. *For a real R-manifold M, the cochain map ψ_M induces a functorial $\mathbb{Z}/(2)$-graded algebra isomorphism*:

$$\psi_M^* : \mathcal{H}_G\big(I_G(M)\big) \xrightarrow{\sim} H\big(\operatorname{Ind}_{G/R}\big(\Omega(M)\big)\big).$$

Definition 2.4. Let M be a real R-manifold. Consider the R-module isomorphism

$$\mathcal{Q}_{\mathfrak{g}} \otimes I_{\Omega(M)} : W(\mathfrak{g}) \otimes \Omega(M) \to \mathcal{W}(\mathfrak{g}) \otimes \Omega(M).$$

Since $\mathcal{Q}_{\mathfrak{g}}$ commutes with the operators i_a ($a \in \mathfrak{g}$) and d and the G-action; in particular, $\mathcal{Q}_{\mathfrak{g}} \otimes I_{\Omega(M)}$ induces a linear isomorphism

$$\mathcal{Q}^M_{G/R} : \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_R \to \operatorname{Ind}_{G/R}\big(\Omega(M)\big)$$

commuting with differentials, where $W(\mathfrak{g}) \otimes \Omega(M)$ is equipped with the standard tensor product R-differential algebra structure.

Define now a cochain map $\Phi_M : \operatorname{Ind}_{G/R}(\Omega(M)) \to (U(\mathfrak{r}) \dot{\otimes} \Omega(M))^R$, making the following diagram commutative:

$$\begin{array}{ccc}
\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_R & \xrightarrow{\hat{\alpha}_M} & \big(S(\mathfrak{r}^*) \otimes \Omega(M)\big)^R\\
\Big\downarrow{\scriptstyle \mathcal{Q}^M_{G/R}} & & \Big\downarrow{\scriptstyle \widehat{\mathcal{Q}}^M_R}\\
\operatorname{Ind}_{G/R}\big(\Omega(M)\big) & \xrightarrow[\Phi_M]{} & \big(U(\mathfrak{r}) \dot{\otimes} \Omega(M)\big)^R,
\end{array}$$

where $\hat{\alpha}_M(P \otimes \omega \otimes \eta) = \varepsilon(\omega) P|_{\mathfrak{r}} \otimes \eta$, for $P \in S(\mathfrak{g}^*)$, $\omega \in \bigwedge(\mathfrak{g}^*)$, $\eta \in \Omega(M)$, and with $\varepsilon : \bigwedge(\mathfrak{g}^*) \to \mathbb{C}$ being the standard augmentation map. (Since $\mathcal{Q}^M_{G/R}$ is a cochain isomorphism, and $\hat{\alpha}_M$, $\widehat{\mathcal{Q}}^M_R$ are both cochain maps, such a Φ_M indeed exists.)

Observe that, in general, Φ_M is not an algebra homomorphism.

Further, Φ_M is functorial in the sense that for any R-equivariant smooth map $f : M \to N$, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Ind}_{G/R}\big(\Omega(N)\big) & \xrightarrow{\Phi_N} & \big(U(\mathfrak{r}) \dot{\otimes} \Omega(N)\big)^R \\ \downarrow & & \downarrow \\ \mathrm{Ind}_{G/R}\big(\Omega(M)\big) & \xrightarrow{\Phi_M} & \big(U(\mathfrak{r}) \dot{\otimes} \Omega(M)\big)^R, \end{array}$$

where the vertical maps are induced canonically from the R-differential algebra homomorphism $f^* : \Omega(N) \to \Omega(M)$.

Theorem 2.5. *For any R-manifold M, the cochain map $\Phi_M : \mathrm{Ind}_{G/R}(\Omega(M)) \to (U(\mathfrak{r}) \dot{\otimes} \Omega(M))^R$ induces an algebra isomorphism in cohomology*:

$$[\Phi_M] : H\big(\mathrm{Ind}_{G/R}\big(\Omega(M)\big)\big) \xrightarrow{\sim} \mathcal{H}_R(M).$$

Thus, by Corollary 2.3, *we have a functorial algebra isomorphism*

$$\mathcal{H}_G\big(I_G(M)\big) \xrightarrow{\sim} \mathcal{H}_R(M).$$

In particular, $\mathcal{H}_G(G/R) \xrightarrow{\sim} U(\mathfrak{r})^R$.

Proof. In the first commutative diagram of (2.4), the cochain maps $\mathcal{Q}^M_{G/R}$, and $\widehat{\mathcal{Q}}^M_R$ are cochain isomorphisms. In particular, they induce vector space isomorphisms in cohomology. By [AM, Theorem 7.1], these induced maps in cohomology are algebra isomorphisms. So, it suffices to prove that $\hat{\alpha}_M$ induces an isomorphism in cohomology. But, this follows from the "polynomial" analogue of [KV, Theorem 33] (for a proof see [DV, Proof of Théorème 24]). □

Remark 2.6. An appropriate analogue of Theorems 2.2 and 2.5, and Corollary 2.3 can be proved for any R-differential algebra B replacing $\Omega(M)$.

Applying the definition of ψ_M as in Theorem 2.2, for the case $R = G$ and a G-manifold M, interestingly we get an explicit expression for the inverse of the isomorphism Θ.

Lemma 2.7. *Take $R = G$ and a G-manifold M. Then, the inverse of the isomorphism*

$$\Theta = \Theta_M : \big(\mathcal{W}(\mathfrak{g}) \otimes \Omega(M)\big)_G \to \big(U(\mathfrak{g}) \dot{\otimes} \Omega(M)\big)^G$$

(*cf. Proposition* 1.2) *is given by the composition*

$$\big(U(\mathfrak{g}) \overset{\bullet}{\otimes} \Omega(M)\big)^G \xrightarrow[\sim]{I\otimes\mu^*} \big(U(\mathfrak{g}) \overset{\bullet}{\otimes} \Omega\big(I_G(M)\big)\big)^G \xrightarrow{\psi_M} \big(\mathcal{W}(\mathfrak{g}) \otimes \Omega(M)\big)_G,$$

where $\mu^*:\Omega(M) \to \Omega(I_G(M))$ *is the* G*-module isomorphism induced from the* G*-equivariant diffeomorphism*

$$\mu : G \times^G M \to M, \quad (g, m) \mapsto g \cdot m.$$

Proof. Since Θ_M is a vector space isomorphism, it suffices to prove that

$$\psi_M \circ (I \otimes \mu^*) \circ \Theta_M = I.$$

From the functoriality of Θ, we have the following commutative diagram:

$$\begin{array}{ccc} \big(\mathcal{W}(\mathfrak{g}) \otimes \Omega(M)\big)_G & \xrightarrow{I\otimes\mu^*} & \big(\mathcal{W}(\mathfrak{g}) \otimes \Omega\big(I_G(M)\big)\big)_G \\ \downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \Theta_{I_G(M)}} \\ \big(U(\mathfrak{g}) \overset{\bullet}{\otimes} \Omega(M)\big)^G & \xrightarrow[I\otimes\mu^*]{} & \big(U(\mathfrak{g}) \overset{\bullet}{\otimes} \Omega\big(I_G(M)\big)\big)^G. \end{array}$$

Take $a = \sum_i x_i \otimes \omega_i \in (U(\mathfrak{g}) \overset{\bullet}{\otimes} \Omega(M))^G$. Then, from the above commutative diagram:

$$\begin{aligned} &\Theta_M \circ \psi_M \circ \Theta_{I_G(M)} \circ (I \otimes \mu^*) \circ \Theta_M^{-1}(a) \\ &\quad = \Theta_M \circ \psi_M \circ (I \otimes \mu^*)(a) = \Theta_M \circ \psi_M \left(\sum_i x_i \otimes (\mu^* \omega_i) \right) = \sum_i x_i \otimes \big((\mu^*\omega_i)|_{1\times M}\big) \\ &\quad = \sum_i x_i \otimes \omega_i = a. \end{aligned}$$

This gives

$$\psi_M \circ \Theta_{I_G(M)} \circ (I \otimes \mu^*) \circ \Theta_M^{-1} = \Theta_M^{-1}.$$

Thus, $\psi_M \circ \Theta_{I_G(M)} \circ (I \otimes \mu^*) = I$ and hence, from the above commutative diagram again, $\psi_M \circ (I \otimes \mu^*) \circ \Theta_M = I$. This proves the lemma. □

3. Cubic Dirac operator and results of Huang–Pandžić and Kostant

We follow the notation and assumptions as in the beginning of Section 2. In particular, (G, R) is a quadratic pair, i.e., G is a real Lie group with complexified Lie algebra $\mathfrak{g}$ and

$R \subset G$ is a closed subgroup with complexified Lie subalgebra $\mathfrak{r} \subset \mathfrak{g}$ such that $\mathfrak{g}$ has a G-invariant nondegenerate symmetric bilinear form $B_{\mathfrak{g}} = \langle \cdot, \cdot \rangle$ which remains nondegenerate restricted to $\mathfrak{r}$.

We now identify the differential of $\mathrm{Ind}_{G/R}(M)$ for M a one-point manifold with Kostant's cubic Dirac operator, where $\mathrm{Ind}_{G/R}(\Omega(M))$ is abbreviated as $\mathrm{Ind}_{G/R}(M)$.

Lemma 3.1. *Let M be the one-point manifold M^o. Then, $\mathrm{Ind}_{G/R}(M^o)$ can canonically be identified with the super-algebra $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^R$ (under the tensor product algebra structure), where the Clifford algebra $\mathrm{Cl}(\mathfrak{p})$ is defined with respect to the nondegenerate form $B_{\mathfrak{g}}|_{\mathfrak{p}}$.*

Moreover, the differential d on $\mathrm{Ind}_{G/R}(M)$ under the above identification is given by

$$d(x) = \mathrm{ad}\,\mathcal{D}^{\mathfrak{p}}(x), \quad \textit{where } \mathcal{D}^{\mathfrak{p}} := \sum u_{p_\ell} \otimes c_{q_\ell} - 1 \otimes \gamma_{\mathfrak{p}},$$

$\{p_\ell\}_\ell$ is any basis of $\mathfrak{p}$ and $\{q_\ell\}_\ell$ is the dual basis with respect to the nondegenerate form $B_{\mathfrak{g}}|_{\mathfrak{p}}$ and $\gamma_{\mathfrak{p}} \in \bigwedge^3(\mathfrak{p}) \simeq \bigwedge^3(\mathfrak{p}^*)$ is the Cartan form

$$\gamma_{\mathfrak{p}}(x, y, z) = \big\langle x, [y, z]\big\rangle, \quad \textit{for } x, y, z \in \mathfrak{p}.$$

It is easy to see that $\mathcal{D}^{\mathfrak{p}}$ is R-invariant, i.e., $\mathcal{D}^{\mathfrak{p}} \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^R$.

Proof. Let $\{r_m\}$ be a basis of $\mathfrak{r}$ and let $\{s_m\}$ be the dual basis of $\mathfrak{r}$ under $B_{\mathfrak{g}|_{\mathfrak{r}}}$. Then, of course, $\{r_m\}_m \cup \{p_\ell\}_\ell$ is a basis of $\mathfrak{g}$ and $\{s_m\}_m \cup \{q_\ell\}_\ell$ is the dual basis of $\mathfrak{g}$. Thus, the element $\mathcal{D} \in \mathcal{W}(\mathfrak{g})$ as in Section 1 is given by

$$\mathcal{D} = \sum_m u_{r_m} \otimes c_{s_m} + \sum_\ell u_{p_\ell} \otimes c_{q_\ell} - 1 \otimes \gamma_{\mathfrak{g}}.$$

Now,

$$\mathcal{W}(\mathfrak{g})_R = \big(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})\big)^R_{\mathrm{hor} R} \cong \big(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})\big)^R.$$

The differential d in $\mathcal{W}(\mathfrak{g})$ is given by $dx = \mathrm{ad}\,\mathcal{D}(x)$. Moreover, d keeps the subspace $\mathcal{W}(\mathfrak{g})_R$ stable. From this it is easy to see that, for $x \in \mathcal{W}(\mathfrak{g})_R$, $dx = \mathrm{ad}\,\mathcal{D}^{\mathfrak{p}}(x)$. This proves the lemma. □

Definition 3.2. As in [Ko1, Section 1.5], the adjoint representation of R on $\mathfrak{p}$ gives rise to the Lie algebra homomorphism

$$\alpha : \mathfrak{r} \to \mathrm{Cl}(\mathfrak{p})^{\mathrm{even}} \quad \text{satisfying} \quad \big[\alpha(x), y\big] = [x, y], \quad \text{for } x \in \mathfrak{r} \text{ and } y \in \mathfrak{p},$$

where the bracket on the left-hand side is commutation in $\mathrm{Cl}(\mathfrak{p})$. Then, α is an R-module map under the adjoint actions. In particular, for $x_1, x_2 \in \mathfrak{r}$,

$$\alpha[x_1, x_2] = x_1 \cdot \alpha(x_2). \tag{1}$$

Thus, we get an algebra homomorphism

$$\xi : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}),$$

so that $\xi(x) = x \otimes 1 + 1 \otimes \alpha(x)$, for $x \in \mathfrak{r}$. It is easy to see that ξ is injective. Moreover, ξ is an R-module homomorphism, where the action of R on $U(\mathfrak{r})$ is induced from the adjoint action and the R-module structure on $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ is as given earlier (obtained from the adjoint actions). For $x \in \mathfrak{r}$ and $a \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$, we have

$$x \cdot a = \xi(x)a - a\xi(x). \tag{2}$$

Let $Z(G)$ (respectively $Z(R)$) be the subalgebra of invariants $U(\mathfrak{g})^G$ (respectively $U(\mathfrak{r})^R$). Then, $Z(G) \otimes 1$ and $\xi(Z(R))$ are subalgebras of $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^R$. Further, for $d = \mathrm{ad}\, \mathcal{D}^{\mathfrak{p}}$,

$$d|_{Z(G)\otimes 1} \equiv 0 \tag{3}$$

and

$$d|_{\xi(U(\mathfrak{r}))} \equiv 0, \tag{4}$$

since $\xi(U(\mathfrak{r}))$ commutes with any element in $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^R$ by (2). Thus, by Theorem 2.5 and Lemma 3.1, we get algebra homomorphisms

$$Z(G) \to H\big(\big(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})\big)^R, \mathrm{ad}\, \mathcal{D}^{\mathfrak{p}}\big) = H\big(\mathrm{Ind}_{G/R}(M^o)\big) \xrightarrow[\sim]{[\Phi_{M^o}]} Z(R),$$

where, as earlier, M^o is the one-point manifold and the first map is induced from the map $z \mapsto z \otimes 1$. Let η_R be the composite algebra homomorphism

$$\eta_R : Z(G) \to Z(R).$$

Define a R-differential algebra homomorphism $F = F_{\mathfrak{r}}^{\mathfrak{g}} : W(\mathfrak{r}) \to W(\mathfrak{g})$ by

$$F(\lambda \otimes 1) = \bar{\lambda} \otimes 1 - 1 \otimes \delta(\lambda) \quad \text{and} \quad F(1 \otimes \lambda) = 1 \otimes \bar{\lambda}, \quad \text{for } \lambda \in \mathfrak{r}^*,$$

where $\bar{\lambda} \in \mathfrak{g}^*$ is defined by $\bar{\lambda}|_{\mathfrak{r}} = \lambda$ and $\bar{\lambda}|_{\mathfrak{p}} \equiv 0$, and $\delta : \mathfrak{r}^* \to \bigwedge^2(\mathfrak{g}^*)$ is defined by

$$\delta(\lambda)(y, z) = \begin{cases} \bar{\lambda}([y, z]) & \text{for } y, z \in \mathfrak{p}, \\ 0 & \text{if at least one of } y, z \in \mathfrak{r}. \end{cases}$$

Similarly, define a R-differential algebra homomorphism $\mathcal{F} = \mathcal{F}_{\mathfrak{r}}^{\mathfrak{g}} : \mathcal{W}(\mathfrak{r}) \to \mathcal{W}(\mathfrak{g})$ by

$$\mathcal{F}(x \otimes 1) = x \otimes 1 + 1 \otimes \alpha(x) \quad \text{and} \quad \mathcal{F}(1 \otimes x) = 1 \otimes x, \quad \text{for } x \in \mathfrak{r}.$$

Clearly,

$$\mathcal{F}|_{U(\mathfrak{r})} = \xi. \tag{5}$$

Then, as proved by Alekseev and Meinrenken [AM1, Section 6], we have:

$$\mathcal{Q}_{\mathfrak{g}} \circ F = \mathcal{F} \circ \mathcal{Q}_{\mathfrak{r}}, \tag{6}$$

i.e., the following diagram is commutative:

$$\begin{array}{ccc} W(\mathfrak{r}) & \xrightarrow{F} & W(\mathfrak{g}) \\ \downarrow{\scriptstyle \mathcal{Q}_{\mathfrak{r}}} & & \downarrow{\scriptstyle \mathcal{Q}_{\mathfrak{g}}} \\ \mathcal{W}(\mathfrak{r}) & \xrightarrow[\mathcal{F}]{} & \mathcal{W}(\mathfrak{g}). \end{array}$$

As a corollary of Theorem 2.5, we get the following. This was conjectured by Vogan (actually Vogan conjectured a slightly weaker version) and proved by Huang and Pandžić [HP, Theorem 3.4] in the case R is a maximal compact subgroup of a connected reductive G. The case when G and R are connected and reductive was proved by Kostant [Ko2, Section 4.1].

Theorem 3.3. *Let (G, R) be a quadratic pair. For the differential $d := \operatorname{ad} \mathcal{D}^{\mathfrak{p}}$ on $(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^R$,*

$$\operatorname{Ker} d = \xi\big(Z(R)\big) \oplus \operatorname{Im} d. \tag{1}$$

In particular, $\xi(Z(R)) \simeq H((U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^R, \operatorname{ad} \mathcal{D}^{\mathfrak{p}})$.

Proof. We first prove that the composite map $\Phi_{M^o} \circ \xi$:

$$Z(R) \xrightarrow{\xi} \big(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})\big)^R \xrightarrow{\Phi_{M^o}} Z(R)$$

is an isomorphism. (In fact, we will see during the proof of the next theorem that $\Phi_{M^o} \circ \xi$ is the identity map.) Let $\{U(\mathfrak{r})^p\}_{p \geqslant 0}$ be the standard filtration of the enveloping algebra $U(\mathfrak{r})$ and let $Z(R)^p := U(\mathfrak{r})^p \cap Z(R)$. By the definition of the map ξ, for $a \in Z(R)^p$,

$$\xi(a) = a \otimes 1 + x,$$

for some $x \in \big(U(\mathfrak{g})^{p-1} \otimes \mathrm{Cl}(\mathfrak{p})\big)^R$. Thus, from the definition of the map Φ_{M^o} and the description of the isomorphism $\widehat{Q}_G^{I_G(M)}$ as in [AM, Proposition 6.5],

$$\Phi_{M^o} \circ \xi(a) = a \mod Z(R)^{p-1}.$$

From this we see that $\Phi_{M^o} \circ \xi$ is an isomorphism.

Since Φ_{M^o} induces an isomorphism in cohomology by Theorem 2.5, we get that the induced cohomology map

$$[\xi]: Z(R) \to H\big(\big(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})\big)^R, \mathrm{ad}\,\mathcal{D}^{\mathfrak{p}}\big)$$

is an isomorphism. From this, of course, (1) follows immediately. □

Theorem 3.4. *The algebra homomorphism $\eta_R : Z(G) \to Z(R)$ is the unique homomorphism making the following diagram commutative*:

$$\begin{array}{ccc} Z(G) & \xrightarrow{\eta_R} & Z(R) \\ \downarrow{\scriptstyle H_G} & & \downarrow{\scriptstyle H_R} \\ S(\mathfrak{g})^G & \xrightarrow{\beta_R} & S(\mathfrak{r})^R, \end{array} \tag{D}$$

where β_R is the restriction map under the identifications $S(\mathfrak{g}) \simeq S(\mathfrak{g}^*)$, $S(\mathfrak{r}) \simeq S(\mathfrak{r}^*)$ induced by the bilinear form $B_{\mathfrak{g}}$, and H_G (respectively H_R) is the inverse of the Duflo isomorphism of $\mathfrak{g}$ (respectively $\mathfrak{r}$) restricted to $S(\mathfrak{g})^G$ (respectively $S(\mathfrak{r})^R$). (Recall that for reductive G, H_G coincides with the Harish-Chandra isomorphism.)

Thus, for $z \in Z(G)$,

$$z \otimes 1 - \xi\big(\eta_R(z)\big) = \mathcal{D}^{\mathfrak{p}} a_z + a_z \mathcal{D}^{\mathfrak{p}}, \tag{1}$$

for some $a_z \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})^{\mathrm{odd}})^R$.

Proof. By [AM, Proposition 6.5],

$$\Phi_{M^o}(z \otimes 1) = D_{\mathfrak{r}}\big((D_{\mathfrak{g}}^{-1}(z))|_{\mathfrak{r}}\big),$$

where $D_{\mathfrak{g}} : S(\mathfrak{g}) \to U(\mathfrak{g})$ is the Duflo isomorphism under the identification $S(\mathfrak{g}) \simeq S(\mathfrak{g}^*)$, and similarly for $D_{\mathfrak{r}}$. This gives that

$$\eta_R(z) = [\Phi_{M^o}](z \otimes 1) = D_{\mathfrak{r}} \circ \beta_R \circ \big(D_{\mathfrak{g}}^{-1}\big)(z),$$

i.e., η_R makes the diagram (D) commutative. Since H_G and H_R are isomorphisms, there is a unique map $Z(G) \to Z(R)$ making the diagram (D) commutative. From this the first part of the theorem follows.

We next prove that

$$\Phi_{M^o} \circ \xi|_{Z(R)} = I, \tag{2}$$

where $\xi : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ is defined in Section 3.2. By the identities (3.2.5), (3.2.6), and the first commutative diagram of (2.4), for $x \in S(\mathfrak{r}^*)^R$,

$$\begin{aligned}\Phi_{M^o} \circ \xi \circ \mathcal{Q}_{\mathfrak{r}}(x) &= \widehat{\mathcal{Q}}_R^{M^o} \circ \hat{\alpha}_{M^o} \circ F(x)\\ &= \mathcal{Q}_{\mathfrak{r}} \circ \hat{\alpha}_{M^o} \circ F(x), \quad \text{since } \mathcal{Q}_{\mathfrak{r}}|_{S(\mathfrak{r}^*)^R} = \widehat{\mathcal{Q}}_R^{M^o}\\ &= \mathcal{Q}_{\mathfrak{r}}(x), \quad \text{from the definition of } F \text{ and } \hat{\alpha}_{M^o}.\end{aligned}$$

Since $\mathcal{Q}_{\mathfrak{r}}|_{S(\mathfrak{r}^*)^R}$ is an isomorphism onto $Z(R)$, this proves (2).

From (2) we easily see that, for $z \in Z(G)$,

$$\Phi_{M^o}(z \otimes 1) = \eta_R(z) = \Phi_{M^o}\big(\xi\big(\eta_R(z)\big)\big),$$

and, moreover, by (3.2.3), (3.2.4), both of $z \otimes 1$ and $\xi(\eta_R(z))$ are cycles under ad $\mathcal{D}^{\mathfrak{p}}$. Thus, they differ by a coboundary, proving (1). This proves the theorem.

Alternatively, we can also obtain (1) in the special (but important) case where G and R are connected reductive groups (and $B_{\mathfrak{g}|_{\mathfrak{r}}}$ is nondegenerate) by using a result of Kostant as follows.

By virtue of Theorem 3.3, define the map $\hat{\eta}_R : Z(G) \to Z(R)$ such that $z \otimes 1 - \xi(\hat{\eta}_R(z)) \in \operatorname{Im} d$. Then, it is easy to see that $\hat{\eta}_R$ is an algebra homomorphism. Moreover, by Kostant [Ko2, Theorem 4.2] (generalizing the corresponding result in the case when R is a maximal compact subgroup of G by Huang and Pandžić [HP, Theorem 5.5]), $\hat{\eta}_R$ replacing η_R also makes the diagram (D) commutative. Thus, from the uniqueness of η_R in the diagram (D), $\hat{\eta}_R = \eta_R$, proving (1). □

Definition 3.5. Fix a space S of spinors for $\mathrm{Cl}(\mathfrak{p})$, i.e., S is a simple module of $\mathrm{Cl}(\mathfrak{p})$. (Recall that if $\mathfrak{p}$ is even dimensional, S is unique up to an isomorphism and there are exactly two nonisomorphic simple modules of $\mathrm{Cl}(\mathfrak{p})$ if $\mathfrak{p}$ is odd dimensional.) Then, for any $U(\mathfrak{g})$-module V, $V \otimes S$ is a $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$-module under the component-wise action. In particular, the element $D^{\mathfrak{p}} \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^R$ defined in Lemma 3.1 acts as a linear endomorphism $D_V^{\mathfrak{p}}$ on $V \otimes S$.

Following Vogan, define the *Dirac cohomology* of V:

$$H_D(\mathfrak{g}, \mathfrak{r}; V) = \frac{\operatorname{Ker} D_V^{\mathfrak{p}}}{\operatorname{Ker} D_V^{\mathfrak{p}} \cap \operatorname{Im} D_V^{\mathfrak{p}}}.$$

Since the element $D^{\mathfrak{p}}$ commutes with $\xi(U(\mathfrak{r}))$ (cf. Section 3.2), both of $\operatorname{Ker} D_V^{\mathfrak{p}}$ and $\operatorname{Im} D_V^{\mathfrak{p}}$ are $\mathfrak{r}$-submodules of $V \otimes S$ via ξ. Thus, $H_D(\mathfrak{g}, \mathfrak{r}; V)$ has a canonical $\mathfrak{r}$-module structure.

Let $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ be an algebra homomorphism, where $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$. Recall that a $U(\mathfrak{g})$-module V is said to have *central character* χ if, for all $v \in V$ and $z \in Z(\mathfrak{g})$,

$$z \cdot v = \chi(z)v.$$

As an immediate consequence of Theorem 3.4, one gets the following corollary. Recall that this corollary was conjectured by Vogan in the case R is a maximal compact subgroup of a real semisimple G and proved in this case by Huang and Pandžić [HP], and proved for general reductive pairs by Kostant [Ko2].

Corollary 3.6. *Let (G, R) be a quadratic pair with G and R connected and let V be a $U(\mathfrak{g})$-module with central character χ. Then, for any $v \in H_D(\mathfrak{g}, \mathfrak{r}; V)$ and $z \in Z(\mathfrak{g})$,*

$$\chi(z)v = \eta_R(z)v.$$

Of course, the homomorphism $\eta_R : Z(\mathfrak{g}) \to Z(\mathfrak{r})$ is completely determined from the commutative diagram (D) *of Theorem* 3.4.

Loosely speaking, the corollary asserts that the central character of any irreducible $\mathfrak{r}$-submodule of $H_D(\mathfrak{g}, \mathfrak{r}; V)$ (*if nonzero*) *determines the central character of V.*

Proof. Since G and R are connected by assumption, $Z(\mathfrak{g}) = Z(G)$ and $Z(\mathfrak{r}) = Z(R)$. By (3.4.1),

$$z \otimes 1 - \xi\big(\eta_R(z)\big) = D^{\mathfrak{p}} a_z + a_z D^{\mathfrak{p}},$$

for some $a_z \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^R$. Thus, for any $v_o \in \operatorname{Ker} D_V^{\mathfrak{p}}$,

$$(z \otimes 1)v_o - \eta_R(z)v_o \in \operatorname{Im} D_V^{\mathfrak{p}} \cap \operatorname{Ker} D_V^{\mathfrak{p}},$$

since $\eta_R(z)v_o \in \operatorname{Ker} D_V^{\mathfrak{p}}$. Thus, $\chi(z)v = \eta_R(z)v$ in $H_D(\mathfrak{g}, \mathfrak{r}; V)$. □

Remark 3.7. After an earlier version of this paper was distributed, E. Meinrenken informed me that he and Alekseev have obtained some results which overlaps with our work (cf. [AM1]). In particular, they also have obtained Theorems 3.3 and 3.4.

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