Proceedings of the International Conference on
Algebraic Groups and Arithmetic, December 2001,
TIFR, Mumbai

A New Realization of the Cohomology of Springer Fibers

Shrawan Kumar and Jesper Funch Thomsen

Dedicated to Professor M.S. Raghunathan

1 Introduction

Fix a positive integer n and consider the algebraic group $G = SL_n(\mathbb{C})$ with its Lie algebra $sl_n(\mathbb{C})$. For any partition σ of n, let $X_\sigma \subset G/B$ be the associated Springer fiber, where B is the standard Borel subgroup consisting of the upper triangular matrices. By the pioneering work of Springer, its cohomology $H^*(X_\sigma)$ with complex coefficients admits an action of the Weyl group S_n. Subsequently, the S_n-algebra $H^*(X_\sigma)$ played a fundamental role in several diverse problems. The aim of this short note is to give a geometric realization of $H^*(X_\sigma)$. More specifically, we prove the following theorem which is the main result of this note.

Theorem *The coordinate ring* $\mathbb{C}[(N_G(T)\cdot\mathcal{N}_{\sigma^\vee})\cap\mathfrak{h}]$ *of the scheme theoretic intersection of* $N_G(T)\cdot\mathcal{N}_{\sigma^\vee}$ *with the Cartan subalgebra* $\mathfrak{h}$ *in* $sl_n(\mathbb{C})$ *is isomorphic to* $\mathrm{H}^*(X_\sigma)$ *as a graded* S_n*-algebra, where* $\sigma^\vee$ *is the dual partition of* σ, T *is the maximal torus consisting of the diagonal matrices with* $\mathfrak{h} :=$ *Lie* T, $N_G(T)$ *is its normalizer in* G, *and* $\mathcal{N}_{\sigma^\vee}$ *is the full nilpotent cone of the Levi component of the parabolic subalgebra of* $sl_n(\mathbb{C})$ *associated to the partition* $\sigma^\vee$.

This theorem should be contrasted with the following theorem of de Concini-Procesi. (A simpler proof of this theorem of de Concini-Procesi was given by Tanisaki [11].)

Theorem ([3], Theorem 4.3) *The cohomology algebra* $\mathrm{H}^*(X_\sigma)$ *is isomorphic, as a graded* S_n*-algebra, with the coordinate ring* $\mathbb{C}[\overline{G\cdot M_{\sigma^\vee}}\cap\mathfrak{h}]$ *of the scheme theoretic intersection of* $\mathfrak{h}$ *with the closure of the* G*-orbit of* $M_{\sigma^\vee}$, *where* $M_{\sigma^\vee}$ *is a nilpotent matrix associated to the partition* $\sigma^\vee$.

The proof of our theorem is based on a certain characterization of the S_n-algebra $\mathrm{H}^*(X_\sigma)$ given in Proposition 3.2, which seems to be of independent interest. The proof of this proposition is based on some works of Bergeron-Garsia, Garsia-Haiman and Garsia-Procesi revolving around the so called $n!$ conjecture.

Finally, it should be mentioned that the direct analogue of our theorem (and also the above theorem of de Concini-Procesi) for other groups does not hold in general. However a partial generalization of the result of de Concini-Procesi is obtained by Carrell [2].

2 Notation and Preliminaries

Fix a positive integer n and consider the algebraic group $G = SL_n(\mathbb{C})$. By $\mathcal{N}$ we denote the full nilpotent cone inside the Lie algebra $sl_n(\mathbb{C})$ of G. The group G acts on $\mathcal{N}$ by the adjoint action with finitely many orbits. An orbit is determined uniquely by the sizes of the Jordan blocks of any element in the orbit, and this sets up a one to one correspondence between the partitions of n and the G-conjugacy classes inside $\mathcal{N}$. For each partition $\sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0$ of n, we let M_σ denote the nilpotent matrix in the Jordan normal form with blocks of sizes $\sigma_0, \sigma_1, \ldots, \sigma_m$ along the diagonal in the stated order and starting from the upper left corner.

Let B denote the Borel subgroup of G consisting of the upper triangular matrices and let T denote the group of diagonal matrices in G. The Lie algebras of B and T will be denoted by $\mathfrak{b}$ and $\mathfrak{h}$ respectively. For any partition σ of n we let X_σ denote the closed subset (called the *Springer fiber*)

$$X_\sigma := \{gB \in G/B : \mathrm{Ad}(g^{-1})M_\sigma \in \mathfrak{b}\}$$

of G/B. This can also be identified with the set of Borel subalgebras of $sl_n(\mathbb{C})$ containing M_σ or with certain fibers of the Springer resolution of the nilpotent cone.

The singular cohomology ring $\mathrm{H}^*(X_\sigma) = \mathrm{H}^*(X_\sigma, \mathbb{C})$ with complex coefficients has an action of the symmetric group S_n on n-letters, the well known Springer representation. It is known that $\mathrm{H}^*(X_\sigma, \mathbb{C})$ is zero in odd degrees, so in the following we will consider it as a (commutative) graded algebra under rescaled grading by assigning degree i to the elements of degree $2i$. By the Springer correspondence, the top degree part $\mathrm{H}^{d_\sigma}(X_\sigma)$ is an irreducible S_n-module.

For the partition $\mu : 1 \geq 1 \geq \cdots \geq 1$ of n, the variety X_μ coincides with G/B. Thus, in this case, one may S_n-equivariantly identify $\mathrm{H}^*(X_\mu)$ with the coinvariant ring $\mathbb{C}[Z_1, \ldots, Z_n]/I$, where I is the ideal generated by the elementary symmetric functions in the variables $Z_1, \ldots, Z_n$. For a general partition σ of n, the natural map

$$\mathrm{H}^*(G/B) \to \mathrm{H}^*(X_\sigma)$$

is a surjective S_n-equivariant map [10, Corollary 2.3]. This also follows from the result of de Concini-Procesi mentioned in the introduction.

2.1 The algebra A_σ

For any partition $\sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0$ of n, let D_σ be the set of pairs of nonnegative integers (i,j) satisfying $i < \sigma_j$. Then D_σ consists of n elements and we fix an ordering $\{(i_s, j_s)\}_{s=1,2,\ldots,n}$ of these. Define the polynomial

$$\Delta_\sigma = \det[X_s^{i_t} Y_s^{j_t}]_{1\leq s,t\leq n} \in R_n,$$

where R_n is the polynomial ring $\mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$. Observe that, up to a sign, Δ_σ does not depend on the choice of the ordering of the elements in D_σ.

The group S_n acts on R_n by acting in the natural way on the two sets of variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ diagonally. We think of R_n as a bigraded S_n-module by counting the degrees in the two sets of variables separately. Define a bigraded S_n-equivariant ideal in R_n by :

$$K_\sigma = \left\{ f \in R_n : f\left(\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}, \frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_n}\right)\Delta_\sigma = 0 \right\},$$

where $\frac{\partial}{\partial X_i}$ and $\frac{\partial}{\partial Y_j}$ are the usual differential operators on R_n.

Define now a bigraded S_n-algebra $A_\sigma := R_n/K_\sigma$. This algebra was introduced by Garsia-Haiman who conjectured that A_σ has dimension $n!$ for any partition σ of n [5]. This conjecture, which was called the *$n!$-conjecture*, is now proved by Haiman [7].

Let d_1 (resp. d_2) be the X-degree (resp. Y-degree) of Δ_σ. Then, it is easy to see that, the bigraded component $A_\sigma^{(d_1,d_2)}$ is one dimensional and, moreover, if $A_\sigma^{(e_1,e_2)} \neq 0$, then $e_1 \leq d_1, e_2 \leq d_2$. Clearly,

$$d_1 = \sum_{(i,j)\in D_\sigma} i, \qquad d_2 = \sum_{(i,j)\in D_\sigma} j = \sum_{s=0}^{\ell} \binom{\sigma_s^\vee}{2}, \tag{2.1}$$

where $\sigma^\vee : \sigma_0^\vee \geq \sigma_1^\vee \geq \cdots \geq \sigma_\ell^\vee > 0$ is the dual partition.

By definition, it follows easily that A_σ is Gorenstein (see, e.g., [4, Exercise 21.7]), and hence that A_σ has a unique minimal nonzero ideal $A_\sigma^{(d_1,d_2)}$.

As explained in [5, Section 3.1], the following theorem follows from the results in [1] and [6].

Theorem 2.1 *The subalgebra of A_σ generated by the images of $X_1, \ldots, X_n$ is S_n-equivariantly isomorphic to $\mathrm{H}^*(X_{\sigma^\vee})$. Similarly, the subalgebra of A_σ generated by the images of $Y_1, \ldots, Y_n$ is S_n-equivariantly isomorphic to $\mathrm{H}^*(X_\sigma)$. If we assign degree 1 to all the elements $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, then both of these isomorphisms are graded algebra isomorphisms.*

3 A Geometric Realization of $\mathrm{H}^*(X_\sigma)$

In this section we give a new geometric realization of $\mathrm{H}^*(X_\sigma)$. Recall that the socle of a ring is defined to be the sum of all its minimal nonzero ideals. Then, with the notation from the previous section, we have the following:

Lemma 3.1 *The top degree d_σ of $\mathrm{H}^*(X_\sigma)$ is equal to d_2. Moreover, the socle of $\mathrm{H}^*(X_\sigma)$ coincides with the top degree part $\mathrm{H}^{d_2}(X_\sigma)$. In particular, the socle of $\mathrm{H}^*(X_\sigma)$ is a graded ideal.*

Proof Let z denote any nonzero homogeneous element in $\mathrm{H}^*(X_\sigma)$. By Theorem 2.1 we may regard z as the image in A_σ of a homogeneous polynomial f in the variables $Y_1, \ldots, Y_n$. As A_σ is Gorenstein, we can find a monomial

$$g = X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_n^{\beta_n}$$

such that the image of $f \cdot g$ in A_σ is nonzero and has the maximal degree, i.e., has bidegree (d_1, d_2). But then the image of $f' := fY_1^{\beta_1} \cdots Y_n^{\beta_n}$ in A_σ is nonzero and, of course, lies in the subalgebra generated by the images of $Y_1, \ldots, Y_n$. In particular, by Theorem 2.1, the image of f' corresponds to a nonzero element in $\mathrm{H}^{d_2}(X_\sigma)$ which equals the product $z \cdot z'$ for some z' in $\mathrm{H}^*(X_\sigma)$.

This proves that d_2 equals the top degree of $\mathrm{H}^*(X_\sigma)$ and that any nonzero element of $\mathrm{H}^*(X_\sigma)$ can be multiplied by an element of $\mathrm{H}^*(X_\sigma)$ to produce a nonzero element in the top degree. This immediately implies the desired result.

□

The above lemma provides us with the following characterization of the algebra $\mathrm{H}^*(X_\sigma)$.

Proposition 3.2 *Let K be a graded algebra with an action of S_n such that there exists a surjective S_n-equivariant graded algebra homomorphism $\phi : \mathrm{H}^*(X_\sigma) \to K$. Assume further that the top degree of K is d_σ. Then ϕ is an isomorphism.*

Proof Assume that ϕ is not injective. Then $\ker(\phi)$ will meet the socle of $\mathrm{H}^*(X_\sigma)$ nontrivially. Thus, by Lemma 3.1, the degree d_σ part $\ker^{d_\sigma}(\phi)$ of $\ker(\phi)$ is nonzero. But $\ker^{d_\sigma}(\phi)$ is a submodule of the irreducible S_n-module $\mathrm{H}^{d_\sigma}(X_\sigma)$ and hence $\ker^{d_\sigma}(\phi)$ coincides with $\mathrm{H}^{d_\sigma}(X_\sigma)$. This is a contradiction, since the top degree of K is equal to d_σ by assumption.

□

Let $Z_T(M_\sigma)$ denote the centralizer of M_σ in T, and let L_σ denote the centralizer in G of the group $Z_T(M_\sigma)$. In other words, L_σ is the set of block diagonal matrices in G with blocks of sizes $\sigma_0, \sigma_1, \ldots, \sigma_m$. Then L_σ is a reductive group with a Borel subgroup $B_\sigma = B \cap L_\sigma$. Moreover, M_σ is a regular nilpotent element in the Lie algebra of L_σ and consequently the full nilpotent cone $\mathcal{N}_\sigma$ of the Lie algebra of L_σ coincides with the closure $\overline{L_\sigma M_\sigma}$ of the L_σ-orbit of M_σ under the adjoint action. In the following, $N_G(T)$ will denote the normalizer of T in G.

Lemma 3.3 *The coordinate ring* $\mathbb{C}[(N_G(T)\cdot\mathcal{N}_\sigma)\cap\mathfrak{h}]$ *of the scheme theoretic intersection of the* $N_G(T)$*-orbit of* $\mathcal{N}_\sigma$ *with* $\mathfrak{h}$ *is a graded* S_n*-algebra.*

Proof The multiplication action of $\mathbb{C}^*$ on the Lie algebra $sl_n(\mathbb{C})$ keeps $N_G(T)\cdot\mathcal{N}_\sigma$ and $\mathfrak{h}$ stable. This defines the desired grading on $\mathbb{C}[(N_G(T)\cdot\mathcal{N}_\sigma)\cap\mathfrak{h}]$. The natural actions of $N_G(T)$ on $N_G(T)\cdot\mathcal{N}_\sigma$ and $\mathfrak{h}$ define an action of $N_G(T)$ on the coordinate ring $\mathbb{C}[(N_G(T)\cdot\mathcal{N}_\sigma)\cap\mathfrak{h}]$. As T acts trivially on $\mathfrak{h}$, this gives the desired S_n-algebra structure by identifying S_n with the Weyl group $N_G(T)/T$.

□

Now we come to the main result of this note.

Theorem 3.4 *The algebra* $\mathbb{C}[(N_G(T)\cdot\mathcal{N}_{\sigma^\vee})\cap\mathfrak{h}]$ *is isomorphic to* $\mathrm{H}^*(X_\sigma)$ *as a graded* S_n*-algebra.*

Proof By the result of de Concini-Procesi mentioned in the introduction, we may identify $\mathrm{H}^*(X_\sigma)$ as a graded S_n-algebra with the coordinate ring $\mathbb{C}[\overline{G\cdot M_{\sigma^\vee}}\cap\mathfrak{h}]$ of the scheme theoretic intersection of $\mathfrak{h}$ with the closure of the G-orbit of $M_{\sigma^\vee}$. The graded S_n-algebra structure on the latter algebra is defined similarly to the graded S_n-algebra structure on $\mathbb{C}[(N_G(T)\cdot\mathcal{N}_\sigma)\cap\mathfrak{h}]$ (as defined in the proof of Lemma 3.3). As $N_G(T)\cdot\mathcal{N}_{\sigma^\vee} = N_G(T)\cdot(\overline{L_{\sigma^\vee}\cdot M_{\sigma^\vee}})$ is a closed subscheme of $\overline{G\cdot M_{\sigma^\vee}}$, we have a surjective morphism of graded S_n-algebras:

$$\mathbb{C}[\overline{G\cdot M_{\sigma^\vee}}\cap\mathfrak{h}]\to\mathbb{C}[(N_G(T)\cdot\mathcal{N}_{\sigma^\vee})\cap\mathfrak{h}].$$

Thus, we get a surjective map of graded S_n-algebras:

$$\phi : \mathrm{H}^*(X_\sigma) \to \mathbb{C}[(N_G(T) \cdot \mathcal{N}_{\sigma^\vee}) \cap \mathfrak{h}].$$

To prove the theorem, in view of Proposition 3.2, it suffices to show that the top degree d of the graded algebra $\mathbb{C}[(N_G(T) \cdot \mathcal{N}_{\sigma^\vee}) \cap \mathfrak{h}]$ is d_σ. As ϕ is surjective, we already know that $d \leq d_\sigma$. For the other inequality, consider the graded surjective map

$$\mathbb{C}[(N_G(T) \cdot \mathcal{N}_{\sigma^\vee}) \cap \mathfrak{h}] \to \mathbb{C}[\mathcal{N}_{\sigma^\vee} \cap \mathfrak{h}],$$

obtained by the $\mathbb{C}^*$-equivariant embedding $\mathcal{N}_{\sigma^\vee} \subset N_G(T) \cdot \mathcal{N}_{\sigma^\vee}$. By a classical result by Kostant [8], $\mathbb{C}[\mathcal{N}_{\sigma^\vee} \cap \mathfrak{h}]$ is isomorphic with the cohomology $\mathrm{H}^*(L_{\sigma^\vee}/B_{\sigma^\vee}, \mathbb{C})$ as graded algebras (in fact, also as modules for the Weyl group $N_{L_{\sigma^\vee}}(T)/T$). But the top degree of $\mathrm{H}^*(L_{\sigma^\vee}/B_{\sigma^\vee}, \mathbb{C})$ coincides with the complex dimension of $L_{\sigma^\vee}/B_{\sigma^\vee}$, which is easily seen to be equal to d_2 (use formula (2.1)). All together, this implies that $d \geq d_2$. But, by Lemma 3.1, $d_2 = d_\sigma$, which ends the proof.

□

References

[1] N. Bergeron, and A. Garsia, *On certain spaces of harmonic polynomials*, in *Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications*, pp. 51–86, Contemp. Math. **138**, American Mathematical Society, 1992.

[2] J. Carrell, *Orbits of the Weyl group and a theorem of de Concini and Procesi*, Compos. Math. **60** (1986), 45–52.

[3] C. de Concini, and C. Procesi, *Symmetric functions, conjugacy classes and the flag variety*, Invent. Math. **64** (1981), 203–219.

[4] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer, 1995.

[5] A. Garsia, and M. Haiman, *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. USA **90** (1993), 3607–3610.

[6] A. Garsia, and C. Procesi, *On certain graded S_n-modules and the q-Kostka polynomials*, Adv. Math. **94** (1992), 82–138.

[7] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14** (2001), 941–1006.

[8] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[9] N. Spaltenstein, *The fixed point set of a unipotent transformation on the flag manifold*, Nederl. Akad. Wetensch. Proc., Ser. A, **79** (1976), 452–456.

[10] R. Hotta, and T. Springer, *A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups*, Invent. Math. **41** (1977), 113–127.

[11] T. Tanisaki, *Defining ideals of the closures of the conjugacy classes and representations of the Weyl groups*, Tôhoku Math. J. **34** (1982), 575–585.

SHRAWAN KUMAR, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA, CHAPEL HILL, NC 27599-3250, U.S.A.
E-mail: `shrawan@email.unc.edu`

JESPER FUNCH THOMSEN, INSTITUT FOR MATEMATISK FAG, AARHUS UNIVERSITET, NY MUNKEGADE, DK-8000 ÅRHUS C, DENMARK
E-mail: `funch@imf.au.dk`