A New Realization of the Cohomology of Springer Fibers

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Dedicated to Professor M.S. Raghunathan

1 Introduction

Fix a positive integer $n$ and consider the algebraic group $G = SL_n(C)$ with its Lie algebra $sl_n(C)$. For any partition $\sigma$ of $n$, let $X_\sigma \subset G/B$ be the associated Springer fiber, where $B$ is the standard Borel subgroup consisting of the upper triangular matrices. By the pioneering work of Springer, its cohomology $H^*(X_\sigma)$ with complex coefficients admits an action of the Weyl group $S_n$. Subsequently, the $S_n$-algebra $H^*(X_\sigma)$ played a fundamental role in several diverse problems. The aim of this short note is to give a geometric realization of $H^*(X_\sigma)$. More specifically, we prove the following theorem which is the main result of this note.

**Theorem** The coordinate ring $C[[N_G(T) \cdot N_{G'}(T) \cap h]]$ of the scheme theoretic intersection of $N_G(T) \cdot N_{G'}(T)$ with the Cartan subalgebra $h = h_0(C)$ is isomorphic to $H^*(X_\sigma)$ as a graded $S_n$-algebra, where $\sigma$ is the dual partition of $\sigma$. $T$ is the maximal torus consisting of the diagonal matrices with $h = Lie T$, $N_G(T)$ is its normalizer in $G$, and $N_{G'}(T)$ is the full nilpotent cone of the Levi component of the parabolic subalgebra of $sl_n(C)$ associated to the partition $\sigma'$.

This theorem should be contrasted with the following theorem of de Concini-Procesi. (A simpler proof of this theorem of de Concini-Procesi was given by Tanisaki [11].)

**Theorem** ([8], Theorem 4.3) The cohomology algebra $H^*(X_\sigma)$ is isomorphic, as a graded $S_n$-algebra, with the coordinate ring $C[[G/T \cdot M_{\sigma'} \cap h]]$ of the scheme theoretic intersection of $h$ with the closure of the $G$-orbit of $M_{\sigma'}$, where $M_{\sigma'}$ is a nilpotent matrix associated to the partition $\sigma'$. 
The proof of our theorem is based on a certain characterization of the $S_n$-algebras $H^n(X_n)$ given in Proposition 3.2, which seems to be of independent interest. The proof of this proposition is based on some works of Bercovici-Garabedian, Garabedian-Haiman and Garabedian-Procesi reviving around the so called n conjecture.

Finally, it should be mentioned that the direct analogue of our theorem [and also the above theorem of de Concini-Procesi] for other groups does not hold in general. However a partial generalization of the result of de Concini-Procesi is obtained by Carroll [2].

2 Notation and Preliminaries

Fix a positive integer n and consider the algebraic group $G = SL_n(C)$. By $N$ we denote the full nilpotent cone inside the Lie algebra $sl_n(C)$ of $G$. The group $G$ acts on $N$ by the adjoint action with finitely many orbits. An orbit is determined uniquely by the sizes of the Jordan blocks of any element in the orbit, and this acts one to one correspondence between the partitions of n and the $G$-conjugacy classes inside $N$. For such partition $\sigma : n_0 \geq n_1 \geq \cdots \geq n_r > 0$ of $n$, we let $M_{\sigma}$ denote the nilpotent matrix in the Jordan normal form with blocks of sizes $n_0, n_1, \ldots, n_r$ along the diagonal in the stated order and starting from the upper left corner.

Let $B$ denote the Borel subgroup of $G$ consisting of the upper triangular matrices and let $T$ denote the group of diagonal matrices in $G$. The Lie algebras of $B$ and $T$ will be denoted by $b$ and $t$, respectively. For any partition $\sigma$ of $n$ we let $X_{\sigma}$ denote the closed subset (called the Springer fiber)

$$X_{\sigma} := \{ b \in G / B : Ad(x^{-1})M_{\sigma} \in b \}$$

of $G / B$. This can also be identified with the set of Borel subalgebra of $sl_n(C)$ containing $M_{\sigma}$ or with certain fibers of the Springer resolution of the nilpotent cone.

The singular cohomology ring $H^*(X_{\sigma}) = H^*(X_n, C)$ with complex coefficients has an action of the symmetric group $S_n$ on $n$-letters, the usual symmetric group which acts by permuting the elements of degree 1. By the Springer correspondence, the top degree part $H^*(X_n)$ is an irreducible $S_n$-module.

For the partition $\mu : 1 \geq \cdots \geq 1$ of $n$, the variety $X_{\mu}$ coincides with $G / B$ itself. In this case, one may $S_n$-equivariantly identify $H^*(X_{\mu})$ with the constant ring $C[z_1, \ldots, z_n] / I$, where $I$ is the ideal generated by the elementary symmetric functions in the variables $z_1, \ldots, z_n$ for a general
partition $\sigma$ of $n$, the natural map

$$H^*(G/B) \to H^*(X_\sigma)$$

is a surjective $S_\sigma$-equivariant map [10, Corollary 2.3]. This also follows from the result of de Concini-Procesi mentioned in the introduction.

2.1 The algebra $A_\sigma$

For any partition $\sigma : \epsilon_0 \geq \epsilon_1 \geq \cdots \geq \epsilon_n > 0$ of $n$, let $D_\sigma$ be the set of pairs of nonnegative integers $(i,j)$ satisfying $i < \epsilon_j$. Then $D_\sigma$ consists of $n$ elements and we fix an ordering $((i_j,j_j))_{j=1,2,\ldots,n}$ of these. Define the polynomial

$$\Delta_\sigma = \det_{i=0}^{n-1} \left[ X_i^j Y_i^j \right],$$

where $R_n$ is the polynomial ring $C[X_1,\ldots,X_n,Y_1,\ldots,Y_n]$. Observe that, up to a sign, $\Delta_\sigma$ does not depend on the choice of the ordering of the elements in $D_\sigma$.

The group $S_\sigma$ acts on $R_n$ by acting in the natural way on the two sets of variables $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ diagonally. We think of $R_n$ as a bigraded $S_\sigma$-module by counting the degrees in the two sets of variables separately. Define a bigraded $S_\sigma$-equivariant ideal in $R_n$ by:

$$K_\sigma = \left\{ f \in R_n : f \left( \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}, \frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_n} \right) \Delta_\sigma = 0 \right\},$$

where $\frac{\partial}{\partial X_i}$ and $\frac{\partial}{\partial Y_j}$ are the usual differential operators on $R_n$.

Define now a bigraded $S_\sigma$-algebra $A_\sigma = R_n/K_\sigma$. This algebra was introduced by Garsia-Haiman who conjectured that $A_\sigma$ has dimension $n!$ for any partition $\sigma$ of $n$ [5]. This conjecture, which was called the conjecture, is now proved by Haiman [7].

Let $d_1$ (resp. $d_2$) be the $X$-degree (resp. $Y$-degree) of $\Delta_\sigma$. Then, it is easy to see that, the bigraded component $A_\sigma^{d_1,d_2}$ is $n$-dimensional and, moreover, if $A_\sigma^{d_1,d_2} \neq 0$, then $e_1 \leq d_1, e_2 \leq d_2$. Clearly,

$$d_2 = \sum_{(i,j) \in D_\sigma} i_j, \quad d_1 = \sum_{(i,j) \in D_\sigma} j + \sum_{i=0}^{n} \sum_{j=1}^{e_i} \frac{\sigma_j^*}{2}, \quad (2.1)$$

where $\sigma^* : \epsilon_0^* \geq \epsilon_1^* \geq \cdots \geq \epsilon_n^* > \epsilon$ is the dual partition.

By definition, it follows easily that $A_\sigma$ is Gorenstein (see, e.g., [6, Exercise 21.7]), and hence that $A_\sigma$ has a unique minimal nonzero ideal $A_\sigma^{d_1,d_2}$.

As explained in [5, Section 3.1], the following theorem follows from the results in [5] and [6].
Theorem 2.1. The subalgebra of $A_\mu$ generated by the images of $X_1, \ldots, X_n$ is $S_n$-equivariantly isomorphic to $H^*(X_\mu)$. Similarly, the subalgebra of $A_\nu$ generated by the images of $Y_1, \ldots, Y_n$ is $S_n$-equivariantly isomorphic to $H^*(X_\nu)$. If we assign degree 1 to all the elements $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, then both of these isomorphisms are graded algebra isomorphisms.

3 A Geometric Realization of $H^*(X_\mu)$

In this section we give a new geometric realization of $H^*(X_\mu)$. Recall that the socle of a ring is defined to be the sum of all its minimal nonzero ideals. Then, with the notation from the previous section, we have the following:

Lemma 3.1. The top degree $d_\mu$ of $H^*(X_\mu)$ is equal to $d_\mu$. Moreover, the socle of $H^*(X_\mu)$ coincides with the top degree part $H^{\leq d_\mu}(X_\mu)$. In particular, the socle of $H^*(X_\mu)$ is a graded ideal.

Proof. Let $z$ denote any nonzero homogeneous element in $H^*(X_\mu)$. By Theorem 2.1 we may regard $z$ as the image in $A_\mu$ of a homogeneous polynomial $f$ in the variables $Y_1, \ldots, Y_n$. As $A_\mu$ is Gorenstein, we can find a monomial

$$g = X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_n^{b_n}$$

such that the image of $f \cdot g$ in $A_\mu$ is nonzero and has the maximal degree, i.e., has bidegree $(d_\mu, d_\mu)$. But then the image of $f' := f Y_1^{b_1} \cdots Y_n^{b_n}$ in $A_\mu$ is nonzero and, of course, lies in the subalgebra generated by the images of $Y_1, \ldots, Y_n$. In particular, by Theorem 2.1, the image of $f'$ corresponds to a nonzero element in $H^{d_\mu}(X_\mu)$ which equals the product $z \cdot z'$ for some $z'$ in $H^*(X_\mu)$.

This proves that $d_\mu$ equals the top degree of $H^*(X_\mu)$ and that any nonzero element of $H^*(X_\mu)$ can be multiplied by an element of $H^{d_\mu}(X_\mu)$ to produce a nonzero element in the top degree. This immediately implies the desired result.

The above lemma provides us with the following characterization of the algebra $H^*(X_\mu)$.

Proposition 3.2. Let $K$ be a graded algebra with an action of $S_n$ such that there exists a surjective $S_n$-equivariant graded algebra homomorphism $\phi : H^*(X_\mu) \to K$. Assume further that the top degree of $K$ is $d_\mu$. Then $\phi$ is an isomorphism.
Proof. Assume that \( \phi \) is not injective. Then \( \ker(\phi) \) will meet the socle of \( H^*(X_\nu) \) nontrivially. Thus, by Lemma 3.1, the degree \( d_\nu \) part \( \ker^{d_\nu}(\phi) \) of \( \ker(\phi) \) is nonzero. But \( \ker^{d_\nu}(\phi) \) is a submodule of the irreducible \( S_\nu \)-module \( H^{d_\nu}(X_\nu) \) and hence \( \ker^{d_\nu}(\phi) \) coincides with \( H^{d_\nu}(X_\nu) \). This is a contradiction, since the top degree of \( K \) is equal to \( d_\nu \) by assumption.

Let \( Z^r(M_\nu) \) denote the centralizer of \( M_\nu \) in \( T \), and let \( L_\nu \) denote the centralizer in \( G \) of the group \( Z^r(M_\nu) \). In other words, \( L_\nu \) is the set of block diagonal matrices in \( G \) with blocks of sizes \( m_0, m_1, \ldots, m_m \). Then \( L_\nu \) is a reductive group with a Borel subgroup \( P_\nu = B \cap L_\nu \). Moreover, \( M_\nu \) is a regular nilpotent element in the Lie algebra of \( L_\nu \) and consequently the full nilpotent cone \( N_\nu \) of the Lie algebra of \( L_\nu \) coincides with the closure \( L_\nu \cdot M_\nu \) of \( M_\nu \) under the adjoint action. In the following, \( N_\nu(G(T)) \) will denote the normalizer of \( T \) in \( G \).

Lemma 3.3 The coordinate ring \( \mathcal{O}(N_\nu(G(T)), N_\nu(G(T)) \cdot h) \) of the scheme theoretic intersection of the \( N_\nu(G(T)) \) orbit of \( N_\nu \) with \( h \) is a graded \( S_\nu \)-algebra.

Proof. The multiplication action of \( C^\ast \) on the Lie algebra \( \mathfrak{sl}_2(C) \) keeps \( N_\nu(G(T)) \cdot M_\nu \) and \( h \) stable. This defines the desired grading on \( \mathcal{O}(N_\nu(G(T)), N_\nu(G(T)) \cdot h) \). The natural actions of \( N_\nu(G(T)) \) on \( N_\nu(G(T)) \cdot N_\nu \) and \( h \) define an action of \( N_\nu(G(T)) \) on the coordinate ring \( \mathcal{O}(N_\nu(G(T)), N_\nu(G(T)) \cdot h) \). As \( T \) acts trivially on \( h \), this gives the desired \( S_\nu \)-algebra structure by identifying \( S_\nu \) with the Weyl group \( N_\nu(G(T))/T \).

Now we come to the main result of this note.

Theorem 3.4 The algebra \( \mathcal{O}(N_\nu(G(T)), N_\nu(G(T)) \cdot h) \) is isomorphic to \( H^*(X_\nu) \) as a graded \( S_\nu \)-algebra.

Proof. The result of de Concini-Procesi mentioned in the introduction, we may identify \( H^*(X_\nu) \) as a graded \( S_\nu \)-algebra with the coordinate ring \( \mathcal{O}(G \cdot M_\nu \cap h) \) of the scheme theoretic intersection of \( h \) with the closure of the \( G \)-orbit of \( M_\nu \). The graded \( S_\nu \)-algebra structure on the latter algebra is defined similarly to the graded \( S_\nu \)-algebra structure on \( \mathcal{O}(N_\nu(G(T)), N_\nu(G(T)) \cdot h) \) as defined in the proof of Lemma 3.3. As \( N_\nu(G(T)) \cdot M_\nu = N_\nu(G(T)) \cdot (L_\nu \cdot M_\nu) \) is a closed subscheme of \( G \cdot M_\nu \), we have a surjective morphism of graded \( S_\nu \)-algebras:

\[ \mathcal{O}(G \cdot M_\nu \cap h) \rightarrow \mathcal{O}(N_\nu(G(T)), N_\nu(G(T)) \cdot h). \]
Thus, we get a surjective map of graded $S_d$-algebras:

$$\phi : H^*(X_d) \to \mathbb{C}(N_0(T) \cdot N_0^*) \cap b.$$ 

To prove the theorem, in view of Proposition 3.2, it suffices to show that the top degree $d$ of the graded algebra $\mathbb{C}(N_0(T) \cdot N_0^*) \cap b$ is $d_0$. As $\phi$ is surjective, we already know that $d \leq d_0$. For the other inequality, consider the graded surjective map

$$\mathbb{C}(N_0(T) \cdot N_0^*) \cap b \to \mathbb{C}(N_0(T) \cdot N_0^*) \cap b,$$

obtained by the $C^*$-equivariant embedding $N_0^* \subset N_0(T) \cdot N_0^*$. By a classical result by Kostant [8], $\mathbb{C}(N_0(T) \cdot N_0^*)$ is isomorphic with the cohomology $H^*(L_0^*/B_0^*, C)$ as graded algebras (in fact, also as modules for the Weyl group $N_0(T)/T$). But the top degree of $H^*(L_0^*/B_0^*, C)$ coincides with the complex dimension of $L_0^*/B_0^*$, which is easily seen to be equal to $d_2$ (use formula (2.1)). All together, this implies that $d \geq d_0$. But, by Lemma 3.1, $d_2 = d_0$, which ends the proof.

References


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