The generalized triangle inequalities for rank 3 symmetric spaces of noncompact type.

Shrawan Kumar, Bernhard Leeb, and John Millson

Dedicated to Robert Osserman on his sixty-seventh birthday.

Abstract. We compute the generalized triangle inequalities explicitly for all rank 3 symmetric spaces of noncompact type. For \(SL(4,C)\) there are 64 inequalities none of them redundant by [Ko]. For both \(Sp(6,C)\) and \(Sp(7,C)\) there are 135 inequalities of which 24 are trivially redundant in the sense that they follow from the inequalities defining the Weyl chamber \(\Delta\). There are 2 more redundant inequalities for each of these two groups. One interesting feature is that these inequalities do not occur for the other systems (and consequently must be redundant because the two polyhedral cones are the same by Theorem 1.6). The two equal polyhedral cones \(D_g(B) = D_h(C)\) have precisely 232 faces and 51 generator-vectors.

1. Introduction

Let \(G\) be a connected semisimple real Lie group with no compact factors and finite center and Lie algebra \(g\). \(K\) be a maximal compact subgroup and \(X = G/K\) be the associated symmetric space. By a symmetric space of noncompact type we will mean a symmetric space \(G/K\) where \(G\) is as above. We let \(\mathfrak{g}\) denote the Lie algebra of \(K\) and let \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\) be the Cartan decomposition. Let \(\mathfrak{a}\) be a Cartan subspace in \(\mathfrak{p}\) (i.e. a maximal subalgebra in \(\mathfrak{p}\) which is necessarily abelian). Let \(\Delta\) be the real points of the split torus in \(G\) corresponding to \(\mathfrak{a}\). Choose an ordering of the restricted roots and let \(\Delta \subset \mathfrak{a}\) be the corresponding (closed) Weyl chamber. Let \(A_\Phi\) be the image of \(\Delta\) under the exponential map \(\exp : \mathfrak{g} \rightarrow G\). Let \(\alpha\) be the point in \(\mathfrak{X}\) that is fixed by \(K\). We will refer to \(\mathfrak{a}\) as the basepoint for \(\mathfrak{X}\). We will need the following theorem, the Cartan decomposition for the group \(G\), see [He], Theorem 1.1, pg. 402.

Theorem 1.1. We have

\[ G = K A_\Phi F. \]

Moreover, for any \(g \in G\), the intersection of the double coset \(K g K\) with \(A_\Phi\) consists of a single point to be denoted \(a(g)\).

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Suppose $\mathcal{E}_2$ is the oriented geodesic segment in $X$ joining the point $x_1$ to the point $x_2$. Then there exists a unique element $y \in G$ which sends $x_1$ to $a$ and $x_2$ to $y = \exp(\delta)$ where $\delta \in \Delta$. Note that the point $\delta$ is uniquely determined by $\mathcal{E}_2$.

We define a map $\sigma$ from $G$-orbits of oriented geodesic segments in $\Delta$ by

$$\sigma(\mathcal{E}_2) = \delta.$$ 

Clearly we have the following consequence of the Cartan decomposition.

**Lemma 1.2.** The map $\sigma$ gives rise to a one-to-one correspondence between $G$-orbits of oriented geodesic segments in $X$ and the points of $\Delta$.

**Remark 1.3.** In the real-rank 1 case $\sigma(\mathcal{E}_2)$ is just the length of the geodesic segment $\mathcal{E}_2$. In general we will call $\sigma(\mathcal{E}_2)$ the $\Delta$-length of $\mathcal{E}_2$ or the $\Delta$-distance between $x_1$ and $x_2$. We will write $d_\Delta(x_1, x_2) = \sigma(\mathcal{E}_2)$.

We note the formula

$$d_\Delta(x_1, x_2) = \log a(x_1^{-1} x_2)$$

where $x_1 = gK, x_2 = g_2K$.

**Remark 1.4.** The delta distance is symmetric in the sense that

$$d_\Delta(x_1, x_2) = -d_\Delta(x_2, x_1),$$

where $v_0$ is the longest element in the restricted Weyl group. However, the naive triangle inequality

$$d_\Delta(x_1, x_2) \leq d_\Delta(x_1, x_3) + d_\Delta(x_3, x_2)$$

does not hold [KLM2]. Here the order is the one defined by the cone $\Delta$. The naive triangle inequality has to be replaced by the inequalities below.

We have the fundamental problem of finding the generalized triangle inequalities for $X$, precisely we have

**Problem 1.5.** Give conditions on a triple $(v_1, v_2, v_3) \in \Delta^3$ that are necessary and sufficient in order that there exist a triangle in $X$ with vertices $x_1, x_2, x_3$ such that $d_\Delta(x_1, x_2) = v_1, d_\Delta(x_2, x_3) = v_2$ and $d_\Delta(x_3, x_1) = v_3$.

We now describe a system of linear inequalities on $\Delta^3$ which will give the required necessary and sufficient conditions. These inequalities will be called the generalized triangle inequalities.

**Remark 1.5.** We will include the inequalities defining $\Delta^3$ in $\mathbb{Z}^3$ in the generalized triangle inequalities.

We will need the following:

**Definition 1.7.** Suppose that $W$ and $W'$ are Coxeter groups acting on $V$ and $V'$ respectively by their natural reflection representations. We define a monomorphism of Coxeter systems to be a pair $(f, \phi)$ where $f : V \to V'$ is an isometric embedding and $\phi$ is a monomorphism $W \to W'$ satisfying $f(\alpha) = \phi(\alpha)f(\alpha)$.

First we reduce to the case in which $G$ is complex by the following:

**Theorem 1.8 ([KLM, KLM1]).** I. The set $D_2(X) \subset \Delta^3$ of triples $(v_1, v_2, v_3)$ for which a triangle in the Problem 1.3 (for $X$) exists is a polyhedral cone.

II. $D_2(X)$ depends only on the spherical Coxeter complex associated to $X$. More precisely, a monomorphism $(f, \phi)$ of the Coxeter system $(a, W)$ to the Coxeter system $(a', W')$ induces an affine embedding $D_2(X) \to D_2(X')$. In particular, if $f$ and $\phi$ are also surjective, then the map $D_2(X) \to D_2(X')$ is an affine isomorphism.
Thus given \( G \) as above we can replace \( G \) by any complex semisimple group of the same rank as \( G \) whose Weyl group coincides with the restricted Weyl group \( W \) of \( G \). Thus it suffices to find the generalized triangle inequalities for the case in which \( G \) is complex.

So from now on we will assume that \( G \) is a connected complex semisimple group. We will accordingly often rewrite \( D_\alpha(X) \) as \( D_\alpha(R) \) where \( R \) is the reduced root system associated to the restricted root system of \( X \).

The system of inequalities for \( D_\alpha(R) \) breaks up into rank 1 subsystems \( \mathfrak{p} \), where \( \mathfrak{p} \) is a standard maximal parabolic subgroup. The subsystem \( \mathfrak{p} \) is controlled by the Schubert calculus in the generalized Grassmannian \( G/P \) in the sense that there is one inequality \( \lambda_{\mathfrak{p}} \leq \lambda \) for each tuple of \( \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \in \mathfrak{p} \) such that

\[
X_{\mathfrak{p}_2}^\mathfrak{p} \cdot X_{\mathfrak{p}_1}^\mathfrak{p} \cdot X_{\mathfrak{p}_3}^\mathfrak{p} = [p]\]

in \( H_*(G/P) \), where \( W_P \) is the set of minimal length coset representatives for the set of cosets \( W/W_P \) (\( W_P \) being the Weyl group of \( P \)) \( \cdot \) is the intersection product and \( X_{\mathfrak{p}_1} \in W_P \) is the Schubert class in \( G/P \). To describe the inequality \( \lambda_{\mathfrak{p}} \leq \lambda \), let \( P \) be the standard maximal parabolic corresponding to a fundamental weight \( \lambda \). Then the action of \( W \) on the weight lattice of \( g \) induces a one-to-one correspondence \( f: W_P \to W \lambda \). Thus we may parameterize the Schubert classes in \( G/P \) by elements of \( \mathfrak{p} \). We let \( h_1 = f(\mathfrak{p}_1) = 1, 2, 3 \). We will sometimes denote the Schubert cycle \( X_{\mathfrak{p}_2}^\mathfrak{p} \) as \( X_{\mathfrak{p}_2}^{(h_1)} \). Then the inequality \( \lambda_{\mathfrak{p}} \leq \lambda \) is given by

\[
\lambda_1(v_1) + \lambda_2(v_2) + \lambda_3(v_3) \leq 0, (v_1, v_2, v_3) \in \Delta^3.
\]

Remark 1.9. The key point is that there is a basis of the algebra \( H_*(G/P) \) parameterized by certain linear functions on \( \mathfrak{a} \).

In order to give an accurate account of the history of work on the problem, we first need to describe the corresponding problem for the infinitesimal symmetric space \( \mathfrak{a} \). The \( A^d \) orbits in \( \mathfrak{a} \) are again parameterized by the points of the cone \( \Delta \). Thus given a tuple of side-lengths \( (v_1, v_2, v_3) \in \Delta^3 \) as above we can look for a triple \( e = (e_1, e_2, e_3) \in \mathfrak{a}^3 \) such that \( e_1 \in A^d v_1, e_2 \in A^d v_2, e_3 \in A^d v_3 \) and \( e_1 + e_2 + e_3 = 0 \).

We let \( D_\alpha(p) \) be the subset of \( (v_1, v_2, v_3) \in \Delta^3 \) such that there is a solution to the above problem. The following theorem (for all connected semisimple groups with no compact factors) was proved in [KL1M1] (however see below).

Theorem 1.10. \( D_\alpha(p) = D_\alpha(X) \).

Many people contributed to finding the generalized triangle inequalities. The reader is urged to consult [KN] and [KL2] for a more complete account. In fact the inequalities were computed for the infinitesimal symmetric space \( p \) for \( G \) complex by [BeS] and [LM] and for general \( G \) (real or complex) in [LM]. An intense interest in recent years in the generalized triangle inequalities was triggered by the paper of Klyachko, [Kly99]. Klyachko proved the generalized triangle inequalities for \( GL(m, \mathbb{C}) \) in the infinitesimal symmetric space case and a refinement was obtained by Belskii [Bel]. In a second paper Klyachko [Kly99], proved the equality \( D_\alpha(p) = D_\alpha(X) \) for certain symmetric spaces of complex simple groups (including \( SL(n) \)). In [ANW], Alekseev, Meinrenken and Woodward proved the corresponding equality for all complex simple \( G \).
The above system of inequalities is not so explicit; in particular, the polyhe-
dra of these systems is not well understood. Thus it is important to compute explicit
examples. We will see that the theorem of Knutson, Tao and Woodward [KTW]
that the inequalities are irredundant for GL(m) is highly exceptional.

**Definition 1.11.** The inequality \(a_{i1}, a_{i2}, a_{i3}\) will be said to be trivially redundant
if it is a consequence of the inequalities defining the cone \(\Delta\) in \(a\).

In the case \(G\) has rank one, the generalized triangle inequalities are precisely
the ordinary triangle inequalities. The rank two examples were worked out in [LM].
For the cases of \(B_2 = C_2\) and \(C_2\) the above system of inequalities was not
minimal. For the case of \(B_2\) there was one trivially redundant inequality. Once it
was removed the remaining inequalities were irredundant. For the case of \(G_2\) there
were no trivially redundant inequalities but three redundant inequalities. The point
of this paper is to work out all the rank three examples. We will see that for \(C_3\) and
\(B_3\) there are 24 trivially redundant inequalities. It is rather surprising that there are
only 9 more redundant inequalities. Furthermore as explained in the abstract these
redundancies were easy to find since the redundant inequalities occurred for one
system and not for the other. Thus we might say that only "obvious" redundancies
occurred. Another interesting consequence is that although the polyhedral cones
\(D_2(B_3)\) and \(D_2(C_3)\) are isomorphic (i.e., there is an affine isomorphism from one
to the other) by Theorem 1.8, the systems of inequalities are different.

In Chapter 3, see Theorem 3.16, we have given a self-contained account of
how one uses the Demazure-B.G.G. operators to realize the duals of the Schubert
homology classes in the Borel model. Our calculations in this paper of the products
of Schubert classes are based on this construction.

This paper is dedicated to Robert Greene on the occasion of his sixtieth birthday.
The third author takes great pleasure in acknowledging many helpful conver-
sations and some hard-fought tennis matches over the years. This paper depends
on [LM] and the two papers [KLM1] and [KLM2]. We have used the computer
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much help and advice in implementing it.

2. Schubert cycles in generalized flag varieties

We continue to assume that \(G\) is a connected complex semisimple algebraic
group. For convenience, we further assume that \(G\) is simply-connected. We fix a
Borel subgroup \(B\) of \(G\). Let \(b\), resp. \(g\), be the Lie algebra of \(B\), resp. \(G\). We
also fix a Cartan subalgebra \(h \subseteq b\). The choice of \((h, h)\) determines the set \(\Pi \subseteq h^*\)
of positive roots and thus the set \(\Phi = \{\alpha_1, \ldots, \alpha_l\} \subseteq \Pi\) of simple roots and also
the fundamental weights \(\omega_1, \ldots, \omega_l\), i.e., being the rank of \(G\). We consider the
real form \(a\) of \(h\) which is the real span of the simple coroots \(\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}\).
Then \(\Pi \subseteq a^*\) and also \(\omega_i \in a^*\). Any algebraic subgroup \(P\) of \(G\) containing \(B\) is called a
standard parabolic subgroup. Let \(\Delta \subseteq a^*\) be the cone on the fundamental weights.
By definition, the dominant weights are the elements of \(\omega_i^\vee \in Z_{\omega_i}\).

To each dominant weight \(\lambda\) we define the associated standard parabolic subgroup
\(P_\lambda\) to be the (connected) subgroup of \(G\) with Lie algebra spanned by \(b\) together
with the root vectors \(X_\alpha\) (corresponding to the root \(-\alpha\)) such that \(\lambda(H_\alpha) = 0\).
Here \(\alpha\) runs through the positive roots \(\Pi\) of \(g\) and \(H_\alpha \in a\) is the corresponding
correct which is equal to \( \frac{2\pi}{\omega_0} \) under the Killing form. We let \( S \) be the set of (simple) reflections in the root hyperplanes defined by the simple roots and let \( W \subset \text{Aut} \mathfrak{g} \) be the Weyl group generated by \( S \). Then \( W \) can canonically be identified with the group \( N(T)/T \), where \( T \) is the maximal torus of \( \mathfrak{g} \) with Lie algebra \( \mathfrak{h} \) and \( N(T) \) is its normalizer in \( G \).

For any standard parabolic subgroup \( P \) of \( G \), let \( W_P \subset W \) be the subgroup consisting of those \( w \) that have representatives in \( P \) and \( S_P = S \cap W_P \). We let \( \ell \) be the length function on \( W \). Each coset \( wW_P \) in \( W/W_P \) has a unique representative, of minimum length \([\text{Hi82}], \text{Ch. I, Corollary 5.4}\). We will denote this element by \( w^* = w^*_P \). The set of such representatives will be denoted by \( W^* \). We have the following criterion for the minimum length element in the coset \( wW_P \), see \([\text{Hi82}], \text{Ch. I, Corollary 5.4}\).

**Lemma 2.1.** \( w^* \in wW_P \) is the minimum length representative if and only if

\[ \ell(w^*s) = \ell(w^*) + 1, \text{ for all } s \in S_P. \]

We will also need the following result, see \([\text{Hi82}], \text{Ch. I, Theorem 5.3}\).

**Lemma 2.2.** Suppose that \( w \in W^* \) and \( v \in W_P \). Then

\[ \ell(wv) = \ell(w) + \ell(v). \]

We now recall the Bruhat decomposition for \( G \), see \([\text{He}], \text{Theorem 1.4, pg. 403}\).

**Theorem 2.3.** For any standard parabolic subgroup \( P \) of \( G \),

\[ G = \bigcup_{w \in W} BwB, \]

\[ G = \bigcup_{w \in W^*} BwP. \]

As a consequence of (2) the generalized flag variety \( G/P \) is the disjoint union of the subsets \( \{C_P^w \mid w \in W^*\} \cup \{wP \}_{w \in W}. \) The subset \( C_P^w \) is birational isomorphic to the affine space \( \mathbb{C}C_P \) and is called a Schubert cell. The closure of \( C_P \), to be denoted \( X_P^w \), is an algebraic subvariety of \( G/P \) and is called a Schubert variety. We will use \( X_P^w \) to denote the integral homology class in \( H_*(G/P) \) carried by \( X_P^w \) and we will often abuse notation and use the same symbol \( X_P^w \) for the variety and its class in homology. We will use \( P\mathbb{Z}(X_P^w) \) to denote the cohomology class of complementary degree to the degree \( d_X \) associated to \( X_P^w \) by the Poincaré dual. In what follows we will let \( N = \dim(G/B) = [\omega_0] \) and \( N_P = \dim(G/P) \).

We recall the following well known:

**Theorem 2.4.** The integral homology \( H_*(G/P) \) is a free \( \mathbb{Z} \)-module with basis \( \{X_P^w \mid w \in W^*\} \).

In particular, the integral homology \( H_*(G/B) \) is a free \( \mathbb{Z} \)-module with basis \( \{X_w \mid w \in W\} \), where \( X_w \) is abbreviated by \( X_w \).

Since \( H_*(G/P) \) is free \( \mathbb{Z} \)-module and we have a distinguished basis (the Schubert classes) it is reasonable to consider the basis for the corresponding cohomology groups that are dual under the Kronecker pairing \( (\cdot, \cdot) \) between homology and cohomology. Let \( \{c_w \mid w \in W^*\} \) denote the dual basis. Thus we have for \( w, w' \in W^* \),

\[ (c_w, X^w_{w'}) = \delta_{w,w'}. \]

It suffices to study \( H^*(G/B) \) because of the following well known theorem.
THEOREM 2.5. Let \( \pi : G/B \to G/P \) be the projection. Then the induced map \( \pi_* : H^*(G/P) \to H^*(G/B) \) is injective with image precisely equal to the \( W_P \)-invariants of \( H^*(G/B) \).

Moreover, for \( w \in W_P \),

\[ \pi_* [c^w_w] = c_w, \]

where again we abbreviate \( c^w_w \) by \( c_w \).

So, from now on, we will identify \( H^*(G/P) \) as a subring of \( H^*(G/B) \) and denote \( c^w_w \) by \( c_w \) itself.

2.1. Poincaré duality in \( G/P \). In the following, for each \( w \in W_P \) we will need to identify \( PD(X^P_w) \) in terms of the basis \( \{ c^w_w \}_{w \in W_P} \).

Define the involutive map \( \theta^P : W \to W \) by \( \theta^P(w) = w_0 w_0 w_0 \) (resp. \( w_0 \) is the longest element of \( W \) (resp. \( W_P \)).

For lack of a precise reference, we give a proof of the following:

PROPOSITION 2.6. For \( w \in W_P \), \( \theta^P(w) \in W_P \) and we have

\[ PD(X^P_w) = c^p_{\theta^P(w)}. \]

The proposition will follow from the next two lemmas.

Lema 2.7. The map \( \theta^P \) carries \( W^P \) into itself. Moreover, we have

\[ \ell(\theta^P(w)) = N_P - \ell(w), \]

where \( N_P \) denotes the complex dimension of \( G/P \). Thus, \( N_P = \ell(w_0) - \ell(w_0, p) \).

Proof. For \( w \in W_P \) and any \( v \in W_P \) we have

\[ \ell(w w_0 w_0, p) = \ell(w_0) - \ell(w_0, p). \]

But, by Lemma 2.2, since \( w \in W_P \) and \( w_0 w_0 \in W_P \) we get

\[ \ell(w_0, p) = \ell(w) + \ell(w_0, p) = \ell(w) + \ell(w_0, p) - \ell(w). \]

Thus,

\[ \ell(w w_0 w_0, p) = (\ell(w) - \ell(w)) + \ell(v). \]

But the above argument shows that the terms in parentheses equal \( \ell(w_0 w_0, p) \).

Thus multiplying \( \theta^P(w) \) by any \( r \in W_P \) increases the length of \( \theta^P(w) \) and accordingly by Lemma 2.1 we have

\[ \theta^P(w) \in W_P. \]

Taking \( v \) to be the identity in the above formula we have

\[ \ell(w w_0 w_0, p) = \ell(w) - \ell(w_0, p) - \ell(w) = N_P - \ell(w). \]

Before proving the proposition we need a general result from algebraic topology.

Let \( M \) be a compact connected oriented manifold of dimension \( n \). For \( a \in H_n(M) \) and \( b \in H_n(M) \) we will let \( a \cdot b \) denote the intersection pairing, \([Br],[Ch],V\), Section 11. We then have \([Br]\), pg. 267.

\[ PD(a, b) = a \cdot b. \]

REMARK 2.8. In our case all the homology is even dimensional so we will not have to worry about the interchange of order.
Suppose now that $H_1(M)$ is free over $Z$ and accordingly we have $H^1(M) \cong \text{Hom}_Z(H_1(M), Z)$. Thus to prove the proposition we have to identify the element $P^1(X'_R)_{\omega}$ of $\text{Hom}_Z(H_{2n-2}(\omega)(G/P), Z)$.

The key point in the proof of the proposition is then

**Lemma 2.9.** For $v, w \in W^P$ with $\ell(v) = \ell(w)$,

$$X'_R \cdot w X'_{P(v)} = \delta_{w, v}.$$  

Proof. Since the action of any element of $G$ by left multiplication on $G/P$ induces the trivial action on $H^*(G/P)$, to prove the lemma it suffices to prove the following equalities at the cycle level:

(3) $$X'_R \cdot w X'_{P(v)} = \delta_{w, v},$$

(4) $$X'_R \cdot w X'_{P(w)} = \delta_{w}. $$

Suppose that the above cycles $X'_R$ and $w X'_{P(v)}$ intersect in a nonempty set. Then (since each cycle is $T$-stable) the intersection is $T$-stable and projective and consequently will contain a $T$-fixed point, say $u \in W^P$. Since the Schubert cell $B u P/P$ contains the unique $P$-fixed point $w$ and $B u P/P = \bigcup B z P/P$ we find $w \leq u$.

Similarly, since $u \in w B u w P(w) P$ and $w u$ is of order 2, we have $w u X'_{P(v)} \in B u P/P$ and as above we find $w u X'_{P(v)} \leq w u P$ and hence $w u P \geq w p u P$. But since $u, v \in W^P$, we get $u \geq v$ (cf. [Ku], Lemma 1.3.18).

Thus $v \leq u \leq w$. But since $\ell(v) = \ell(w)$, we obtain $v = u = w$. By the above argument the intersection $X'_R \cap w X'_{P(v)} = \{w\}$ set theoretically. The proof of (4) is completed by observing that the intersection is transverse at $w$ or alternatively by using the Poincaré duality and (3).

Thus as operators on $H_{2n-2}(\omega)(G/P)$ we have $P^1(X'_R)_{\omega} = \delta_{w} \omega$ and the proposition follows. □

3. A formula for $\omega$  

3.1. The Borel model. We continue to assume that $G$ is a connected, simply-connected complex semisimple algebraic group. As earlier let $w_0$ denote the 1-st fundamental weight. Recall that the Borel model for $H^*(G/B)$ is obtained through the Borel homomorphism

$$\beta : Z[w_1, w_2, \ldots, w_r] \rightarrow H^*(G/B),$$

which is the unique algebra homomorphism taking $w_0$ to the first Chern class of the line bundle on $G/B$ associated to the character $-w_0$ of $B$. It is easy to see that the homomorphism $\beta$ commutes with the Weyl group actions and thus for any standard parabolic subgroup $P$ of $G$, the $W$-invariant $Z[w_1, w_2, \ldots, w_r]^P$ is mapped to $H^*(G/P)$ under $\beta$.

Let $I \subset Z[w_1, w_2, \ldots, w_r]$ be the ideal generated by the $W$-invariant polynomials with zero constant term. Then by extending the scalars to the real numbers, $\beta$ induces a surjective homomorphism (still denoted by $\beta : Z[w_1, w_2, \ldots, w_r] \rightarrow \mathbb{P}^*(G/B, \mathbb{R})$, with kernel precisely equal to $I \otimes_{\mathbb{Z}} \mathbb{R}$.
In a subsequent subsection we will use the divided-difference operators of Demazure and Bernstein-Gelfand-Gelfand to find a polynomial $p_w \in \mathbb{R}[x_1, \ldots, x_n]$ such that $\beta(p_w)$ is the cohomology class $e_w$ for $w \in W$.

### 3.2. The Demazure–BGG operators

For more details on this subsection the reader is urged to consult [HI82], Chapter IV and [Ku], Chapter XI. We will set $V = \mathbb{C}^n$ henceforth. Let $a_1$ be a simple root and let $a_2$ be the corresponding simple reflection. We define the divided difference operator $A_n : S^1(V) \to S^{n-1}(V)$ by

$$A_n(f) = \frac{f - a_1 f}{a_1}.$$

We note that $A_n \circ A_n = 0$. It is also important to note (and simple to prove) that $A_n$ is a twisted derivation in the following sense.

**Lemma 3.1.** $A_n(pq) = A_n(p)q + (ap)A_n(q)$.

From the definition and the above lemma, it is easy to see that $A_n$ keeps the integral form $\mathbb{Z}[a_1, a_2, \ldots, a_n] \subset S^1(V)$ stable and, moreover, it also keeps $A_n$ stable.

For any $w \in W$ we further define $A_w$ by

$${A}_w := A_n \circ a_{w_1} \circ \cdots \circ a_{w_n}$$

where $w = w_1 \cdots w_n$ is a reduced decomposition of $w$ as a product of simple reflections. We have [HI82], Chapter IV, Proposition 1.7.

**Proposition 3.2.** The operators $A_n$ are well-defined, i.e., they do not depend upon the choice of the reduced decomposition of $w$. Moreover, we have $A_w \circ A_n = A_{w_0}$ if $f(w_0) = f(w) + f(\gamma)$ and $A_w \circ A_n = 0$ otherwise.

### 3.3. The topological Demazure–BGG operators

As earlier, for any topological space $X$, $H^* (X)$ denotes the singular cohomology of $X$ with integral coefficients.

There is an analogue of the Demazure–BGG operator $D_n$, on $H^*(G/B)$ (for any simple root $a_1$) defined directly as follows.

Let $\pi_0 : G/B \to G/P_1$ be the locally trivial fibre with fibre (over $eP_1$) $P_1/B = \mathbb{P}^1$. It is easy to show that the restriction map $\gamma : H^*(G/B) \to H^*(P_1/B)$ is surjective. In fact, $e_n$ maps to the generator of $H^2(P_1/B)$. Choose a $Z$-module splitting $\delta : H^*(P_1/B) \to H^*(G/B)$ of $\delta$. Then by the Leray-Hirsch Theorem, the map

$$\Phi : H^*(G/R) \otimes H^*(P_1/B) \to H^*(G/B), \quad \sigma \otimes \tau \mapsto (\pi_0^* \sigma) \cup \delta(\tau),$$

is an isomorphism. Hence, $\pi_0^*$ is injective and $H^*(G/B)$ is a free module over $H^*(G/P_1)$ (under $\pi_0^*$) with basis 1 and $\sigma(e)$, where $e := e_1$ is in $H^2(P_1/B)$ is the generator, i.e.,

$$H^*(G/B) \cong H^*(G/P_1) \oplus \sigma(e) H^{n-2}(G/P_1),$$

for any $n \geq 0$.

Write, for any $u \in H^*(G/B)$,

$$u = \pi_0^* u_1 + \sigma(e) \pi_0^* u_2,$$

where $u_1 \in H^2(G/P_1)$ and $u_2 \in H^{n-2}(G/P_1)$ are uniquely determined by the above equation.

Now define

$$D_n u := \pi_0^* u_2 \in H^{n-2}(G/B).$$
It is easy to see that $D_\sigma$ does not depend upon the choice of the splitting $\sigma$.

Clearly, $D_\sigma^0 = 0$.

**Proposition 3.3.** The Borel homomorphism $\beta$ intertwines $A_\sigma$ and $D_\sigma$ for any simple reflection $\sigma_i$.

**Proof.** Let $R$ be the ring $R[\omega_1, \omega_2, \ldots, \omega_n]$ and $R_1$ the subring of $\omega_i$-invariants. Then $R$ is generated as an $R_1$-module by $1$ and $\omega_i$. Further, $R(R_1) \subset H^*(G/B)$ (cf. subsection 3.1). By definition, $A_\sigma$ commutes with the multiplication by $R_1$ and also $D_\sigma$ commutes with $H^*(G/B)$. Moreover, $D_\sigma(\omega_i) = A_\sigma(\omega_i) = 0$. Thus, to prove the proposition, it suffices to observe that

$$D_\sigma(\beta(\omega_i)) = \beta(D_\sigma(\omega_i)) = 1.$$

Now, it is easy to see that $\beta(\omega_i)$ is the generator $e_i$ which proves ($\ast$).

Thus the operators $D_\sigma$ again satisfy the braid relations and we may extend $D_\sigma$ to $D_\sigma$ by taking a reduced decomposition of $w$. Moreover, $D_\sigma$ satisfies the twisted derivation property:

$$D_\sigma(xy) = D_\sigma(x)y + (x,y)D_\sigma(y), \text{ for } x, y \in H^*(G/B).$$

We also recall the following well-known result due to Chevalley, cf. [BGG], Theorem 3.14.

**Lemma 3.4.** For any simple reflection $\sigma_i$ and any $w \in W$, the cup product

$$e_{\sigma_i} \cdot e_w = \sum_{\beta \in B} \langle w, \beta \rangle^* e_{\beta},$$

where the notation $w^< \beta$ means $w \leq v$, $\ell(v) = \ell(w) + 1$, $\beta \in B$ and $v = s_\beta w$; and $s_\beta \in W$ is defined by $s_\beta x = x - (x, \beta^\vee)\beta$, for $x \in B$.

The following result is of basic importance.

**Proposition 3.5.** For any simple reflection $\sigma_i$ and any $w \in W$,

$$D_\sigma e_w = e_{\sigma_i} \cdot e_w, \text{ if } w_{\sigma_i} < w,$$

and

$$D_\sigma e_w = 0, \text{ otherwise}.$$

**Proof.** We first consider the case $w_{\sigma_i} > w \Leftrightarrow w \in W^\alpha$. By Theorem 2.5, $e_\sigma \in H^*(G/B)$, hence $D_\sigma e_w = 0$.

So assume now that $w_{\sigma_i} \leq w$. By the Chevalley formula 3.4,

$$e_{\sigma_i} \cdot e_w = e_w + \sum_{w_{\sigma_i} \leq w} \langle w_{\sigma_i}, \beta \rangle^* e_{\beta}.$$

By a standard property of Coxeter groups [Ku], Corollary 1.3.19, any $v$ appearing in the above sum satisfies $w_{\sigma_i} > v$. Hence applying $D_\sigma$ to the above equation and using the previous case, we get

$$D_\sigma(e_{\sigma_i} \cdot e_w) = D_\sigma(e_w).$$

By using the twisted derivation property and the previous case again, we get

$$D_\sigma(e_{\sigma_i} \cdot e_w) = D_\sigma(e_{\sigma_i}) e_w = e_{\sigma_i}.$$
since $D_{n}(e_{n}) = 1$ (because $e_{n}$ restricted to $F_{1}/B$ equals $e$). Combining the above two equations, we get the proposition.

3.4. The polynomials $p_{m}$. In this section we will polynomials $p_{m}$ such that

$$
\beta(p_{m}) = e_{m}.
$$

First we will find $p_{m}$. We define $p_{m}$ for $m$ the longest element in $W$ as follows. Let $d$ be the product of the positive roots. Then define

$$
(6) \quad p_{m} = \frac{d}{W}.
$$

**Proposition 3.6.**

$$
\beta(p_{m}) = e_{m}.
$$

As we will see that Proposition 3.6 will be an almost immediate consequence of Lemma 3.6. However we need a preliminary general lemma from algebra.

Let $G$ be a finite group and $\rho: G \to \text{Aut}(V)$ be a faithful representation. Let $R = \text{End}(V^{*})$ be the algebra of regular functions on $V$ and $F = Q(V^{*})$ be the quotient field. The representation $\rho$ induces a representation $\bar{\rho}$ from $G$ into $\text{Aut}(F)$ where we consider $F$ as a vector space over the fixed field $L = F^{0}$. We have

$$
\rho(g)(v)(\bar{\cdot}) = \psi(g)(\bar{v}), \quad v \in V.
$$

**Lemma 3.7.** The set $(\rho(g) : g \in G)$ is an independent subset of the $F$ vector space $\text{End}_{L}(F)$, where $F$ acts on $\text{End}_{L}(F)$ via its multiplication on the range.

*Proof.* Suppose we have a minimal dependence relation

$$
\sum_{g \in G} q_{g}(g)(v) = 0, \quad v \in F.
$$

Write $q_{g} = \frac{q_{g}}{\lambda_{g}}$ with $\lambda_{g}, \lambda_{h} \in R$ and relatively prime. For $f \in R$ we let $V(f)$ denote the zero locus of $f$. For each $g \in G$, let $F(g)$ be the fixed subspace of $\rho(g)$. For $g \neq 1$ the subspace $F(g)$ is proper hence $\bigcup_{g \neq 1} F(g) \subseteq V$. Choose $v_{0} \in V \setminus \bigcup_{g \neq 1} F(g) \cup \bigcup_{g \neq 1} V(v_{g}) \cup \bigcup_{g \neq 1} V(v_{h})$. Then $G \to G \cdot v_{0}$ is an embedding. Fix $v_{0} \in G$. Find a polynomial $p_{m}$ such that

1. $p_{m}(\rho(g)(v_{0})) = 1$.
2. $p_{m}(v) = 0, v \in G \cdot v_{0} \setminus \rho(g)(v_{0})^{-1}(v_{0})$.

Now $0 = \sum_{g \in G} q_{g}(v_{0})(g)(p_{m}(v)) = \sum_{g \in G} q_{g}(v_{0})p_{m}(\rho(g)^{-1}v_{0}) = q_{g_{0}}(v_{0}) \neq 0$. This is a contradiction.

**Lemma 3.8.** $A_{m}p_{m} = 1$.

*Proof.* Take a reduced decomposition $u_{0} = s_{1} \cdots s_{k}$, then it is standard that

$$
(M_{u_{0}, 1}, s_{1}, \cdots, s_{k}, s_{k+1})
$$

is an enumeration of the positive roots. For $q \in F$, let $M_{q}$ be the operation of multiplication by $q$. Write $A_{m} = A_{m_{1}}(I - s_{1}) \circ \cdots \circ A_{m_{k+1}}(I - s_{k+1})$. It is then easy to see that for any $p \in R$ we may write (see the next paragraph)

$$
(7) \quad A_{m}p = \sum_{w \in W} \theta_{w}w \cdot p.
$$

Moreover we have

$$
A_{m_{1}}(I - s_{1}) \circ \cdots \circ A_{m_{k}}(I - s_{k})
$$

Let $s_i$ be a simple reflection. Since $\ell(s_i w) < \ell(w)$ we have, by Proposition 3.2, $A_{s_i}A_{w} = 0$. From this we see that for all $p \in R$ we have $\sum_{w} a_{w} \cdot p = \sum_{w} (s_i - q_{w}j)(s_i w - p)$. Apply Lemma 3.7 to conclude

$$s_i - q_{w}j = q_{w}$$

Combining this with (b) we obtain

$$q_{w} = (-1)^{\ell(w) + N} q_{w} = (-1)^{\ell(w) + 1} q_{w}$$

Hence by (7) we obtain

$$A_{s_i}p_{w} = \sum_{w \in X} (-1)^{\ell(w) + 1} w \cdot p_{w} = 1$$

Now we can complete the proof of Proposition 3.6. From $\text{div}(G/B) = N$ we deduce that the $N$-th graded component of $S(V^*)/f_{k}$ is a one-dimensional vector space over $R$ with basis $(e_{w})$. We will now prove that $\ell$ (or $p_{w}$) is another basis element.

We prove that $\ell \not\in f_{k}$. Suppose $\ell \in f_{k}$. Then $\ell = \sum_{i} a_{i}$, where $q_{w} \in S(V^*)^W$ are homogeneous with zero constant term. By applying the alternator $\sum_{w \in X} (-1)^{\ell(w)} q_{w}$ to both sides we see that we may assume that the $q_{w}$ are antilinear elements of $S(V^*)$. But any antilinear element vanishes on all the root hyperplanes and consequently is divisible by $d$. We conclude that all the $\ell(w)$ are zero. This implies that $d = 0$, a contradiction.

Since $d$ has degree $N$ we find that $d$ (and hence $p_{w}$) is another basis vector of the $N$-graded component of $S(V^*)/f_{k}$. Hence there exists $c \in R$ such that $p_{w} = c e_{w}$ (mod $f_{k}$). But, by Proposition 3.5, we have $A_{w} e_{w} = e_{w}$. Hence $c = A_{w}p_{w}$. Then, by Lemma 3.8, we have $c = 1$ and Proposition 3.6 is proved.

We obtain as a first consequence the following

**Lemma 3.9.** The Weyl group $W$ acts on the top graded component (of degree $\ell(w)$) of the graded ring $S(V)/f_{k}$ by the sign representation.

We next define $p_{w}$ for a general $w$. Express $w$ as a left segment of $w_{0}$. Precisely we find $u$ such that $w_{0} = uw$ and $\ell(w_{0}) = \ell(w) + \ell(u)$. Then we define

$$p_{w} = A_{w} p_{w_{0}} = A_{w_{0}} - A_{w_{0}} p_{w_{0}}$$

As a consequence of Propositions 3.3, 3.6 and 3.5, we get the desired realization of the duals of the Schubert homology classes in the Borel model.

**Theorem 3.10.**

$$\beta(p_{w}) = e_{w}$$

We will need the following lemma.
Lemma 3.11. Suppose that we have a factorization \( w = w_1 w_2 \) with \( \ell(w) = \ell(w_1) + \ell(w_2) \). Then \( A_{w_1} A_{w_2} = A_w \).

Proof. Suppose we have realized \( w \) as a left segment of \( w_2 \) by \( w_0 = w_0 \). Then \( w_0 = w_0 (w_1) \) identifies \( w_0 \) as a left segment of \( w_2 \) with \( w_2^1 w_0 = w_1 w_0 \) and \( A_{w_0} A_{w_2} = A_w \) by Proposition 3.2.

4. The inequalities for the rank 3 root systems

In this section we describe the inequalities for the rank 3 root systems \( A_3 \), \( C_3 \) and \( B_3 \). Since there are many inequalities in each case we will give only a system of representatives modulo the action of the symmetric group \( S_3 \) and leave to the reader the task of symmetrizing the inequalities. We note that the polytopes for \( C_3 \) and \( B_3 \) are isomorphic by Theorem 1.5, though we will see below that the systems are different. We will also see that there are many trivially redundant inequalities labeled as \((*)\).

In each of the three cases there are 3 standard maximal parabolics, hence the system breaks up into three subsystems. We let \( r, s \) and \( t \) be the simple reflections associated to the nodes from left to right of the Dynkin diagram following the Bourbaki convention (so \( t \) corresponds to the long simple root in the case of \( C_3 \) and the short simple root in the case of \( B_3 \)). In what follows \( w_0 \) will denote the longest element in \( W \) and \( w_0^1 \) will denote the longest element in \( W^P \). Let \( \lambda \) be a fundamental weight. We will also use the notation \( X_{w^0} \) for the Schubert cycle \( X_{w^0}^{P} \) with \( w \in W^P \) and \( P \) the standard maximal parabolic subgroup associated to \( \lambda \).

In general the Weyl chamber \( \Delta \) is a simplicial cone, hence in rank 3 it is defined by 3 linear inequalities. These inequalities will contribute 9 inequalities in \((v_1, v_2, v_3)\) after symmetrization.

4.1. The inequalities for \( A_3 \).

In this case the quotients \( G/P \) for maximal parabolics \( P \) are Grassmannians and the cohomology rings are well-known. In particular, all the structure constants are 1 or 0. We will merely record the inequalities for the three subsystems. The Weyl chamber \( \Delta \) is given by

\[
\Delta = \{(x, y, z, w) : x + y + z + w = 0, x \geq y \geq z \geq w\}.
\]

We give below the inequalities in terms of triples \((v_1, v_2, v_3) \in \Delta^2 \) with \( v_i = (x_i, y_i, z_i, w_i), i = 1, 2, 3 \). But, to get the full set of inequalities, we need to symmetrize these with respect to the action of \( S_3 \) diagonally permuting the variables \( x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, w_1, w_2, w_3 \).

4.1.1. The subsystem associated to \( H^*(G/P_1) \). In this case the quotient \( G/P_1 \) is \( \mathbb{CP}^3 \). Thus, we obtain the subsystem (before symmetrization):

\[
x_1 + x_2 + x_3 \leq 0,
y_1 + y_2 + y_3 \leq 0,
z_1 + z_2 + z_3 \leq 0.
\]

Hence there are 10 inequalities after symmetrization.
4.1.2. The subsystem associated to $H^*(G/P_3)$. In this case the quotient $G/P_3$ is the Grassmannian of 3-planes in $C^4$. We obtain the subsystem (before symmetrization):

\[
\begin{align*}
&x_1 + y_1 + z_2 + u_2 + z_3 + w_3 \leq 0 \\
&x_2 + y_2 + z_2 + u_2 + z_3 + w_3 \leq 0 \\
&x_3 + y_3 + z_2 + u_2 + z_3 + w_3 \leq 0 \\
&y_1 + x_1 + z_2 + z_3 + w_3 + y_3 \leq 0 \\
&y_2 + x_2 + z_2 + z_3 + w_3 + y_3 \leq 0 \\
&y_3 + x_3 + z_2 + z_3 + w_3 + y_3 \leq 0 \\
\end{align*}
\]

Hence, there are 51 inequalities after symmetrization.

4.1.3. The subsystem associated to $H^*(G/P_3)$. This subsystem is dual to the first subsystem. In this case the quotient is the Grassmannian of 2-planes in $C^4$. We obtain the subsystem (dual to the first):

\[
\begin{align*}
&x_1 + y_1 + z_1 + y_2 + z_2 + u_2 + w_2 + z_3 + w_3 \leq 0 \\
&x_2 + y_1 + z_1 + y_2 + z_2 + u_2 + w_2 + z_3 + w_3 \leq 0 \\
&x_3 + y_1 + z_1 + y_2 + z_2 + u_2 + w_2 + z_3 + w_3 \leq 0 \\
\end{align*}
\]

Hence again there are 10 inequalities after symmetrization.

Thus there are altogether 50 + 9 inequalities defining $D_3(A_3)$ and the system is minimal. [KTW], where 9 in 41 + 9 accounts for 9 inequalities defining $\Delta^3$ in $C^3$.

4.2. The inequalities for $C_3$. In this subsection we take simply-connected $G$ of type $C_3$, i.e. $G = Sp(6)$. We note that the Weyl chamber $\Delta$ is given by the triples $x, y, z$ of real numbers satisfying

\[
x \geq y \geq z \geq 0.
\]

Here $x, y, z$ are the coordinates relative to the standard basis $e_1, e_2, e_3$ in the notation of [Bo], pg. 254 - 265. The inequalities will now be in terms of $(v_1, v_2, v_3) \in \Delta^3$ with $v_i = (x_i, y_i, z_i), i = 1, 2, 3$. We will need to symmetrize the inequalities with respect to the action of $S_3$ diagonally permuting the variables $x_1, x_2, x_3; y_1, y_2, y_3; z_1, z_2, z_3$.

In what follows one will often need to verify that an expression of an element $w \in W$ in the generators $e_i, i$ of minimal length. One can do this by finding the word as a connected subword of a minimal length expression of the longest word $w_0$. Thus one needs a plentiful supply of such expressions. The first part of the following lemma follows from [Bo], Proposition 1.2, pg. 121. Also, in the reflection representation, the coordinate sign changes are given by $r_{si}$ (first coordinate), $s_{si}$ (second coordinate) and $t$ (third coordinate) and $w_0 = -1$. From this the second part of the following lemma follows easily.

**Lemma 4.1.**

1. Let $w_0$ be a product of the simple generators (in any order). Then $\langle w_0 \rangle = w_0$. 

(2) A product of the three sign-changes \( x, y, z \) in any order is equal to \( \alpha_0 \).

We will also need the following proposition.

**Proposition 4.2.**

\[ P_{\alpha_0} = z^t y^r (x y^r) \mod I, \]

where \( I \) is the ideal generated by the \( W \)-invariant polynomials in \( \mathbb{Z}[x, y, z] \) with zero constant term.

**Proof.** By the definition of \( P_{\alpha_0} \) (cf. equation (6)), we have

\[ P_{\alpha_0} = \frac{(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)(2x)(2y)(2z)}{48} \]

In the ring \( \mathbb{Z}[x, y, z] \) we have \( (x - y)(x - z)(y - z) = 6X^2Y^2Z \) mod \( J \), where \( J \) is the ideal of \( \mathbb{Z}[x, y, z] \) generated by the symmetric polynomials with zero constant term. Now the proposition follows from the above by taking \( X = x^2, Y = y^2, Z = z^2 \). \( \square \)

We will also need the following simple fact about the Weyl group \( W \) of \( G_3 \). Let \( w \in W \) and choose a reduced decomposition of \( w \). Let \( n(w) \) be the number of times \( t \) appears in this decomposition.

**Lemma 4.3.** \( n(w) \) is independent of the reduced decomposition of \( w \).

**Proof.** From the Coxeter group property of \( W \), any two reduced decompositions of \( w \in W \) can be obtained from another by using the Artin (i.e., generalized braid) relations, that is by replacing \( r_i r_j \) by \( s_i s_j r_i \) or \( r_i s_j r_i \) by \( r_i s_j r_i \).

We leave the proof of the following lemma to the reader.

**Lemma 4.4.** \( W^{P_3} = \{ e, r, s, t, \sigma, \tau \} \),

We will abbreviate the classes \( c_i \) for \( u \in W^{P_3} \) by \( a_i \), \( i = 1, 2, 3, 4 \), according to the following table. Moreover, in the following table we list the elements \( u \) of \( W^{P_3} \), their lengths, the minimally singular weight \( \lambda_u \) and associated to the Schubert cycle \( X_u = X^u \) and the notation for the cohomology class \( PD(X_u) \),

\[
\begin{array}{|c|c|c|c|}
\hline
u & \ell(u) & \lambda_u & PD(X_u) \\
\hline
e & 0 & (1, 0, 0) & a_0 \\
\hline
r & 1 & (0, 1, 0) & a_1 & a_0 \\
\hline
s & 2 & (0, 0, 1) & a_2 & a_0 \\
\hline
\sigma & 3 & (0, 0, -1) & a_3 & a_2 \\
\hline
\tau & 4 & (1, -1, 0) & a_4 & a_1 \\
\hline
\rho & 5 & (1, 0, 0) & a_5 & 1 \\
\hline
\end{array}
\]

We now give the corresponding subsystem leaving to the reader the task of symmetrizing the inequalities below. We label each inequality with the ordered partition of 5 that it corresponds to. For example the label \((3, 2, 0)\) means the inequality corresponds to the formula \( a_3 \cdot a_2 \cdot 1 = a_5 \cdot a_1 \) in \( H^*(G/P_1) \).
We refer to the above chart to obtain $PD(a_2) = X^1_{xy} = X_{(0,0),1}, PD(a_3) = X^0_{xy} = X_{(0,0),0}$, and $PD(1) = X^0_{(0,0)} = X_{(0,0),0}$. We have

$X_{(0,0),1} - X_{(0,0),0} - X_{(0,0),0} = [x]$

Applying the linear functional $(0, 0, 1), (0, 0, -1)$ and $(-1, 0, 0)$ to $v_1, v_2, v_3$ respectively we get the inequality $z_1 - z_2 - z_3 \leq 0$.

So the system of inequalities (before symmetrization) is given by:

\[
\begin{align*}
 z_1 &\leq z_2 + z_3 & (5, 0, 0) \\
 y_1 &\leq y_2 + z_3 & (4, 1, 0) \\
 z_1 &\leq z_2 + z_3 & (3, 2, 0) \\
 z_1 &\leq y_2 + y_3 & (3, 1, 1) \\
 (-) &- z_1 - z_2 - y_3 \leq 0 & (2, 2, 1)
\end{align*}
\]

The three inequalities in this subsystem generated by $(-)$ corresponding to the ordered partition $(2, 2, 1)$ are trivially redundant and do not occur in the system for $B_3$.

Thus, there are 21 inequalities in this subsystem after symmetrization which includes 3 trivially redundant inequalities. There are no other redundant inequalities in this subsystem.

4.2. The subsystem associated to $H^*(B_3)$. The space $G_1/P_3$ is the space of totally-isotropic 2-planes. The group $W_{P_3}$ is generated by the commuting simple reflections $r$ and $t$. We leave the proof of the following lemma to the reader.

**Lemma 4.5.** $W_{P_3} = \{e, s, r, t, rs, st, srs, std, trs, tdr, trts, stt, sttst\}$.

We will abbreviate the classes $w_i$ for $w \in W_{P_3}$ to $a_i$ or $a_i'$ as indicated in the next table. For the benefit of the reader, we also list for the elements $w$ in $W_{P_3}$, the corresponding maximally singular weight $w \cdot \omega = w \cdot (1, 1, 0)$ and the Poincaré dual class $PD(X^1_{xy})$:

<table>
<thead>
<tr>
<th>$w$</th>
<th>$\lambda_w$</th>
<th>$\rho(w)$</th>
<th>$\omega \cdot PD(X^1_{xy})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$(1,1,0)$</td>
<td>$0$</td>
<td>$a_1$</td>
</tr>
<tr>
<td>$s$</td>
<td>$(0,1,1)$</td>
<td>$1$</td>
<td>$a_1$</td>
</tr>
<tr>
<td>$rs$</td>
<td>$(0,1,1)$</td>
<td>$2$</td>
<td>$a_2$</td>
</tr>
<tr>
<td>$st$</td>
<td>$(1,0,-1)$</td>
<td>$2$</td>
<td>$a_3$</td>
</tr>
<tr>
<td>$ts$</td>
<td>$(1,0,-1)$</td>
<td>$3$</td>
<td>$a_4$</td>
</tr>
<tr>
<td>$sts$</td>
<td>$(0,0,-1)$</td>
<td>$3$</td>
<td>$a_5$</td>
</tr>
<tr>
<td>$sts$</td>
<td>$(0,0,-1)$</td>
<td>$4$</td>
<td>$a_6$</td>
</tr>
<tr>
<td>$rst$</td>
<td>$(-1,1,0)$</td>
<td>$4$</td>
<td>$a_7$</td>
</tr>
<tr>
<td>$rst$</td>
<td>$(-1,1,0)$</td>
<td>$5$</td>
<td>$a_8$</td>
</tr>
<tr>
<td>$rst$</td>
<td>$(-1,1,0)$</td>
<td>$6$</td>
<td>$a_9$</td>
</tr>
<tr>
<td>$rst$</td>
<td>$(-1,1,0)$</td>
<td>$7$</td>
<td>$a_{10}$</td>
</tr>
</tbody>
</table>

We list the polynomials $p_n$ (mod $f$) in the next table, which is obtained by applying Lemma 3.11 and observing that both $x^2y^2 + x^3y^2$ and $x^4 + y^2 + x^2y^3$ belong to $f$. 


In the following table it will be convenient to use $g_n$ to denote the finite geometric series

$$g_n = \sum_{i=0}^{n} x^i = x^n.$$ 

<table>
<thead>
<tr>
<th>$w$</th>
<th>$w'$</th>
<th>$w = w'w$</th>
<th>$p_w \pmod{I}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>$r_1$</td>
<td>$w_1$</td>
<td>$(xy)^2$</td>
</tr>
<tr>
<td>$r_1$</td>
<td>$r_0$</td>
<td>$w_0$</td>
<td>$(xy)^3$</td>
</tr>
<tr>
<td>$r_0$</td>
<td>$r_1$</td>
<td>$w_0$</td>
<td>$x^2(x+y)$</td>
</tr>
<tr>
<td>$r_1$</td>
<td>$r_0$</td>
<td>$w_1$</td>
<td>$y^2(x+y)$</td>
</tr>
<tr>
<td>$rs$</td>
<td>$rs_1$</td>
<td>$w_1$</td>
<td>$g_4$</td>
</tr>
<tr>
<td>$rs$</td>
<td>$rs_1$</td>
<td>$w_0$</td>
<td>$g_3$</td>
</tr>
<tr>
<td>$r_1s$</td>
<td>$r_0s$</td>
<td>$w_1$</td>
<td>$xy$</td>
</tr>
<tr>
<td>$r_0s$</td>
<td>$r_1s$</td>
<td>$w_0$</td>
<td>$x+y$</td>
</tr>
</tbody>
</table>

Now computing the products of $p_w \pmod{I}$ and using the Chevalley formula Lemma 3.4, we obtain the following.

**Theorem 4.6.** The cohomology ring $H^*(G/P_2)$ is given by the following table.

<table>
<thead>
<tr>
<th>$H^*(G/P_2)$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$a_1^2 + a_2^2$</td>
<td>$a_1a_2$</td>
<td>$a_1a_3$</td>
<td>$a_1a_4$</td>
<td>$a_1a_5$</td>
<td>$a_1a_6$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$a_2^2$</td>
<td>$2a_1a_2$</td>
<td>$a_2a_3$</td>
<td>$a_2a_4$</td>
<td>$a_2a_5$</td>
<td>$a_2a_6$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$a_3^2$</td>
<td>$2a_1a_3$</td>
<td>$2a_2a_3$</td>
<td>$a_3a_4$</td>
<td>$a_3a_5$</td>
<td>$a_3a_6$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$a_4^2$</td>
<td>$2a_1a_4$</td>
<td>$2a_2a_4$</td>
<td>$2a_3a_4$</td>
<td>$a_4a_5$</td>
<td>$a_4a_6$</td>
</tr>
<tr>
<td>$a_5$</td>
<td>$a_5^2$</td>
<td>$2a_1a_5$</td>
<td>$2a_2a_5$</td>
<td>$2a_3a_5$</td>
<td>$2a_4a_5$</td>
<td>$a_5a_6$</td>
</tr>
<tr>
<td>$a_6$</td>
<td>$a_6^2$</td>
<td>$2a_1a_6$</td>
<td>$2a_2a_6$</td>
<td>$2a_3a_6$</td>
<td>$2a_4a_6$</td>
<td>$2a_5a_6$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H^*(G/P_2)$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$a_1^2 + a_2^2$</td>
<td>$a_1a_2$</td>
<td>$a_1a_3$</td>
<td>$a_1a_4$</td>
<td>$a_1a_5$</td>
<td>$a_1a_6$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$a_2^2$</td>
<td>$2a_1a_2$</td>
<td>$a_2a_3$</td>
<td>$a_2a_4$</td>
<td>$a_2a_5$</td>
<td>$a_2a_6$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$a_3^2$</td>
<td>$2a_1a_3$</td>
<td>$2a_2a_3$</td>
<td>$a_3a_4$</td>
<td>$a_3a_5$</td>
<td>$a_3a_6$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$a_4^2$</td>
<td>$2a_1a_4$</td>
<td>$2a_2a_4$</td>
<td>$2a_3a_4$</td>
<td>$a_4a_5$</td>
<td>$a_4a_6$</td>
</tr>
<tr>
<td>$a_5$</td>
<td>$a_5^2$</td>
<td>$2a_1a_5$</td>
<td>$2a_2a_5$</td>
<td>$2a_3a_5$</td>
<td>$2a_4a_5$</td>
<td>$a_5a_6$</td>
</tr>
<tr>
<td>$a_6$</td>
<td>$a_6^2$</td>
<td>$2a_1a_6$</td>
<td>$2a_2a_6$</td>
<td>$2a_3a_6$</td>
<td>$2a_4a_6$</td>
<td>$2a_5a_6$</td>
</tr>
</tbody>
</table>

We now write down the submatrix (before symmetrization).
We explain how an inequality corresponds to a decorated ordered partition by the example of \((4', 2', 1)\). The decorated partition corresponds to the three fold product \(a_1 \cdot a_2 \cdot a_3 = a_0 = \text{top class in } N^3(G/P_2)\). Taking the Poincaré dual cycle, the above product corresponds to the intersection product \(X^0_{a_1} \cdot X^0_{a_2} \cdot X^0_{a_3} = X^0_{a_0}\). Using the correspondence between Weyl group elements in \(W_{P_2}\) and linear functionals arising from the Weyl group orbit of \((1, 1, 0)\), we find that the three Weyl group elements indexing the cycle in the intersection product correspond to the linear functionals \((1, 1, -1), (-1, 0, 1), (-1, 0, -1)\). Applying these linear functionals to \(a_1, a_2, a_3\) respectively and collecting terms one obtains the inequality \(a_0 = a_1 - a_2 + a_3\). This generates \((a_1 + a_2 + a_3 + a_0) \leq 0\) equivalently \(a_0 = a_1 - a_2 + a_3\).

The inequalities in the subspaces labelled \((v)\) corresponding to the ordered partitions \((1', 1', 3), (1', 2', 2), (1', 2', 2')\) are trivially redundant. They correspond to the ordered partitions \((5', 1', 1), (4', 2', 1)\) and \((3', 2', 2)\) do not occur in the system for \(P_2\). Consequently they too must be redundant. We now check this directly.

First the three inequalities corresponding to the decorated ordered partition \((3', 2', 2')\) are trivially redundant. In order to check that the three inequalities corresponding to \((5', 1', 1)\) are redundant, we observe that we have \(z_1 \leq z_1 + z_3\) from the first subsystem (corresponding to \(G/P_1\)). As a consequence we have \(z_1 \leq z_1 + z_3 + z_1 + z_2\). Finally to check that the six inequalities corresponding to \((4', 2', 1)\) are redundant, we observe that we have \(z_1 \leq z_1 + z_2\) from the first subsystem. Hence \(a_0 \leq a_1 + a_2 + a_3 + a_0\).
This subproblem (corresponding to $G(P_1)$) after symmetrization consists of 78 inequalities of which 21 are trivially redundant (marked by (**)). There are 5 more redundant inequalities marked by (**). These 9 inequalities do not occur in the system for $B_3$.

4.2.3. The subproblem associated to $H^*(G(P_3))$. The space $G(P_3)$ is the space of totally-isotropic 3-planes, i.e., the Lagrangian Grassmannian. The group $W_{P_3}$ is generated by the simple reflections $r$ and $s$. We leave the proof of the following lemma to the reader.

**Lemma 4.7.**

$W/P_3 = \{e, t, st, stt, stt\}$

We again abbreviate the classes $a_n$ to $a_0, a'_0$ or $a_0'$ according to the following table.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$\lambda_w$</th>
<th>$a_0$</th>
<th>$\text{PD}(X_{a_0})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$(1,1,1)$</td>
<td>1</td>
<td>$a_0$</td>
</tr>
<tr>
<td>$t$</td>
<td>$(1,1,-1)$</td>
<td>$a_1$</td>
<td></td>
</tr>
<tr>
<td>$st$</td>
<td>$(1,-1,1)$</td>
<td>$a_3$</td>
<td></td>
</tr>
<tr>
<td>$stt$</td>
<td>$(1,-1,-1)$</td>
<td>$a_1$</td>
<td></td>
</tr>
<tr>
<td>$trt$</td>
<td>$(-1,1,-1)$</td>
<td>$a_3$</td>
<td></td>
</tr>
<tr>
<td>$ttrt$</td>
<td>$(-1,1,-1)$</td>
<td>$a_3$</td>
<td></td>
</tr>
</tbody>
</table>

We again give the formulas for $p_n$ (mod 1). In the following table we define a symmetric cubic polynomial $f(x,y,z)$ by $f(x,y,z) = x^3 + y^3 + z^3 + x^2 y + y^2 z + y z^2$.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$v$</th>
<th>$w v w$</th>
<th>$p_n$ (mod 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$trt$</td>
<td>$trt$</td>
<td>$a_3$</td>
<td>$(y^2 z)$</td>
</tr>
<tr>
<td>$ttrt$</td>
<td>$ttrt$</td>
<td>$a_3$</td>
<td>$(y^2 z(x + y + z))$</td>
</tr>
<tr>
<td>$stt$</td>
<td>$stt$</td>
<td>$a_3$</td>
<td>$(y^2 z(x + y + z))$</td>
</tr>
<tr>
<td>$stt$</td>
<td>$ttrt$</td>
<td>$a_3$</td>
<td>$(y^2 z(x + y + z))$</td>
</tr>
<tr>
<td>$t$</td>
<td>$st$</td>
<td>$x + y + z$</td>
<td></td>
</tr>
</tbody>
</table>

We now have the following table constructed by multiplying the polynomials $p_n$ modulo 1.

**Theorem 4.8.** The cohomology ring $H^*(G(P_3))$ is given by the following table.

<table>
<thead>
<tr>
<th>$B^*(G(P_3))$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$2a_2$</td>
<td>$2a_3 + a_1$</td>
<td>$a_4$</td>
<td>$2a_5$</td>
<td>$2a_3$</td>
<td>$a_6$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$2a_4$</td>
<td>$a_5$</td>
<td>$a_3$</td>
<td>$3a_3$</td>
<td>$a_6$</td>
<td>$0$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$0$</td>
<td>$a_6$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Using the above tables, the reader can easily verify that we have the following inequalities (which have to be symmetrized).
\[ v_1 + y_1 + z_1 \leq x_2 + y_2 + z_2 + x_3 + y_3 + z_3 \quad (6, 0, 0) \]
\[ v_1 + y_1 + z_2 \leq x_1 + x_2 + y_1 + x_3 + y_2 + z_3 \quad (5, 1, 0) \]
\[ x_1 + x_2 + y_2 \leq v_1 + z_1 + x_2 + y_3 + z_3 \quad (4, 2, 0) \]
\[ x_1 + z_1 + x_2 \leq y_1 + x_2 + x_3 + y_2 + z_3 \quad (3, 3, 0) \]
\[ y_1 + y_2 + z_1 \leq v_1 + y_1 + x_2 + x_2 + z_3 \quad (3', 2, 1) \]

This gives that the subsystem corresponding to \( G/P_3 \) consists of 27 inequalities. None of them are trivially redundant.

The 27 inequalities above can be rewritten in a very simple way. Let \( S = \sum_{i=1}^{3} x_i + y_i + z_i \). Then the 27 inequalities are just the inequalities

\[ x_i + y_j + z_k \leq \frac{S}{3}, \quad i, j, k = 1, 2, 3. \]

Thus finally we see that for \( C_3 \), there are 155 = 126 + 9 inequalities of which 24 are trivially redundant (9 in 126 + 9 coming from the inequalities defining \( A^3 \) inside \( A^3 \)). There are 9 more redundant inequalities. These 9 inequalities do not occur in the system for \( B_3 \). Hence the subsystem for \( C_3 \) can be brought down to altogether 102 inequalities. Moreover, a computer calculation shows that the polyhedral cone \( \Delta_{B_3}(C_3) \) has exactly 102 faces and thus these 102 inequalities are irredundant.

4.3. The inequalities for \( B_3 \). In this subsection, we take simply-connected \( G \) of type \( B_3 \), i.e., \( G = \text{Spin}(7) \).

We note that the Weyl chamber \( \Delta \) is given by triples \( x, y, z \) of real numbers satisfying

\[ v \geq y \geq z \geq 0. \]

The inequalities will now be in terms of \( (v_1, v_2, v_3) \) \( \in \Delta^3 \) where \( v_1 = (x_1, y_1, z_1), i = 1, 2, 3. \)

The Weyl groups for \( \text{Sp}(6) \) and \( \text{Spin}(7) \) are isomorphic. In fact, in the standard coordinates \( (x, y, z) \) they are identical. Hence the sets \( W_i, i = 1, 2, 3 \) will be identical. However, the generators \( A_{ij} \) and the polynomials \( p_{ij} \) will be different (but proportional) as we now see. To distinguish, we will denote them by \( p_{ij}^{\text{Spin}(7)} \) and \( p_{ij}^{\text{Sp}(6)} \), respectively.

We will need the following proposition.

**Proposition 4.9.**

\[ p_{ij}^{\text{Spin}(7)} = \frac{x^i y^j (xy)}{8} \quad \text{mod} \ \mathbb{Z}. \]

**Proof.** By equation (6) we have

\[ p_{ij}^{\text{Spin}(7)} = xy(x^2 - y^2)(x^2 - z^2)(y^2 - z^2). \]

Hence the proposition follows from the corresponding result for \( \text{Sp}(6) \).

The next lemma tells us how to read off the polynomials \( p_{ij} \) for the case of \( \text{Spin}(7) \) from the corresponding polynomials for \( \text{Sp}(6) \). We will temporarily use the
notation $A^{2w}(7)$ and $A^{2n}(6)$ for the Demazure-BGG operators $A_w$ for the groups Spin(7) and Spin(6) respectively.

**Lemma 4.10.** Let $v \in W$ and let $n(v; t)$ be the number of times the simple reflection $t$ occurs in some reduced decomposition of $v$. Then we have

$$A^{2n(v; t)}(7) = 2^{n(v; t)} A^{2n(v; t)}(6).$$

**Proof.** $A^{2n(v; t)}(7) = A^{2n(v; t)}(6)$ and $A^{2n(v; t)}(6) = A^{2n(v; t)}(6)$. □

**Corollary 4.11.** For any $w \in W$, $p_w^{2\operatorname{Spin}(7)} = 2^{-n(w; v)} p_w^{2\operatorname{Spin}(6)} = 2^{-n(w; v)} p_w^{2\operatorname{Spin}(6)}$.

**Proof.** Note that $n(w; v) = n(v; t)$, for $v = v^{-1}$.

**4.3.1.** The subsystem associated to $H^*(G/P)$. We note that $W_t$ is the group generated by $e$ and $t$. We have (since $w$ what we had for $Sp(6)$).

**Lemma 4.22.** $W_t = \{ e, r, s, st, star, star, star \}$.

We have $G/P \approx Q_8$, the smooth quadric hypersurface in $CP^6$ so the inequalities will be parametrized by a subset of the ordered partitions of 5. We will abbreviate the classes $e_w$ to $e_w$ according to the following table. In addition, we list the elements of $W_t$, their lengths, the maximally singular weight $\lambda_0$ associated to the Schubert cycle $X_{\lambda_0} = X_{\lambda_0}$ and the notation for the cohomology class $PD(X_{\lambda_0})$.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$l(w)$</th>
<th>$\lambda_0$</th>
<th>$e_w$</th>
<th>$PD(X_{\lambda_0})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>0</td>
<td>$e$</td>
<td>1</td>
<td>$a_0$</td>
</tr>
<tr>
<td>$r$</td>
<td>1</td>
<td>$(0,0,0)$</td>
<td>$a_1$</td>
<td>$a_1$</td>
</tr>
<tr>
<td>$s$</td>
<td>2</td>
<td>$(0,0,0)$</td>
<td>$a_1$</td>
<td>$a_1$</td>
</tr>
<tr>
<td>$st$</td>
<td>3</td>
<td>$(0,0,-1)$</td>
<td>$a_1$</td>
<td>$a_2$</td>
</tr>
<tr>
<td>$star$</td>
<td>4</td>
<td>$(0,-1,0)$</td>
<td>$a_4$</td>
<td>$a_4$</td>
</tr>
<tr>
<td>$star$</td>
<td>5</td>
<td>$(-1,0,0)$</td>
<td>$a_0$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

The following theorem follows easily from Corollary 4.11 by using the corresponding theorem for $Sp(6)$.

**Theorem 4.13.** The multiplication table for $H^*(G/P)$ is given by the following table.

$$H^*(G/P) \cong \begin{array}{cccccc} a_0 & a_2 & a_0 & a_4 & a_4 & a_0 \\ a_0 & 2a_2 & a_0 & a_4 & a_4 & a_0 \\ a_2 & a_2 & 2a_4 & a_4 & a_4 & a_0 \\ a_4 & a_4 & a_4 & 2a_0 & a_0 & a_0 \\ a_0 & a_4 & a_4 & a_0 & 2a_2 & a_0 \\ a_0 & a_0 & a_0 & a_0 & a_0 & 2a_0 \end{array}$$

We now give the corresponding subsystem, leaving to the reader the task of symmetrizing the inequalities below.

$$x_1 \leq x_2 + x_3 \quad (5,0,0)$$
$$x_1 \leq x_2 + x_3 \quad (4,1,0)$$
$$x_1 \leq x_2 + x_3 \quad (3,2,0)$$
$$x_1 \leq x_2 + x_3 \quad (3,1,1)$$

**After symmetrizing there are 18 inequalities, none are trivially redundant.**
4.3.2. The subsystem associated to $H^*(G/P_2)$. The space $G/P_2$ is the space of totally isotropic 2-planes. We have that $W_P$ is the group generated by the commuting simple reflections $r$ and $t$. We have, as for $Sp(6)$,

**Lemma 4.14.** $W_P = \{ e, s, r_s, t_s, r_s t_s, r_t s, r_t s t_s, r_0 t_s, s t s, r_0 t_s r_t s, s t s r_t s, r_t s t s, r_0 t_s r_t s t_s \}$.

We will abbreviate the classes $e_u$ by $b_0$ or $b'$ or $b''$ according to the following table.

The following table follows easily from the corresponding tables for $Sp(6)$ and Corollary 4.11.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$\lambda_w$</th>
<th>$\xi(w)$</th>
<th>$\pi_u (\text{mod } 1)$</th>
<th>$PD(X_{\lambda_w})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$(1, 1, 0)$</td>
<td>6</td>
<td>1</td>
<td>$b_0$</td>
</tr>
<tr>
<td>$s$</td>
<td>$(1, 0, 1)$</td>
<td>1</td>
<td>$a + y$</td>
<td>$b_0$</td>
</tr>
<tr>
<td>$r_s$</td>
<td>$(0, 1, 1)$</td>
<td>2</td>
<td>$x_1$</td>
<td>$b'$</td>
</tr>
<tr>
<td>$t_s$</td>
<td>$(1, 0, 1)$</td>
<td>2</td>
<td>$1/2 x_2$</td>
<td>$b'$</td>
</tr>
<tr>
<td>$r_t s$</td>
<td>$(0, 1, 0)$</td>
<td>3</td>
<td>$1/2 x y(a + y)$</td>
<td>$b_0$</td>
</tr>
<tr>
<td>$s t s$</td>
<td>$(0, -1, 0)$</td>
<td>3</td>
<td>$1/2 x y_2$</td>
<td>$b''$</td>
</tr>
<tr>
<td>$a r t_s t_s$</td>
<td>$(0, 0, 0)$</td>
<td>4</td>
<td>$1/2 (c y)^2$</td>
<td>$b_0$</td>
</tr>
<tr>
<td>$r a r t_s t_s$</td>
<td>$(0, -1, 0)$</td>
<td>5</td>
<td>$1/4 x y_2 (c + y)$</td>
<td>$b''$</td>
</tr>
<tr>
<td>$r t a r t_s t_s$</td>
<td>$(0, 0, 0)$</td>
<td>6</td>
<td>$1/4 (c y)^2$</td>
<td>$b_0$</td>
</tr>
<tr>
<td>$a r t_s a r t_s$</td>
<td>$(0, -1, 0)$</td>
<td>7</td>
<td>$1/4 (c y)^2 (a + y)$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

From these formulae one can easily construct the multiplication table for $H^*(G/P_2)$.

**Theorem 4.15.** The cohomology ring $H^*(G/P_2)$ is given by the following table.

<table>
<thead>
<tr>
<th>$H^*(G/P_2)$</th>
<th>$b_0$</th>
<th>$b'_0$</th>
<th>$b''_0$</th>
<th>$b_0 + b'_0$</th>
<th>$b_0 + b''_0$</th>
<th>$b_0 + b'_0 + b''_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_0$</td>
<td>$b_0 + 2b_0'$</td>
<td>$b_0 + b'_0$</td>
<td>$b_0 + b''_0$</td>
<td>$b_0 + b'_0 + b''_0$</td>
<td>$b_0 + 2b_0'$</td>
<td></td>
</tr>
</tbody>
</table>

We then read off the inequalities.
\[ x_1 + y_1 \leq x_2 + y_2 + x_3 + y_3 = (7,0,0) \]
\[ x_1 + x_2 \leq x_2 + x_3 + y_3 = (6,1,0) \]
\[ y_1 + z_1 \leq y_2 + z_2 + x_3 + y_3 = (5',2',0) \]
\[ x_1 + x_2 \leq x_1 + x_3 + z_3 + y_3 = (5',3',0) \]
\[ x_1 + y_2 \leq x_1 + z_2 + x_3 + y_3 = (4',3',0) \]
\[ y_1 + z_1 \leq y_2 + z_2 + x_3 + z_3 = (5',1,1) \]
\[ y_1 + z_2 \leq y_1 + z_2 + x_3 + z_3 = (4',2',1) \]
\[ z_1 + z_2 \leq y_1 + z_2 + x_3 + z_3 = (4',3',1) \]
\[ (\times) \quad y_1 + z_2 \leq x_1 + z_2 + y_3 + z_3 = (3',2',2') \]
\[ (\times) \quad x_1 + z_2 \leq y_1 + z_2 + y_3 + z_3 = (3',2',2') \]
\[ (\times) \quad x_1 + z_2 + 2z \leq y_1 + z_2 + x_2 + z_3 = (3',2',2') \]
\[ (\times) \quad y_1 + z_2 + 2z \leq x_1 + z_2 + x_2 + z_3 = (3',2',2') \]

The inequalities corresponding to the ordered partitions \((3',3,1)\), \((3',3',2'),\) \((3',2',2'),\) and \((3',2',2')\) are trivially redundant. After symmetrising they give rise to 24 trivially redundant inequalities. The trivially redundant inequalities corresponding to the decorated ordered partitions \((3',2',2')\) and \((3',2',2')\) do not occur in the system for \(Sp(6)\). There are 72 inequalities after symmetrising, of which 24 are trivially redundant.

4.33. The subsystem associated to \(H^3(G/P_8)\). The space \(G^1/P_8\) is the space of totally-isotropic 3-planes in \(G^1\). The group \(W_{20}\) is generated by the simple reflections \(s\) and \(t\). We have the following from the corresponding results for \(Sp(6)\).

**Lemma 4.16**

\[ W_{20} = \{ e, f, s, s f, t, t sf, t s f, t s r f, t r f, t r s f \}. \]

We again abbreviate the chains \(s f\) to \(b_1\)'s or \(b_2\)'s or \(b_3\)'s according to the following table.
Theorem 4.17. The cohomology ring is given by the following table.

<table>
<thead>
<tr>
<th>H^i(G/P_1)</th>
<th>b_i</th>
<th>b_j</th>
<th>b_k</th>
<th>b_l</th>
<th>b_m</th>
<th>b_n</th>
<th>b_o</th>
</tr>
</thead>
<tbody>
<tr>
<td>b_1</td>
<td>b_2</td>
<td>b_3</td>
<td>b_4</td>
<td>b_5</td>
<td>b_6</td>
<td>b_7</td>
<td>b_8</td>
</tr>
<tr>
<td>b_9</td>
<td>b_10</td>
<td>b_11</td>
<td>b_12</td>
<td>b_13</td>
<td>b_14</td>
<td>b_15</td>
<td>b_16</td>
</tr>
<tr>
<td>b_17</td>
<td>b_18</td>
<td>b_19</td>
<td>b_20</td>
<td>b_21</td>
<td>b_22</td>
<td>b_23</td>
<td>b_24</td>
</tr>
</tbody>
</table>

In this case we find the following subsystem of linear inequalities:

\[ x_1 + y_1 + z_1 \leq x_2 + y_2 + z_2 + x_3 + y_3 + z_3 \quad (6,6,0) \]

\[ x_1 + y_1 + z_2 \leq x_3 + y_3 + z_3 + x_4 + y_4 + z_4 \quad (5,1,6) \]

\[ z_1 + y_1 + z_3 \leq y_3 + z_3 + x_3 + z_3 + x_3 + y_3 \quad (4,2,0) \]

\[ x_1 + x_2 + x_3 \leq y_1 + x_1 + x_2 + x_3 + z_3 + y_3 \quad (3,3,0) \]

\[ \text{(**) } x_1 + x_2 + x_3 + x_4 \leq x_5 + y_5 + z_5 + x_5 + y_5 + z_5 \quad (4,1,1) \]

\[ x_1 + y_1 + x_1 \leq y_1 + x_1 + z_1 + x_2 + y_2 + z_2 \quad (3',2,1) \]

\[ \text{(**) } y_1 + y_2 + z_3 + x_1 + x_2 + y_2 + z_3 + y_3 \quad (3',2,1) \]

After symmetrizing there are 36 inequalities. None are trivially redundant. However the 9 inequalities corresponding to the ordered partitions \((4,1,1)\) and \((3',2,1)\) and marked by (**) above do not occur for Sp(6) and consequently they must be redundant.

We now check this directly.

In order to see that the 9 inequalities corresponding to \((4,1,1)\) are redundant, we observe (from the first subsystem corresponding to G/P_1) that \(x_1 \leq x_2 + x_3\). Furthermore, we have the inequalities for \(\Delta_1\) given by \(x_1 \leq y_1, 1 \leq i \leq 3\). Hence \(x_1 + x_2 + x_3 \leq y_1 + x_2 + y_2 + x_3 + y_3\). As for the 6 inequalities corresponding to \((3',2,1),\) we have (from the first subsystem) \(x_1 \leq x_2 + x_3\) and the inequalities (for \(\Delta_1\)) \(y_1 \leq x_1, y_2 \leq x_2,\) and \(x_3 \leq y_3\). Hence \(x_1 + y_1 + x_2 + y_2 + x_2 + y_2 + x_3 + y_3\).

To summarize, for \(\Delta_1\), there are altogether 135 = 126 + 9 inequalities (including 9 needed to define \(\Delta^2\) in \(\mathbb{Z}^3\)) of which 24 are trivially redundant and there are 9 more redundant inequalities. These 9 inequalities do not occur in the system for Sp(6).

Hence the subsystem for \(\Delta_2\) can be brought down to altogether 102 inequalities. Moreover, a computer calculation shows that the polyhedral cone \(D_1(\Delta_2)\) has exactly 102 faces and thus these 102 inequalities are irredundant. Of course, by Theorem 1.8, \(D_1(\Delta_2) = D_1(G_2)\).
5. Generators of the cone

In the previous section we have described the irredundant system of linear inequalities defining the polyhedral cones $D_3(A_3)$ and $D_3(C_3) = D_3(B_3)$. We now give a system of generators for the cone $D_3(C_3)$. The components of each of the 51 generators are arranged in the order $z_1, z_2, z_3, y_1, y_2, y_3, x_1, x_2, x_3$ whereas the coordinates of the corresponding generator $(v_1, v_2, v_3) \in D_3(C_3)$ satisfy

\[ v_i = (x_i, y_j, z_k), 1 \leq i \leq 3. \]

**Theorem 5.1.** The following 51 vectors are a set of generators of the polyhedral cone $D_3(C_3) = D_3(B_3)$ in the 9-dimensional space $\mathbb{R}^9$.

\[
\begin{align*}
(1) & \quad 110110000 \\
(2) & \quad 101111000 \\
(3) & \quad 110101000 \\
(4) & \quad 111010100 \\
(5) & \quad 111011000 \\
(6) & \quad 101101000 \\
(7) & \quad 101101100 \\
(8) & \quad 110101000 \\
(9) & \quad 110110000 \\
(10) & \quad 111001000 \\
(11) & \quad 110101100 \\
(12) & \quad 110011000 \\
(13) & \quad 110110000 \\
(14) & \quad 101101000 \\
(15) & \quad 111010100 \\
(16) & \quad 101000000 \\
(17) & \quad 111001100 \\
(18) & \quad 111011100 \\
(19) & \quad 111100000 \\
(20) & \quad 101111100 \\
(21) & \quad 110110000 \\
(22) & \quad 111011000 \\
(23) & \quad 111100000 \\
(24) & \quad 110000000 \\
(25) & \quad 111110110 \\
(26) & \quad 111111100 
\end{align*}
\]
References


MATHEMATISCHES INSTITUT, UNIVERSITÄT MÜNCHEN, THIERSBERGSTRASSE 29 D-80333 MÜNCHEN, DEUTSCHLAND

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742, USA.

E-mail addresses: shramov@math.umd.edu