Equivariant Analogue of Grothendieck's Theorem for Vector Bundles on $\mathbb{P}^1$

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To Professor C.S. Seshadri on his seventieth birthday

ABSTRACT. Let T be a complex torus acting algebraically on $\mathbb{P}^1(\mathbb{C})$. In this note we prove a T-equivariant analogue of a theorem of Grothendieck. More specifically, we show that any T-equivariant vector bundle on $\mathbb{P}^1(\mathbb{C})$ is a direct sum of T-equivariant line subbundles.

The aim of this note is to prove the following equivariant analogue of Grothendieck's theorem, existence of which was asked by W. Fulton.

Let T be a complex (connected) torus acting algebraically on $X = \mathbb{P}^1(\mathbb{C})$.

THEOREM. *Let E be a T-equivariant algebraic vector bundle on X. Then there exist T-equivariant line subbundles $L_1, \cdots, L_m \subset E$ such that we have a T-equivariant vector bundle isomorphism:*

$$\begin{array}{ccc} L_1 \oplus \cdots \oplus L_m & \xrightarrow{\sim} & E \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^1, & \end{array}$$

under $v_1 \oplus \cdots \oplus v_m \mapsto v_1 + \cdots + v_m$.

PROOF. As is well known (cf. [2; Exercise 6.6, Chapter I]), the group of algebraic automorphisms of X can be identified with the projective linear group PGL(2) via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [z_1, z_2] = [az_1 + bz_2, cz_1 +$

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$dz_2]$. Let $D \subset \mathrm{PGL}(2)$ be the standard diagonal subgroup. Since D fixes the point at infinity $[1, 0]$, the action of D on X lifts to give a D-equivariant line bundle structure on $\mathcal{O}_X(1)$ (and hence on any $\mathcal{O}_X(n), n \in \mathbb{Z}$). Thus $\mathcal{O}_X(n)$ acquires a T-equivariant line bundle structure.

By virtue of Grothendieck's theorem (cf. [1; Théorème 2.1]), one can decompose $E \cong \mathcal{O}_X(n_1) \oplus \cdots \oplus \mathcal{O}_X(n_m)$ with $n_1 \geq n_2 \geq \cdots \geq n_m$. Tensoring E with $\mathcal{O}_X(-n_1)$ and putting a T-equivariant line bundle structure on $\mathcal{O}_X(-n_1)$, we can assume that $E \simeq \mathcal{O}_X \oplus \mathcal{O}_X(n_2) \oplus \cdots \oplus \mathcal{O}_X(n_m)$ with each $n_2, \cdots, n_m \leq 0$. Thus any $0 \neq \sigma \in H^0(X, E)$ is nowhere vanishing. Further, $H^0(X, E) \neq 0$. Of course, $H^0(X, E)$ is a T-module. Take a T-eigenvector $0 \neq \sigma \in H^0(X, E)$. Then $\{\mathbb{C}\sigma(x)\}_{x \in \mathbb{P}^1}$ is a T-equivariant line subbundle, denoted L_1, of E.

We get an exact sequence of T-equivariant bundles:

$$(*) \qquad 0 \to L_1 \to E \to E/L_1 \to 0.$$

We claim that this sequence is T-equivariantly split:

Consider the T-module $\mathrm{Hom}_{\mathcal{O}}(E/L_1, E)$ of bundle morphisms. Then we get a natural T-module map

$$\pi : \mathrm{Hom}_{\mathcal{O}}(E/L_1, E) \twoheadrightarrow \mathrm{Hom}_{\mathcal{O}}(E/L_1, E/L_1).$$

Observe that all the line bundles, occurring in E/L_1 as direct summands, have degrees ≤ 0. This can be seen, e.g., by tensoring the sequence $(*)$ with $\mathcal{O}_X(-1)$ and then considering the associated long exact cohomology sequence. Thus $(*)$ splits as vector bundles by [2; Propositions 6.3 and 6.7, Chap. III] (without regarding the T-equivariance). In particular, π is surjective. Thus π induces a surjective map

$$\pi^T : \mathrm{Hom}_{\mathcal{O}}(E/L_1, E)^T \twoheadrightarrow \mathrm{Hom}_{\mathcal{O}}(E/L_1, E/L_1)^T,$$

where the superscript T means T-invariants. Take a preimage f of the identity homomorphism $\mathbf{I}$ under π^T. This f provides a T-equivariant splitting of $(*)$. Thus $E \simeq L_1 \oplus E/L_1$ as T-equivariant bundles. So, by induction on rank E, the theorem follows. □

References

[1] Grothendieck, A.: Sur la classification des fibres holomorphes sur la sphere de Riemann, *Amer. J. Math.* *79* (1957), 121–138.

[2] Hartshorne, R.: *Algebraic Geometry*, GTM 52, Springer-Verlag, 1977.

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