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A CONJECTURAL GENERALIZATION OF THE $n!$ RESULT TO ARBITRARY GROUPS

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Dedicated to Prof. M. S. Raghunathan on his sixtieth birthday

Abstract. The aim of this paper is to formulate a conjecture for an arbitrary simple Lie algebra $\mathfrak{g}$ in terms of the geometry of principal nilpotent pairs. When $\mathfrak{g}$ is specialized to sl_n, this conjecture readily implies the $n!$ result and it is very likely that, in fact, it is equivalent to the $n!$ result in this case. In addition, this conjecture can be thought of as generalizing an old result of Kostant. In another direction, we show that to prove the validity of the $n!$ result for an arbitrary n and an arbitrary partition of n, it suffices to show its validity only for the staircase partitions.

1. Introduction

To any partition $\sigma : \sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_m > 0$ of a positive integer n, one associates an element Δ_σ of the complex polynomial ring $R_n := \mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ in $2n$-variables by defining it (up to a sign) to be the determinant of the matrix $(X_s^{i_t} Y_s^{j_t})_{1 \leqslant s,t \leqslant n}$ for some enumeration $\{(i_1, j_1), \ldots, (i_n, j_n)\}$ of the elements in $D_\sigma := \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : i < \sigma_j\}$. (We could think of Δ_σ as a generalization of the Vandermonde determinant.) Let A_σ be the quotient of the ring Diff_{2n} of differential operators on R_n with constant coefficients by the ideal I_σ consisting of the elements mapping Δ_σ to zero.

A. Garsia and M. Haiman [12] conjectured that A_σ should have complex dimension $n!$. This conjecture later became known as the *$n!$-conjecture*. The interest in the $n!$-conjecture came from its connection with the positivity conjecture by Macdonald, stating that the Kostka–Macdonald coefficients $K_{\lambda,\mu}(q,t)$ should be polynomials in q and t with nonnegative integer coefficients. Relating the coefficients of $K_{\lambda,\mu}(q,t)$ to the dimension of certain bigraded parts of A_σ, Garsia and Haiman found that the $n!$-conjecture would imply the above positivity conjecture.

In spite of its simple appearance, the $n!$-conjecture turned out to be much harder to prove than first anticipated, and only recently M. Haiman [16] was able to give the first, and until now the only proof of the conjecture using the geometry of the Hilbert scheme of n-points in $\mathbb{C}^2$ (following some suggestions of C. Procesi). Unfortunately, his proof is quite complicated which makes it less accessible.

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The aim of this paper is to relate the $n!$-conjecture with the geometry of principal nilpotent pairs $\mathbf{e}$. More specifically, let $\mathfrak{g}$ be a simple Lie algebra (corresponding to a simple adjoint group G) with a Cartan subalgebra $\mathfrak{h}$ and let $\mathbf{e} \in \mathfrak{g}^d$ be a principal nilpotent pair in $\mathfrak{g}^d := \mathfrak{g} \oplus \mathfrak{g}$ (a notion defined recently by Ginzburg, cf. Definition 1). We associate to $\mathbf{e}$ a subgroup $\mathcal{G}^{\mathbf{e}}$ of finite index of the full automorphism group $\mathcal{G}$ of $\mathfrak{g}$ and consider the scheme-theoretic intersection of the $\mathcal{G}^{\mathbf{e}}$-orbit closure $\bar{\mathcal{O}}_{\mathbf{e}}$ of $\mathbf{e}$ in $\mathfrak{g}^d$ with $\mathfrak{h}^d$, where $\mathcal{G}$ acts on $\mathfrak{g}^d$ diagonally via its adjoint action. Then one of the main conjectures of this paper asserts that the scheme $\bar{\mathcal{O}}_{\mathbf{e}} \cap \mathfrak{h}^d$ is Gorenstein. In addition, we conjecture that the affine coordinate ring $\mathfrak{D}_{\mathbf{e}} := \mathbb{C}[\bar{\mathcal{O}}_{\mathbf{e}} \cap \mathfrak{h}^d]$ supports the regular representation of W; in particular, it is of dimension $|W|$ (cf. Conjecture 2). It is shown (cf. Proposition 3) that the above conjecture for $\mathfrak{g} = sl_n$ readily implies the $n!$ result by using a result of de Concini and Procesi on the cohomology of Springer fibers and a simple characterization of the algebra A_σ in terms of the cohomology of Springer fibers (cf. Theorems 3, 4). In fact, by using the deformations of unstable orbits to semistable orbits (cf. Section 4), we show in Proposition 3 that the (conjectural) Gorenstein property of the ring $\mathfrak{D}_{\mathbf{e}}$ in the case of $\mathfrak{g} = sl_n$ already implies the $n!$ result provided we use a certain variant of a conjecture by Ginzburg (cf. Conjecture 1). Thus our conjecture provides a generalization of the $n!$ result to an arbitrary simple $\mathfrak{g}$ (the $n!$ result being viewed as a result for the case of the Lie algebra sl_n).

In another direction, using the geometry of Hilbert schemes and Borel's fixed point theorem, we show that the validity of the $n!$ result for an arbitrary n and an arbitrary partition of n follows from its validity for just the staircase partitions $\mathfrak{s}_m : m \geqslant m-1 \geqslant \ldots \geqslant 1 > 0$ for any positive integer m (cf. Theorem 7).

Let $\mathbb{D}_n^{\mathbb{C}}(\triangle_n)$ be the subspace of the polynomial ring $P_n := \mathbb{C}[X_1, \ldots, X_n]$ in n variables over $\mathbb{C}$ obtained by applying all the constant coefficient differential operators to the Vandermonde determinant Δ_n. Then it is a well-known result that $\mathbb{D}_n^{\mathbb{C}}(\triangle_n)$ is of dimension $n!$. Now let p be a prime such that $n \leqslant p^2$ and let k be a field of char. p. Then we conjecture that the space $\mathbb{D}_n^k(\triangle_n)$ over k obtained by applying all *divided* differential operators with constant coefficients to Δ_n is again of dimension $n!$ (cf. Conjecture 3). This conjecture immediately implies the validity of $n!$ result for $n = p^2$ and the box partition $\mathfrak{b}_p : p \geqslant \ldots \geqslant p$ of n.

For $n = tm + r$ with $t > r \geqslant 0, t > 1$ and $m \geqslant 1$, consider the *box plus one row partition* $\sigma = \sigma(t, m, r) : \sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_m > 0$ of n with $\sigma_i = t$ for $0 \leqslant i \leqslant m-1$ and $\sigma_m = r$. Finally, in Section 8, we realize the algebra A_σ for any such partition as the associated graded algebra of the cohomology algebra with complex coefficients $H^*(\mathrm{SL}_n/B, \mathbb{C})$ with respect to a certain specific filtration depending solely on t (cf. Corollary 2), where B is a Borel subgroup of SL_n consisting of the upper triangular matrices of determinant one. The proof of this corollary relies on a description of the cohomology of Springer fibers due to de Concini and Procesi which we have included in the appendix. In addition, we have included a different description of the cohomology of Springer fibers due to Tanisaki in the appendix, which has been used in the proof of Theorem 4.

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2. Preliminaries

Let n be a positive integer and let $\sigma : \sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_m > 0$ be a partition of n. The *diagram* of σ is the array of lattice points:

$$D_\sigma := \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : i < \sigma_j\},$$

where $\mathbb{Z}_+ := \{0, 1, \ldots\}$ is the set of nonnegative integers. We order the elements of D_σ in any manner as $\{1, \ldots, n\}$ and let $s \in \{1, \ldots, n\}$ correspond to the lattice point (i_s, j_s). Define a generalization of the Vandermonde determinant as the polynomial in $2n$ variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ with integer coefficients:

$$\triangle_\sigma := \det\big[X_s^{i_t} Y_s^{j_t}\big]_{1 \leqslant s,t \leqslant n}.$$

Clearly a different choice of the ordering of D_σ will give rise to the same $\triangle_\sigma$ up to a sign.

Let Diff_{2n} be the algebra $\mathbb{C}[\partial_{X_1}, \ldots, \partial_{X_n}, \partial_{Y_1}, \ldots, \partial_{Y_n}]$ of all differential operators in $2n$ variables with the constant coefficients in the field of complex numbers $\mathbb{C}$, where $\partial_{X_i} := \partial/\partial X_i, \partial_{Y_i} := \partial/\partial Y_i$. Let $\mathrm{Diff}_{2n}(\triangle_\sigma)$ be the space $\{D(\triangle_\sigma) : D \in \mathrm{Diff}_{2\mathrm{n}}\}$. The following celebrated conjecture was made by Garsia and Haiman [12] and proved recently by Haiman [16].

We will refer to the following result as $n!$ *result for* σ.

Theorem 1. ($n!$ result for σ) *For any positive integer n and partition σ of n,*

$$\dim_{\mathbb{C}} \mathrm{Diff}_{2\mathrm{n}}(\triangle_\sigma) = \mathrm{n}!.$$

2.1. Hilbert schemes

The Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$ of n points in $\mathbb{C}^2$ parameterizes the set of ideals I in $\mathbb{C}[X,Y]$ whose associated quotient ring has complex dimension n. By a result of Fogarty [11], the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$ is a smooth irreducible variety of dimension $2n$. Any partition σ of n defines a point $I_\sigma \in \mathrm{Hilb}^n(\mathbb{C}^2)$ by letting I_σ denote the ideal spanned by the monomials $\{X^i Y^j : (i,j) \notin D_\sigma\}$. Thanks to Procesi and Haiman, the $n!$ result can be reformulated in geometric terms using $\mathrm{Hilb}^n(\mathbb{C}^2)$ as follows.

There is a "tautological" vector bundle $\mathfrak{T}$ with the scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$ as base, where the fiber over any $I \in \mathrm{Hilb}^n(\mathbb{C}^2)$ is the finite-dimensional $\mathbb{C}$-vector space (in fact, a $\mathbb{C}$-algebra) $\mathbb{C}[X,Y]/I$. Now consider the map of sheaves $\mathfrak{T}^{\otimes n} \otimes \mathfrak{T}^{\otimes n} \to \wedge^n(\mathfrak{T})$, defined by $(a_1 \otimes \ldots \otimes a_n) \otimes (b_1 \otimes \ldots \otimes b_n) \mapsto (a_1 b_1) \wedge \ldots \wedge (a_n b_n)$ for any $a_i, b_i \in \mathbb{C}[X,Y]/I$ (which is the fiber over I). This map clearly induces the map

$$\beta : \mathfrak{T}^{\otimes n} \to (\mathfrak{T}^{\otimes n})^* \otimes \wedge^n(\mathfrak{T}).$$

Identifying the polynomial ring $R_n := \mathbb{C}[X_1, \dots, X_n, Y_1, \dots, Y_n]$ with the ring $\mathbb{C}[X, Y]^{\otimes n}$, we can view the fiber $(\mathbb{C}[X, Y]/I)^{\otimes n}$ of the vector bundle $\mathfrak{T}^{\otimes n}$ over I as a quotient of R_n. Let $\bar{K}_\sigma$ be the kernel of β over I_σ and let K_σ denote the associated preimage in R_n. An easy calculation then tells us (see [16]):

$$K_\sigma = \{f \in R_n : f(\partial_{X_1}, \dots, \partial_{X_n}, \partial_{Y_1}, \dots, \partial_{Y_n})(\triangle_\sigma) = 0\}.$$

In particular, defining the algebra

$$A_\sigma := R_n/K_\sigma,$$

we find that $A_\sigma \simeq \mathrm{Diff}_{2\mathrm{n}}(\triangle_\sigma)$ as vector spaces. Hence the $n!$ result is equivalent to the statement that the map β restricted to the fiber over I_σ has rank $n!$ for all the partitions σ of n.

It is not difficult to see that the rank of β over the fiber of any $I \in \mathrm{Hilb}^n_{\mathrm{vg}}(\mathbb{C}^2)$ is equal to $n!$, where $\mathrm{Hilb}^n_{\mathrm{vg}}(\mathbb{C}^2) \subset \mathrm{Hilb}^n(\mathbb{C}^2)$ is the open dense subset consisting of those ideals whose zero set consists of n distinct points. Using that the rank of β along fibers is a lower semi-continuous function on $\mathrm{Hilb}^n(\mathbb{C}^2)$, we conclude that the rank of β along any fiber is at most $n!$. Furthermore, any point in $\mathrm{Hilb}^n(\mathbb{C}^2)$, by using the natural action of the torus $(\mathbb{C}^*)^2$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$, can be deformed into a point of the form I_σ for some partition σ of n. Applying semi-continuity once more, this provides us with the following reformulation of the $n!$ result.

Theorem 2. *For any positive integer n and partition σ of n, the rank of β over the fiber of I_σ is equal to $n!$. Thus,* $\dim A_\sigma = n!$.

Further, the validity of the $n!$ result for all partitions of n is equivalent to the assertion that the rank of β over any fiber of $\mathrm{Hilb}^n(\mathbb{C}^2)$ *is constant* (*which is automatically equal to $n!$*).

The algebra A_σ will play a fundamental role in the paper. Observe that the algebra A_σ is a bigraded algebra, where the two gradings come respectively from the total degree in the variables $X_1, \dots, X_n$ and the total degree in the variables $Y_1, \dots, Y_n$. Moreover, there is a S_n-action on A_σ preserving the bidegrees coming from the (diagonal) action of S_n on the polynomial ring $\mathbb{C}[X_1, \dots, X_n, Y_1, \dots, Y_n]$, given by $\theta(X_i) = X_{\theta(i)}$ and $\theta(Y_i) = Y_{\theta(i)}$, for any $\theta \in \mathrm{S}_n$.

We say that a point $I \in \mathrm{Hilb}^n(\mathbb{C}^2)$ (or alternatively the algebra $\mathbb{C}[X, Y]/I$; or alternatively the scheme Spec $\mathbb{C}[X, Y]/I$) satisfies the *maximal rank condition* if the rank of β over I is equal to $n!$ (which is the maximum possible rank over $\mathrm{Hilb}^n(\mathbb{C}^2)$). Thus the set of points $I \in \mathrm{Hilb}^n(\mathbb{C}^2)$ satisfying the maximal rank condition is open in $\mathrm{Hilb}^n(\mathbb{C}^2)$.

3. A characterization of the algebra A_σ

Fix a positive integer n. Let $\sigma : \sigma_0 \geqslant \sigma_1 \geqslant \dots \geqslant \sigma_m > 0$ be a partition of n and let $\sigma^\vee$ be the dual partition. Let $G = \mathrm{SL}_n(\mathbb{C})$ and let X_σ be the Springer fiber (over complex numbers) corresponding to a fixed nilpotent matrix M_σ (of size $n \times n$) with Jordan blocks of sizes $\sigma_0, \sigma_1, \dots, \sigma_m$. Recall that X_σ consists of those Borel subalgebras $\mathfrak{b}$ of $sl_n(\mathbb{C})$ such that $M_\sigma \in \mathfrak{b}$. Cohomology of a space X with coefficients in $\mathbb{C}$ will be denoted by $H^*(X,\mathbb{C})$ or simply by $H^*(X)$. In this paper, we only encounter spaces X such that $H^*(X)$ is concentrated in even degrees. *We will consider them as graded algebras under rescaled grading by assigning degree i to the elements of $H^{2i}(X)$.*

For the definition of Gorenstein rings and their general properties, the reader is referred to, e.g., [1].

Theorem 3. *For any partition σ, there exists an algebra T_σ satisfying the following:*

(1) *T_σ is a S_n-equivariant graded quotient of $H^*(X_\sigma \times X_{\sigma^\vee})$, under the diagonal Springer action of S_n on $H^*(X_\sigma \times X_{\sigma^\vee})$.*

(2) *$T_\sigma^{d_\sigma}$ is one-dimensional and, moreover, it is the sign representation of S_n, where T_σ^m denotes the graded component of degree m of T_σ and d_σ is the complex dimension of the variety $X_\sigma \times X_{\sigma^\vee}$.*

(3) *T_σ is a Gorenstein algebra. (Since T_σ is a finite-dimensional graded algebra and its top graded component is one-dimensional, the condition that T_σ is Gorenstein is equivalent to the condition that the pairing*

$$T_\sigma^m \times T_\sigma^{d_\sigma - m} \to T_\sigma^{d_\sigma}$$

induced by the multiplication map is perfect for all $m \geqslant 0$.)

Further, such a T_σ (satisfying the above three properties) is unique up to an isomorphism. More specifically, if $\varphi_\sigma : H^*(X_\sigma \times X_{\sigma^\vee}) \to T_\sigma$ and $\varphi'_\sigma : H^*(X_\sigma \times X_{\sigma^\vee}) \to T'_\sigma$ are two quotients satisfying the above properties, then their kernels are, in fact, equal.

Proof. By the Springer correspondence (see, e.g., [7], 13.3 or [18], §4.4), the top cohomologies of X_σ and $X_{\sigma^\vee}$ are irreducible representations of S_n which differ by the sign representation ϵ. Hence, there is a unique copy of ϵ in $S_\sigma^{d_\sigma}$, where we abbreviate $H^*(X_\sigma \times X_{\sigma^\vee})$ by S_σ. Let $(S_\sigma^{d_\sigma})^\epsilon$ be the corresponding isotypical component, which is a one-dimensional space. Now, consider the symmetric bilinear form γ obtained as the composition

$$S_\sigma \times S_\sigma \to S_\sigma^{d_\sigma} \to (S_\sigma^{d_\sigma})^\epsilon \simeq \mathbb{C}\,,$$

where the first map is induced by the multiplication and the second map is induced by the S_n equivariant projection. Now define

$$T_\sigma := S_\sigma / \operatorname{rad}\gamma\,,$$

where $\operatorname{rad}\gamma$ is the radical of γ. It is easy to see that T_σ satisfies all three properties listed above.

To prove the uniqueness, let $J := \ker\varphi_\sigma$ and $J' := \ker\varphi'_\sigma$. Then if J' is not a subset of J, take a nonzero homogeneous element $x \in J + J'/J \subset T_\sigma$. Then, since T_σ is Gorenstein, there exists an element $y \in T_\sigma$ such that $xy \neq 0$ in $T_\sigma^{d_\sigma}$. We can take a homogeneous preimage $\bar{x}$ of x in $J' \subset S_\sigma$. But J' being an ideal, $\bar{x}\bar{y} \in J'$ for any $\bar{y} \in S_\sigma$. Since $(J' \cap S_\sigma^{d_\sigma})^\epsilon = 0$, this leads to a contradiction, proving that $J' \subset J$. Similarly, $J \subset J'$, and hence $J = J'$. □

Remark 1. (a) It is easy to see that the Künneth decomposition of $H^*(X_\sigma \times X_{\sigma^\vee})$ gives rise to a bigrading on T_σ and this bigrading respects the S_n-action.

(b) It is well known that $H^*(X_\sigma)$ is concentrated in even degrees (cf. [8], Theorem 3.9). Moreover, it can be easily shown that $T_\sigma^1 = H^2(X_\sigma \times X_{\sigma^\vee})$.

(c) It is natural to ask if there exists a compact topological manifold M_σ of (real) dimension $2d_\sigma$ admitting a S_n-action on $H^*(M_\sigma)$ and a continuous map $\phi : M_\sigma \to X_\sigma \times X_{\sigma^\vee}$ such that the induced map ϕ^* in cohomology is S_n-equivariant and surjective. If so, $H^*(M_\sigma) \simeq T_\sigma$.

We give such a construction for the partition $\sigma : 2 \geqslant 1$ of $n = 3$:[1]

In this case $\sigma = \sigma^\vee$ and X_σ is nothing but two copies of $\mathbb{P}^1$ identified at ∞. Thus $X_\sigma \times X_{\sigma^\vee}$ can be identified with the subset $(\mathbb{P}^1 \times \infty \times \mathbb{P}^1 \times \infty) \cup (\infty \times \mathbb{P}^1 \times \infty \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \infty \times \infty \times \mathbb{P}^1) \cup (\infty \times \mathbb{P}^1 \times \mathbb{P}^1 \times \infty)$ of $(\mathbb{P}^1)^4$. Now take the union of the last two subsets which is the join of two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ along a point. Finally, take M_σ to be the manifold-join of these two (i.e., remove a small disc from these two separately and join them along the boundary). Then $H^*(M_\sigma)$ is a surjective image of that of $H^*(X_\sigma \times X_\sigma^\vee)$. The Weyl group S_n acts on $H^*(M_\sigma)$ making the above map S_n-equivariant (but S_n does not act in any natural manner on M_σ itself).

(d) It is well known (see, e.g., [21], Section 5) that $d_\sigma = \sum_{(i,j)\in D_\sigma} (i + j)$, where D_σ is the diagram of σ as in Section 2.

Theorem 4. *For any partition σ of n, there exists a bigraded S_n-equivariant isomorphism of algebras:*

$$T_\sigma \simeq A_\sigma,$$

where A_σ is defined in Section 2.

As a preparation to prove the above theorem, we prove the following two lemmas.

Let $\sigma : \sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_m > 0$ be a partition of n and let $\sigma^\vee : \sigma'_0 \geqslant \ldots \geqslant \sigma'_{m'} > 0$ be the dual partition. For any $m' < i \leqslant n$, set $\sigma'_i = 0$, and define the integer (for any $1 \leqslant k \leqslant n$):

$$d_k(\sigma) := n - \textstyle\sum_{s=0}^{n-k-1} \sigma'_s \ . \tag{1}$$

Let S be a subset of the diagram D_σ of cardinality k, together with a total ordering on S written as $S = \{z_1, \ldots, z_k\}$. Define i_s, j_s by $z_s = (i_s, j_s)$. Now define the monomial

$$P(S) = \textstyle\prod_{s=1}^{k} X_s^{i_s} Y_s^{j_s} \in \mathbb{C}[X_1, \ldots, X_k, Y_1, \ldots, Y_k].$$

For each integer $1 \leqslant t \leqslant k$, let S_t denote the subset $\{z_s : s > t\}$ of S.

Lemma 1. *Let S be as above and let r be a positive integer satisfying $k - d_k(\sigma) < r \leqslant k$. For any sequence $\mathbf{s} : 1 \leqslant s_1 < s_2 < \cdots < s_r \leqslant k$, define*

$$P(S)_{\mathbf{s}} = (\partial_{Y_{s_1}} \partial_{Y_{s_2}} \ldots \partial_{Y_{s_r}}) P(S).$$

Then there exist integers $1 \leqslant t' < t \leqslant k$ such that $P(S)_{\mathbf{s}}$ is symmetric in the variables (X_t, Y_t) and $(X_{t'}, Y_{t'})$ (i.e., it is invariant under the involution of $\mathbb{C}[X_1, \ldots, X_k, Y_1, \ldots, Y_k]$ given by $X_t \mapsto X_{t'}, Y_t \mapsto Y_{t'}$).

[1] R. MacPherson mentioned to us that he also constructed such a (possibly different) M_σ in this case.

Proof. Notice first of all that if $n-k \geqslant \sigma_0$, then $d_k(\sigma)=0$ and hence there is no r satisfying $k<r\leqslant k$. In the following we therefore assume that $n-k<\sigma_0$. By symmetry, we may assume $s_i=i$, $i=1,\dots,r$. Assume that the statement is not true.

If $j_s=0$ for an integer $s\leqslant r$, then clearly $P(S)_{\mathbf{s}}=0$ and the lemma follows in this case. Hence, assume that $j_s>0$ for all $s\leqslant r$. Consider $(i_s,j_s)\in S_r$ and assume $(i_s,j_s+1)\in S$. Let s' denote the integer such that $(i_s,j_s+1)=(i_{s'},j_{s'})$. If $s'\leqslant r$ then the statement follows with $t'=s'$ and $t=s$. Hence we may assume that, for each $(i_s,j_s)\in S_r$, either $(i_s,j_s+1)\notin S$ or $(i_s,j_s+1)\in S_r$.

Consider the set

$$M=\{0\leqslant i<\sigma_0 : (i,j)\in S \text{ for all } j=0,1,\dots,\sigma'_i-1\}.$$

For each $i\in M$ the element $(i,0)$ is then contained in S and therefore also in S_r. Hence, $(i,1)\in S_r$ if $(i,1)\in S$, and then $(i,2)\in S_r$ if $(i,2)\in S$ etc. This way we find that $(i,j)\in S_r$ for $i\in M$ and any $0\leqslant j<\sigma'_i$. In particular, the number of elements r in $S\setminus S_r$ is bounded by

$$r=|S\setminus S_r|\leqslant \textstyle\sum_{s\notin M, s<\sigma_0}(\sigma'_s-1)\leqslant \sum_{s=0}^{\sigma_0-|M|-1}(\sigma'_s-1).$$

As the number $n-k$ of elements in $D_\sigma\setminus S$ is bounded below by

$$n-k=|D_\sigma\setminus S|\geqslant \sigma_0-|M|,$$

we conclude that

$$r\leqslant \textstyle\sum_{s=0}^{n-k-1}(\sigma'_s-1)=k-d_k(\sigma),$$

which contradicts the assumption. □

Lemma 2. *Let σ be any partition of n. Let $k\leqslant n$ and r be positive integers such that $k-d_k(\sigma)<r\leqslant k$. Then, for any sequence $\mathbf{s}: 1\leqslant s_1<s_2<\ldots<s_k\leqslant n$,*

$$e_r(\partial_{Y_{s_1}},\dots,\partial_{Y_{s_k}})(\Delta_\sigma)=0\,,$$

where e_r denotes the r-th elementary symmetric polynomial.

Proof. By symmetry, we may assume $s_i=i$ for all $1\leqslant i\leqslant k$. Expanding Δ_σ as a sum of monomials, we can write (for some index set A)

$$\Delta_\sigma=\textstyle\sum_{\alpha\in A} f_\alpha(X_{k+1},\dots,X_n,Y_{k+1},\dots,Y_n)P(S_\alpha),$$

where $P(S_\alpha)$ are monomials in the variables $X_1,\dots,X_k,Y_1,\dots,Y_k$ of the form described earlier for some subsets $S_\alpha\subseteq D_\sigma$ of cardinality k together with some total ordering on them. Now, by Lemma 1,

$$e_r(\partial_{Y_1},\dots,\partial_{Y_k})P(S_\alpha)$$

is a sum of monomials each of which is symmetric in two distinct variables (X_t,Y_t) and $(X_{t'},Y_{t'})$ for $1\leqslant t<t'\leqslant k$. In particular, the polynomial

$$P:=e_r(\partial_{Y_1},\dots,\partial_{Y_k})(\Delta_\sigma)$$

can be written as a sum of monomials in the variables $X_1,Y_1,\dots,X_n,Y_n$ each of which is symmetric in the variables (X_t,Y_t) and $(X_{t'},Y_{t'})$ for some $1\leqslant t<t'\leqslant k$. The pair $t<t'$ could be different for different monomials occurring in P. If $P\neq 0$, fix a monomial M occurring in P and a pair $t<t'$ such that M is symmetric in (X_t,Y_t) and $(X_{t'},Y_{t'})$. Let $\pi\in \mathrm{S}_n$ be the permutation (t,t'). As Δ_σ is S_n-antisymmetric, $\pi P=-P$. But $\pi(M)=M$, which is a contradiction. Thus $P=0$. □

Now we are ready to prove Theorem 4.

Proof. By virtue of Lemma 2 and Theorem 9 (of the appendix), we get a S_n-equivariant morphism

$$\phi_\sigma : H^*(X_\sigma) \to A_\sigma$$

induced from the natural map $k[Y_1, \dots, Y_n] \to A_\sigma$.

From the definition of Δ_σ, it is easy to see that

$$\tau\Delta_\sigma = \pm\Delta_{\sigma^\vee},$$

where τ is the algebra automorphism of R_n taking $X_i \mapsto Y_i, Y_i \mapsto X_i$. Thus, using the identification

$$H^*(X_{\sigma^\vee}) \simeq k[X_1, \dots, X_n]/J(\sigma^\vee),$$

where $J(\sigma^\vee)$ is the ideal defined in Theorem 9, we get a morphism

$$\phi_{\sigma^\vee} : H^*(X_{\sigma^\vee}) \to A_\sigma$$

induced from the natural map $k[X_1, \dots, X_n] \to A_\sigma$. Combining the two, we get a surjective graded S_n-equivariant morphism

$$\phi : H^*(X_\sigma) \otimes H^*(X_{\sigma^\vee}) \to A_\sigma,$$

where $\phi(y \otimes x) := \phi_\sigma(y) \cdot \phi_{\sigma^\vee}(x)$.

By Remark 1 (d), $A_\sigma^{d_\sigma} = \mathbb{C}\Delta_\sigma$. Thus, $A_\sigma^{d_\sigma}$ transforms via the sign character of S_n. We next show that A_σ is Gorenstein: For any $0 \leqslant m \leqslant d_\sigma$, take a nonzero $x \in A_\sigma^m$ and take a homogeneous lift $\hat{x} \in R_n$. Viewing $\hat{x}$ as a differential operator $D(\hat{x})$ (under the identification $\mathrm{Diff}_{2n} \simeq R_n$, cf. Section 2), we get $D(\hat{x})\Delta_\sigma \neq 0$. Now take $\hat{y} \in R_n^{d_\sigma - m}$ such that

$$D(\hat{y})(D(\hat{x})\Delta_\sigma) = D(\hat{y}\hat{x})\Delta_\sigma = 1.$$

This shows that the pairing $A_\sigma^m \times A_\sigma^{d_\sigma - m} \to A_\sigma^{d_\sigma}$ (obtained by the multiplication) is perfect. Thus, A_σ satisfies the hypotheses of Theorem 3 and hence the isomorphism $A_\sigma \simeq T_\sigma$ follows. □

Remark 2. As pointed out by M. Haiman, the map ϕ_σ (and similarly $\phi_{\sigma^\vee}$) defined above is injective. Their existence (and injectivity) can also be obtained from the results in [2] and [13] as explained in [12], Section 3.1. However, for the sake of completeness, we have included a complete proof.

4. Principal nilpotent pairs and deformation

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and let $G = G_{\mathrm{ad}}$ be the corresponding semisimple linear algebraic group of adjoint type. Then, as is well known, G can be identified with the identity component of the group $\mathrm{Aut}(\mathfrak{g})$ of Lie algebra automorphisms of $\mathfrak{g}$. We denote the rank of $\mathfrak{g}$ by $\mathrm{rk}(\mathfrak{g})$.

We let $\mathfrak{g}^d$ denote the direct sum $\mathfrak{g} \oplus \mathfrak{g}$ and think of G as acting diagonally on $\mathfrak{g}^d$ by the adjoint action. The following definition is due to Ginzburg.

Definition 1. ([14], Definition 1.1) A pair $\mathbf{e} = (e_1, e_2) \in \mathfrak{g}^d$ is called a *principal nilpotent pair* if the following three conditions hold:

(1) $[e_1, e_2] = 0$.
(2) The common centralizer of e_1 and e_2 in $\mathfrak{g}$ is of dimension $\mathrm{rk}(\mathfrak{g})$.
(3) For any pair $t_1, t_2 \in \mathbb{C}^*$, there exists $g \in G$ such that

$$\mathrm{Ad}(g)\mathbf{e} = (t_1 e_1, t_2 e_2).$$

(Observe that by (3), e_1 and e_2 are ad-nilpotent elements.)

We need the following result due to Ginzburg.

Theorem 5. ([14], Theorem 1.2) *Let $\mathbf{e} = (e_1, e_2)$ be a principal nilpotent pair. Then there exists a pair $\mathbf{h} = (h_1, h_2) \in \mathfrak{g}^d$ satisfying the following:*
(1) *h_1 and h_2 are semisimple.*
(2) $[h_1, h_2] = 0$.
(3) $[h_i, e_j] = \delta_{ij} e_j$.
(4) *The common centralizer of h_1 and h_2 is a Cartan subalgebra in $\mathfrak{g}$.*
(5) *All the eigenvalues of $ad_{h_i} : \mathfrak{g} \to \mathfrak{g}$ are integral for $i = 1, 2$.*

Such a pair $\mathbf{h}$, called an *associated semisimple pair*, is determined by the principal nilpotent pair $\mathbf{e}$ uniquely up to a G-conjugacy.

Let $\mathbf{e} = (e_1, e_2)$ be a principal nilpotent pair. As e_1 and e_2 are ad-nilpotent and commute, the sum $e_1 + e_2$ is also ad-nilpotent. For any $t \in \mathbb{C}$ we can form the following automorphism of $\mathfrak{g}$:

$$g(t) = \exp(t\ \mathrm{ad}(e_1 + e_2)),$$

or, alternatively, consider $g(t)$ as an element of G. The action of $g(t)$ on the associated semisimple pair $\mathbf{h} = (h_1, h_2)$ is then described by

$$g(t)h_i = h_i - te_i.$$

From this one immediately concludes:

Lemma 3. *All the elements of the form $\mathbf{h}(t) = (h_1 - te_1, h_2 - te_2)$ lie in the G-orbit $O_{\mathbf{h}}$ of $\mathbf{h}$.*

The *induced cone* $\mathfrak{C}O_{\mathbf{h}}$ in $\mathfrak{g}^d$ is, by definition, equal to $\cup_{t\in\mathbb{C}}\ t \cdot O_{\mathbf{h}} \subseteq \mathfrak{g}^d$. By the above result, we know that $(th_1 + e_1, th_2 + e_2)$ is contained in $\mathfrak{C}O_{\mathbf{h}}$ when $t \neq 0$. Hence, (e_1, e_2) is contained in the closure $\overline{\mathfrak{C}O_{\mathbf{h}}}$.

4.1. Deformation

For a complex vector space V, let $\mathbb{C}[V]$ denote the affine coordinate ring of V. This acquires a natural grading coming from the identification $\mathbb{C}[V] = S(V^*)$, where $S(V^*)$ denotes the symmetric algebra of the vector space dual V^* of V. If X is any subset of V, we denote by $I(X)$ the ideal of functions in $\mathbb{C}[V]$ vanishing identically on X. The vector space, spanned by the leading terms (i.e., the highest degree components) of all the elements in $I(X)$, forms an ideal itself. This ideal is denoted by $\mathrm{gr}(I(X))$. For any ideal $I \in \mathbb{C}[V]$, we denote by $\mathcal{V}(I) := \{x \in V : f(x) = 0 \text{ for all } f \in I\}$ the corresponding closed subvariety of V.

In the following, H will denote a reductive group over $\mathbb{C}$. We also consider a finite-dimensional complex representation V of H. For a H-stable vector subspace $M \subset \mathbb{C}[V]$, $\mathrm{gr}(M)$ is again H-stable, where $\mathrm{gr}(M)$ denotes the vector space spanned by the leading terms of the elements in M. Moreover,

$$\mathrm{gr}(M) \simeq M, \quad \text{as } H\text{-modules}.$$

Definition 2. An H-orbit Hv in V is called *semistable* if 0 is not contained in its closure. If Hv is not semistable it is said to be *unstable.*

For any subset $U \subset V$, we define the *induced cone* (as earlier) $\mathfrak{C}U := \cup_{t\in\mathbb{C}}\, t\cdot U$, and the *associated cone* $\mathcal{K}(U)$ as the closed (reduced) subvariety $\mathcal{V}(\mathrm{gr}(I(U)))$ of V.

Let Hv be a semistable orbit in V. It is said that an unstable orbit Hu *can be deformed into the semistable orbit* Hv if Hu is contained in the associated cone $\mathcal{K}(Hv)$ of Hv and, moreover, Hu has the same dimension as Hv.

With this notation, we recall the following result from [4], Satz (3.4).

Proposition 1. *Let V be a finite-dimensional representation of H and let Hv be a semistable orbit. Then*

$$\mathcal{K}(Hv) = (\overline{\mathfrak{C}U} \setminus \mathfrak{C}U) \cup \{0\},$$

where $U := \overline{Hv}$.

Moreover, if Hu is an unstable H-orbit such that there exists a deformation of Hu into the (*semistable*) *orbit Hv, then*

$$I(\overline{Hu}) \supset \mathrm{gr}(I(\overline{Hv})).$$

4.2. Deforming principal nilpotent pairs

We now turn to certain deformations inside the finite-dimensional representation $\mathfrak{g}^d$ of G.

Lemma 4. *Let $\mathbf{e} = (e_1, e_2)$ be a principal nilpotent pair. Then the G-orbit $O_{\mathbf{e}} \subset \mathfrak{g}^d$ is unstable.*

Proof. By Definition 1, each element of the form (te_1, te_2), $t \in \mathbb{C}^*$, is contained in $O_{\mathbf{e}}$. Hence 0 must be contained in the closure. $\square$

Lemma 5. *Let $\mathbf{h} = (h_1, h_2)$ be a semisimple pair associated to a principal nilpotent pair $\mathbf{e}$. Then the G-orbit $O_{\mathbf{h}}$ is closed and hence semistable.*

Proof. As h_1 and h_2 commute, one can find a Cartan subalgebra of $\mathfrak{g}$ containing both h_1 and h_2. Hence, the common centralizer of h_1 and h_2 in G contains a maximal torus. This implies that the orbit is closed. $\square$

Lemma 6. *The orbit $O_{\mathbf{e}}$ can be deformed into $O_{\mathbf{h}}$.*

Proof. As observed earlier, the element $\mathbf{e}$ lies in $\overline{\mathfrak{C}O_{\mathbf{h}}}$ and hence, by Proposition 1, also lies in the associated cone $\mathcal{K}(O_{\mathbf{h}})$ of $O_{\mathbf{h}}$ (since all the elements of $\mathfrak{C}O_{\mathbf{h}}$ are semisimple pairs). It remains to show that the dimensions of $O_{\mathbf{e}}$ and $O_{\mathbf{h}}$ are the same. For this it is enough to show that the dimension $d_{\mathbf{e}}$ of the common centralizer of e_1, e_2 in $\mathfrak{g}$ is the same as the dimension $d_{\mathbf{h}}$ of the common centralizer of h_1, h_2 in $\mathfrak{g}$. But, by definition, $d_{\mathbf{e}} = d_{\mathbf{h}} = \mathrm{rk}(\mathfrak{g})$. $\square$

4.3. Orbits of the automorphism group of $\mathfrak{g}$

For the rest of this section we assume, for simplicity, that $\mathfrak{g}$ is simple (and, as earlier, G is the associated adjoint group). We fix a maximal torus T of G and a set of simple roots Δ. The full automorphism group $\mathcal{G}$ of the Lie algebra $\mathfrak{g}$ is then isomorphic to the semidirect product of G with the group Γ of the diagram automorphisms of $\mathfrak{g}$ (for the Dynkin diagram of $\mathfrak{g}$ with respect to Δ).

When e is a nilpotent element in $\mathfrak{g}$, by $\mathcal{G}^e$ we mean the subgroup of those elements in $\mathcal{G}$ leaving the G-orbit of e invariant. In particular, $G \subseteq \mathcal{G}^e \subseteq \mathcal{G}$. The following lemma gives a complete description of $\mathcal{G}^e$.

Lemma 7. *When $\mathfrak{g}$ is of type D_l with even $l > 4$, and e corresponds to a very even partition (with the usual classification of nilpotent classes [7]), then $\mathcal{G}^e$ coincides with G. When $\mathfrak{g}$ is of type D_4 and e corresponds to a very even partition or one of the partitions $(1^3 5)$ or $(1^5 3)$, then $\mathcal{G}^e$ is a subgroup of $\mathcal{G}$ of index 3. In the remaining cases $\mathcal{G}^e$ coincides with $\mathcal{G}$.*

Proof. Up to G-conjugacy we can embed e into a standard sl_2-triple (h, e, f) with h contained in the Lie algebra $\mathfrak{h}$ of T and such that $\alpha(h)$ is positive for all $\alpha \in \Delta$. The weighted Dynkin diagram of e is then the Dynkin diagram of $\mathfrak{g}$ with respect to Δ, with values $\alpha(h)$, $\alpha \in \Delta$, attached to the vertices.

Fix a diagram automorphism g and consider the induced sl_2-triple $(g \cdot h, g \cdot e, g \cdot f)$. Then $\alpha(g \cdot h)$ is positive for all $\alpha \in \Delta$, and hence the associated weighted Dynkin diagram of $g \cdot e$ is obtained by applying the diagram automorphism g to the weighted Dynkin diagram of e. As the weighted Dynkin diagram of e determines the G-conjugacy class of e, we see that $g \in \mathcal{G}^e$ if and only if the weighted Dynkin diagram of e is invariant under g.

We thus conclude that $\mathcal{G}^e/G$ is the subgroup of Γ consisting of those diagram automorphisms of the Dynkin diagram of $\mathfrak{g}$ which leave the weighted Dynkin diagram of e invariant. Now, the result follows by the inspection of the possible weighted Dynkin diagrams as found in [7]. □

Spaltenstein introduced an involution, called the *Spaltenstein involution*, on the class of "special" nilpotent orbits in $\mathfrak{g}$ (cf. [20]).

Lemma 8. *Let $e \in \mathfrak{g}$ be a nilpotent element such that the orbit $O_e := G \cdot e$ is special, and let e' denote a nilpotent element in the special nilpotent orbit dual to $G \cdot e$ under the Spaltenstein involution. Then, for any $\gamma \in \mathcal{G}$, $\gamma(O_e)$ is special and, moreover, $\gamma(O_{e'})$ is dual to the orbit $\gamma(O_e)$. In particular,*

$$\mathcal{G}^e = \mathcal{G}^{e'}.$$

Proof. The first part is known. However we did not find a precise reference, so we give a brief outline. The Spaltenstein involution is the restriction of a map d defined on the set of nilpotent orbits ([20], Chap. III). When $\mathfrak{g}$ is of classical type, the uniqueness statement Théorème 1.5, Chap. III in [20] implies that d commutes with any automorphism of $\mathfrak{g}$. Combining this with the fact that the range of d coincides with the special nilpotent orbits, the result follows for the classical types. When $\mathfrak{g}$ is of type E_6 it follows from the diagrams in [7], p. 441, that there exists at most one order reversing involution on the set of special nilpotent orbits. In particular, the Spaltenstein involution must commute with any automorphism of $\mathfrak{g}$, as it is known to be order reversing, and the result follows. In the remaining cases $\mathcal{G}$ coincides with G and the statement is trivial.

Alternatively, as pointed out by G. Lusztig, the first part follows from the fact that the Spaltenstein involution corresponds to tensoring the associated Springer representation by the sign representation.

To prove the "In particular" statement, take $\gamma \in \mathcal{G}^e$. Then $\gamma(O_{e'})$ is dual to $\gamma(O_e) = O_e$, and hence $\gamma(O_{e'}) = O_{e'}$. This proves the lemma. □

By [14], Lemma 4.13, for a principal nilpotent pair $\mathbf{e} = (e_1, e_2)$ in $\mathfrak{g}$, the G-orbits of e_1 and e_2 are Spaltenstein duals of each other. In particular, by Lemma 8, the groups $\mathcal{G}^{e_1}$ and $\mathcal{G}^{e_2}$ coincide. In the following, we use the notation $\mathcal{G}^{\mathbf{e}}$ for this common group.

Now we conjecture the following variant of a conjecture by Ginzburg [14], Conjecture (2.5). The original conjecture of Ginzburg is false, as seen in Proposition 4.

Conjecture 1. *Let $\mathbf{e} = (e_1, e_2) \in \mathfrak{g}^d$ be a principal nilpotent pair. Then there is a G-module isomorphism:*

$$\mathbb{C}[\bar{\mathcal{O}}_{\mathbf{e}}] \simeq \mathbb{C}[G/T],$$

where $T \subset G$ is a maximal torus and $\mathcal{O}_{\mathbf{e}}$ denotes the $\mathcal{G}^{\mathbf{e}}$ orbit of $\mathbf{e}$ in $\mathfrak{g}^d$.

Lemma 9. *Let $\mathbf{e} = (e_1, e_2)$ be a principal nilpotent pair with an associated semisimple pair $\mathbf{h} = (h_1, h_2)$. Then the ideal $\mathrm{gr}(I(G \cdot \mathbf{h}))$ is invariant under the action of $\mathcal{G}^{\mathbf{e}}$.*

Proof. When $\mathcal{G}^{\mathbf{e}}$ coincides with G there is nothing to prove. By Lemma 7, this takes care of any $\mathfrak{g}$ of type B_l, C_l, E_7, E_8, G_2, F_4 or A_1 and also the cases when e_1 (and hence also e_2) corresponds to a very even partition in type D_l, even $l > 4$.

Suppose now that $\mathfrak{g}$ is of type A_l ($l > 1$), E_6 or D_l (l odd). Then $\mathcal{G}^{\mathbf{e}}/G$ coincides with $\mathcal{G}/G$ which is the group of order 2. By Planche I-IX in [5] it follows that the group $\mathcal{G}/G$ is generated by an automorphism of $\mathfrak{g}$, which acts on a chosen Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ via multiplication by -1. Choosing $\mathfrak{h}$ to contain h_1 and h_2, since the ideal $\mathrm{gr}(I(G \cdot \mathbf{h}))$ is homogeneous, the result follows in these cases as well.

We are left with the situation when $\mathfrak{g}$ is of type D_l, with l even. Notice that if $g \in \mathcal{G}^{\mathbf{e}}$ leaves the G-orbit of $\mathbf{e}$ invariant, then by [14], Theorem 1.2(iv), the G-orbit of $\mathbf{h}$ is also left invariant by g. We may hence concentrate on the situation when $g \in \mathcal{G}^{\mathbf{e}}$ does not leave the G-orbit of $\mathbf{e}$ invariant. Now referring to the classification of principal nilpotent pairs as in [10], Proposition 5 and Theorem 2, it follows that any element $g \in \mathcal{G}^{\mathbf{e}}$ leaves the G-orbit of $\mathbf{e}$ invariant unless $\mathbf{e}$ corresponds to a non-integral rectangular skew-graph (see [10] for the notation), which is the same as saying that e_1 and e_2 correspond to very even partitions. Thus, when $l > 4$, this situation is already handled above. This leaves us with type D_4 and principal nilpotent pairs $\mathbf{e} = (e_1, e_2)$ consisting of elements e_1 and e_2 corresponding to very even partitions. In this situation there exists an automorphism of $\mathfrak{g}$ carrying e_1 and e_2 to the nilpotent elements corresponding to the partitions $(1^3 5)$ and $(1^5 3)$. As the partitions $(1^3 5)$ and $(1^5 3)$ are not very even, this reduces the problem to a situation which has already been handled. $\square$

Theorem 6. *Let $\mathbf{h}$ be a semisimple pair associated to a principal nilpotent pair $\mathbf{e}$. Assume the validity of the above Conjecture 1 for $\mathbf{e}$. Then*

$$I(\bar{\mathcal{O}}_{\mathbf{e}}) = \mathrm{gr}(I(O_{\mathbf{h}})),$$

where $\mathcal{O}_{\mathbf{e}}$ denotes the $\mathcal{G}^{\mathbf{e}}$-orbit of $\mathbf{e}$ and $O_{\mathbf{h}}$ denotes the G-orbit of $\mathbf{h}$ (which is closed by Lemma 5).

Proof. By Proposition 1, Lemma 6 and Lemma 9,

$$I(\bar{\mathcal{O}}_{\mathbf{e}}) \supset \mathrm{gr}(I(O_{\mathbf{h}})),$$

which induces a surjective map of G-modules

$$\mathbb{C}[\mathfrak{g}^d]/(\mathrm{gr}(I(O_{\mathbf{h}}))) \twoheadrightarrow \mathbb{C}[\bar{\mathcal{O}}_{\mathbf{e}}] \simeq \mathbb{C}[G/T], \tag{2}$$

where the last isomorphism follows from Conjecture 1. Moreover, $\mathrm{gr}(I(O_{\mathbf{h}})) \simeq I(O_{\mathbf{h}})$, as G-modules, so that the left side of (2) is isomorphic with $\mathbb{C}[O_{\mathbf{h}}]$ (as G-modules). But the G-isotropy of $\mathbf{h}$ can easily be seen to be equal to a maximal torus T and hence $\mathbb{C}[O_{\mathbf{h}}] \simeq \mathbb{C}[G/T]$. This forces the surjective map (2) to be an isomorphism, thus the theorem follows. □

Remark 3. Using the classification of principal nilpotent pairs by Elashvili and Panyushev (see [10] and Appendix to [14]), one can work out the following description of the irreducible components of $\bar{\mathcal{O}}_{\mathbf{e}}$ for a principal nilpotent pair $\mathbf{e}$.

- A_l : $\bar{\mathcal{O}}_{\mathbf{e}}$ is the union of the G-orbit closures of $\mathbf{e}$ and its transpose $\mathbf{e}^t := (e_1^t, e_2^t)$. These two components coincide precisely when $\mathbf{e} = \mathbf{e}_\sigma$ for a box partition σ (see the proof of Proposition 3 for the notation $\mathbf{e}_\sigma$).
- $B_l, C_l, E_l, G_2, F_4, D_l$ (l even): $\bar{\mathcal{O}}_{\mathbf{e}}$ coincides with the G-orbit closure of $\mathbf{e}$ (cf. Remark on p. 560 in [14]).
- D_l (l odd): If $\mathbf{e}$ is defined by a single skew-diagram (see [10], Prop. 5(i)), then $\bar{\mathcal{O}}_{\mathbf{e}}$ consists of two distinct G-orbit closures of principal nilpotent pairs (those defined by the same skew-graph). Otherwise, $\bar{\mathcal{O}}_{\mathbf{e}}$ coincides with the G-orbit closure of $\mathbf{e}$.

5. A conjectural generalization of $n!$ result to arbitrary groups

In this section $\mathfrak{g}$ will denote an arbitrary simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Let G be the associated complex adjoint group with maximal torus T such that Lie $T = \mathfrak{h}$. As in the last section, $\mathfrak{g}^d$ denotes the direct sum $\mathfrak{g} \oplus \mathfrak{g}$ and also $\mathfrak{h}^d := \mathfrak{h} \oplus \mathfrak{h}$. Let $W := N/T$ be the Weyl group of G; N being the normalizer of T in G. For a principal nilpotent pair $\mathbf{e}$, recall the definition of $\mathcal{G}^{\mathbf{e}}$ from the last section.

Proposition 2. *Let $\mathbf{e} = (e_1, e_2)$ be a principal nilpotent pair in $\mathfrak{g}$ and let $\mathcal{O}_{\mathbf{e}}$ be the diagonal $\mathcal{G}^{\mathbf{e}}$-orbit of $\mathbf{e}$ in $\mathfrak{g}^d$. Consider the affine coordinate ring $\mathfrak{D}_{\mathbf{e}} := \mathbb{C}[\bar{\mathcal{O}}_{\mathbf{e}} \cap \mathfrak{h}^d]$, where $\bar{\mathcal{O}}_{\mathbf{e}} \cap \mathfrak{h}^d$ denotes the scheme-theoretic intersection of $\bar{\mathcal{O}}_{\mathbf{e}}$ with $\mathfrak{h}^d$ inside $\mathfrak{g}^d$. Then $\mathfrak{D}_{\mathbf{e}}$ is a bigraded finite-dimensional (commutative) algebra, which admits a natural W-module structure compatible with the bigrading.*

Proof. Consider the action of the two-dimensional torus $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathfrak{g}^d$ by $(z, t).(h_1, h_2) = (zh_1, th_2)$, for $z, t \in \mathbb{C}^*$ and $h_1, h_2 \in \mathfrak{g}$. By the definition of principal nilpotent pairs, $\mathbb{C}^* \times \mathbb{C}^*$ keeps $\mathcal{O}_{\mathbf{e}}$ (and thus $\bar{\mathcal{O}}_{\mathbf{e}}$) stable. Of course, $\mathbb{C}^* \times \mathbb{C}^*$ also keeps $\mathfrak{h}^d$ stable. This gives the desired bigrading on $\mathfrak{D}_{\mathbf{e}}$. Further, under the diagonal action, N keeps $\mathfrak{h}^d$ stable and, of course, it keeps $\bar{\mathcal{O}}_{\mathbf{e}}$ stable. Thus we get an action of N on $\mathfrak{D}_{\mathbf{e}}$. But T acts trivially on $\mathfrak{h}^d$; in particular, it acts trivially on $\mathfrak{D}_{\mathbf{e}}$ giving rise to an action of W (on $\mathfrak{D}_{\mathbf{e}}$).

To prove that $\mathfrak{D}_{\mathbf{e}}$ is finite-dimensional, it suffices to prove that set theoretically the scheme $\bar{\mathcal{O}}_{\mathbf{e}} \cap \mathfrak{h}^d$ is a singleton. But this follows immediately since the only element of $\mathfrak{g}^d$ which is both nilpotent and semisimple is 0, where we call an element of $\mathfrak{g}^d$ nilpotent (resp. semisimple) if both of its components are nilpotent (resp. semisimple). □

We come to one of the main conjectures of the paper.

Conjecture 2. *Let the notation and assumptions be as in the above Proposition 2. Assume further that $\mathbf{e}$ is non-exceptional ([14], Definition 4.1). Then the scheme $\bar{\mathcal{O}}_{\mathbf{e}} \cap \mathfrak{h}^d$*

is Gorenstein. Moreover, $\mathfrak{D}_{\mathbf{e}} = \mathbb{C}[\bar{\mathcal{O}}_{\mathbf{e}} \cap \mathfrak{h}^d]$ *supports the regular representation of* W*; in particular, it is of dimension* $|W|$.

Remark 4. (a) When one of the entries e_1 or e_2 of $\mathbf{e} = (e_1, e_2)$ is zero, then $\mathcal{O}_{\mathbf{e}}$ coincides with the G-orbit of $\mathbf{e}$. Hence, in this case, Conjecture 2 reduces to a well-known result of Kostant.

(b) An old preprint of Bezrukavnikov–Ginzburg contained a variant of Conjecture 2 (cf. [3], Section 8.3). In addition, M. Haiman has informed us that he also conjectured around 1993 (though unpublished) a variant of Conjecture 2 in the special case of $\mathfrak{g} = sl_n$.

(c) In an earlier version of our paper, Conjecture 2 was formulated with $\bar{\mathcal{O}}_{\mathbf{e}}$ replaced by the G-orbit closure $\bar{O}_{\mathbf{e}}$ of $\mathbf{e}$. This turns out to be false in general (as pointed out by S. Strømme), since Ginzburg's original conjecture [14], Conjecture 2.5 is false (cf. Proposition 4). In the same version we had conjectured that $\bar{O}_{\mathbf{e}} \cap \mathfrak{h}^d$ is even a complete intersection in $\mathfrak{h}^d$, but it is false in general (as pointed out by Haiman).

(d) By computer calculations, using SINGULAR [15], we have checked the above conjecture for $sl_n, n \leqslant 4$.

Proposition 3. *The validity of the above Conjecture* 2 *for* $\mathfrak{g} = sl_n$ *(and every principal nilpotent pair* $\mathbf{e}$*) implies the* $n!$ *result for any partition of* n.

Alternatively, assume the validity of Conjecture 1 *and assume, in addition, that* $\mathfrak{D}_{\mathbf{e}}$ *is Gorenstein. Then the* $n!$ *result follows for any partition of* n.

Proof. Take a partition $\sigma : \sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_m > 0$ of n. Let $\{v_1, \ldots, v_n\}$ be the standard basis of $\mathbb{C}^n$. Define a nilpotent transformation $e' = e'_\sigma \in sl_n$ which takes $v_1 \mapsto v_2 \mapsto v_3 \mapsto \ldots \mapsto v_{\sigma_0} \mapsto 0, v_{\sigma_0+1} \mapsto v_{\sigma_0+2} \mapsto \ldots \mapsto v_{\sigma_0+\sigma_1} \mapsto 0, v_{\sigma_0+\sigma_1+1} \mapsto v_{\sigma_0+\sigma_1+2} \mapsto \ldots \mapsto v_{\sigma_0+\sigma_1+\sigma_2} \mapsto 0, \ldots$. Similarly, define $e'' = e''_\sigma \in sl_n$ which takes $v_1 \mapsto v_{\sigma_0+1} \mapsto v_{\sigma_0+\sigma_1+1} \mapsto \ldots \mapsto v_{\sigma_0+\ldots+\sigma_{m-1}+1} \mapsto 0, v_2 \mapsto v_{\sigma_0+2} \mapsto \ldots \mapsto v_{\sigma_0+\ldots+\sigma_{m-1}+2} \mapsto 0 \ldots$. Then, by [14], $\mathbf{e}_\sigma := (e', e'')$ is a principal nilpotent pair (cf. also [19]).

Consider the maps

$$\begin{aligned} H^*(X_{\sigma^\vee}) \otimes H^*(X_\sigma) &\simeq \mathbb{C}[\overline{G \cdot e'} \cap \mathfrak{h}] \otimes \mathbb{C}[\overline{G \cdot e''} \cap \mathfrak{h}] \\ &\simeq \mathbb{C}[\overline{\mathcal{G}^{e'} \cdot e'} \cap \mathfrak{h}] \otimes \mathbb{C}[\overline{\mathcal{G}^{e''} \cdot e''} \cap \mathfrak{h}] \\ &\simeq \mathbb{C}[(\overline{(\mathcal{G}^{\mathbf{e}_\sigma} \times \mathcal{G}^{\mathbf{e}_\sigma}) \cdot \mathbf{e}_\sigma}) \cap \mathfrak{h}^d] \\ &\twoheadrightarrow \mathfrak{D}_{\mathbf{e}_\sigma}, \end{aligned}$$

where the first isomorphism is due to de Concini and Procesi [9] and is bigraded and $W \times W$-equivariant and the last map is surjective (bigraded and W-equivariant), being induced by the closed immersion

$$\bar{\mathcal{O}}_{\mathbf{e}_\sigma} \cap \mathfrak{h}^d \subset (\overline{(\mathcal{G}^{\mathbf{e}_\sigma} \times \mathcal{G}^{\mathbf{e}_\sigma}) \cdot \mathbf{e}_\sigma}) \cap \mathfrak{h}^d.$$

By our Conjecture 2, $\mathfrak{D}_{\mathbf{e}_\sigma}$ is Gorenstein; in particular, its top graded component is one-dimensional (since its zeroth graded component is one-dimensional). Moreover, since it supports the regular representation of W (by Conjecture 2), the top graded component will have to be the sign representation (being the only representation of dimension one apart from the trivial representation and the trivial representation occurs in degree 0). But since $\mathfrak{D}_{\mathbf{e}_\sigma}$ is a quotient of $H^*(X_{\sigma^\vee}) \otimes H^*(X_\sigma)$, and the latter has a unique copy of the sign representation which occurs in the top degree d_σ (cf. [18], Section 4.4), we conclude that $\mathfrak{D}^{d_\sigma}_{\mathbf{e}_\sigma}$ is nonzero (and is the sign representation of W). Thus, by Theorems 3, 4, we get that

$$\mathfrak{D}_{\mathbf{e}_\sigma} \simeq A_\sigma \,.$$

But by Conjecture 2 again, dim $\mathfrak{D}_{\mathbf{e}_\sigma} = n!$ and thus the $n!$ result follows for the partition σ (cf. Theorem 2).

Now we prove the “Alternatively” part of the proposition. Let $\mathbf{h}_\sigma$ be a semisimple pair associated to $\mathbf{e}_\sigma$. Then, by Theorem 6, $I(\bar{\mathcal{O}}_{\mathbf{e}_\sigma}) = \mathrm{gr}(I(O_{\mathbf{h}_\sigma}))$. Thus

$$I(\bar{\mathcal{O}}_{\mathbf{e}_\sigma}) + I(\mathfrak{h}^d) \subset \mathrm{gr}(I(O_{\mathbf{h}_\sigma}) + I(\mathfrak{h}^d)),$$

and hence the S_n-module $\mathbb{C}[O_{\mathbf{h}_\sigma} \cap \mathfrak{h}^d]$ is a quotient of the S_n-module $\mathfrak{D}_{\mathbf{e}_\sigma}$. But the scheme $O_{\mathbf{h}_\sigma} \cap \mathfrak{h}^d$ is reduced and is isomorphic with the S_n-orbit $\mathrm{S}_n \cdot \mathbf{h}_\sigma$. It is easy to see that S_n acts freely on $\mathbf{h}_\sigma$, thus $\mathbb{C}[O_{\mathbf{h}_\sigma} \cap \mathfrak{h}^d]$ is the regular representation of S_n. In particular, the sign representation occurs in $\mathbb{C}[O_{\mathbf{h}_\sigma} \cap \mathfrak{h}^d]$ and hence also in $\mathfrak{D}_{\mathbf{e}_\sigma}$. But then, by the argument given above, $\mathfrak{D}_{\mathbf{e}_\sigma}$ has a unique copy of the sign representation which occurs in the top degree d_σ. Moreover, $\mathfrak{D}_{\mathbf{e}_\sigma}$ being Gorenstein (by assumption), $\mathfrak{D}^{d_\sigma}_{\mathbf{e}_\sigma}$ is one-dimensional. Thus, by Theorems 3 and 4, we again get that $\mathfrak{D}_{\mathbf{e}_\sigma} \simeq A_\sigma$, and hence A_σ has dimension at least $n!$. Finally, since dim $A_\sigma \leqslant n!$ (from the semicontinuity argument as in Section 2), we get that dim $A_\sigma = n!$. □

The following was pointed out to us independently by D. Panyushev and S. Strømme for the principal nilpotent pair $\mathbf{e}_\sigma$ in sl_3 associated to the partition $\sigma : 2 \geqslant 1$.

Proposition 4. *Ginzburg's conjecture* [14]*, Conjecture* 2.5*, is false in general.*

Proof. Let $\mathbf{e} = \mathbf{e}_\sigma$ denote the principal nilpotent pair in sl_n corresponding to a partition σ of n (as in the proof of Proposition 3) and let $\mathbf{h}$ be an associated semisimple pair. By choosing a suitable basis, one can assume that $-\mathbf{h}$ is an associated semisimple pair of the transposed principal nilpotent pair $\mathbf{e}^t$. But if Ginzburg's conjecture were true for both of $\mathbf{e} = \mathbf{e}_\sigma$ and $\mathbf{e}^t$, then, by a variant of Theorem 6, we would get that $I(\overline{G \cdot \mathbf{e}}) = I(\overline{G \cdot \mathbf{e}^t})$. This would imply that $G \cdot \mathbf{e} = G \cdot \mathbf{e}^t$. This is true however only for the principal nilpotent pairs $\mathbf{e}_\sigma$ defined by box partitions σ. Thus, for any partition σ of n which is not a box partition, Ginzburg's conjecture is false. □

6. Reduction to the staircase partition

Let $T \subset \mathrm{GL}_2(\mathbb{C})$ be the maximal torus consisting of the diagonal matrices (of nonzero determinant) and let B be the Borel subgroup consisting of the upper triangular matrices (of nonzero determinant). Let $v_1 = (1, 0)$ and $v_2 = (0, 1)$ be the standard basis of $\mathbb{C}^2$. The algebra of regular functions on $\mathbb{C}^2$ is identified with $\mathbb{C}[X, Y]$, where X and Y are the linear maps determined by

$$X(v_1) = 1 \ , \ X(v_2) = 0 \ , \ Y(v_1) = 0 \ , \ Y(v_2) = 1.$$

Consider the standard representation $\mathbb{C}^2$ of $\mathrm{GL}_2(\mathbb{C})$. This induces a natural representation of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}[X, Y]$. The space of homogeneous polynomials $\mathbb{C}[X, Y]_d$ of degree d in $\mathbb{C}[X, Y]$ forms an irreducible $\mathrm{GL}_2(\mathbb{C})$-submodule of dimension $d + 1$. The following lemma is well known.

Lemma 10. *Let $V \subseteq \mathbb{C}[X,Y]_d$ be a nonzero B-stable subspace. Then there exists an integer $i \geqslant 0$ such that*

$$V = \mathrm{span}_{\mathbb{C}}\{Y^d, Y^{d-1}X, \ldots, Y^{d-i}X^i\}.$$

Conversely, for any $i \geqslant 0$, the vector space

$$\mathrm{span}_{\mathbb{C}}\{Y^d, Y^{d-1}X, \ldots, Y^{d-i}X^i\}$$

is B-stable.

The action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}[X,Y]$ clearly gives rise to its natural action on $\mathrm{Hilb}^n(\mathbb{C}^2)$ leaving the *punctual Hilbert scheme* $\mathrm{Hilb}^n_0(\mathbb{C}^2) \subset \mathrm{Hilb}^n(\mathbb{C}^2)$ invariant, where $\mathrm{Hilb}^n_0(\mathbb{C}^2)$ is the projective subvariety consisting of ideals containing some power of the maximal ideal (X,Y).

Recall from Section 2 that any partition σ of n gives rise to the element $I_\sigma \in \mathrm{Hilb}^n(\mathbb{C}^2)$. Furthermore, it is clear from the definition that I_σ lies in the punctual Hilbert scheme $\mathrm{Hilb}^n_0(\mathbb{C}^2)$. Moreover, the T-stable points of $\mathrm{Hilb}^n(\mathbb{C}^2)$ are precisely the points of the form I_σ, for some partition σ of n. Since $\mathrm{Hilb}^n_0(\mathbb{C}^2)$ is a $\mathrm{GL}_2(\mathbb{C})$-stable projective subvariety, the following result follows readily from the Borel's fixed point Theorem.

Lemma 11. *The closure $\overline{BI}$ of the B-orbit of any $I \in \mathrm{Hilb}^n_0(\mathbb{C}^2)$ inside $\mathrm{Hilb}^n(\mathbb{C}^2)$ contains a B-fixed point (of the form I_σ).*

Lemma 12. *For a partition $\sigma : \sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_m > 0$ of n, the corresponding element I_σ is a B-invariant point of $\mathrm{Hilb}^n(\mathbb{C}^2)$ if and only if $\sigma_i > \sigma_{i+1}$ for all $i = 0, 1, \ldots, m-1$.*

Thus, for a partition σ as above such that I_σ is B-invariant, we have either $\sigma_0 > m+1$ or else $m = \sigma_0 - 1$ and σ is the staircase partition $\mathfrak{s}_{m+1} : m+1 > m > \ldots > 1$ (i.e., $\sigma_i = m+1-i$ for $i = 0, 1, \ldots, m$).

Proof. By Lemma 10, I_σ is B-invariant if and only if

$$X^iY^j \in I_\sigma \ , \ i > 0 \ \Rightarrow \ X^{i-1}Y^{j+1} \in I_\sigma.$$

By the definition of I_σ this is equivalent to the condition:

$$i \geqslant \sigma_j \ , \ i > 0 \ \Rightarrow \ i - 1 \geqslant \sigma_{j+1}.$$

This is clearly equivalent to $\sigma_i > \sigma_{i+1}$ for all $i = 0, 1, \ldots, m-1$. This proves the lemma. □

For $f \in \mathbb{C}[X,Y]$ we let $\mathrm{Hilb}^n(\mathbb{C}^2)_f$ denote the subset of $\mathrm{Hilb}^n(\mathbb{C}^2)$ defined by

$$\mathrm{Hilb}^n(\mathbb{C}^2)_f = \{I \in \mathrm{Hilb}^n(\mathbb{C}^2) : f \notin I\}.$$

We prove the following.

Lemma 13. $\mathrm{Hilb}^n(\mathbb{C}^2)_f$ *is an open subset of* $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Proof. Consider the tautological vector bundle $\mathfrak{T}$ over the base $\mathrm{Hilb}^n(\mathbb{C}^2)$ of rank n as in Section 2, where the fiber over any $I \in \mathrm{Hilb}^n(\mathbb{C}^2)$ is the $\mathbb{C}$-algebra $\mathbb{C}[X,Y]/I$. Then the multiplication by f on each fiber induces a map of vector bundles $m_f : \mathfrak{T} \to \mathfrak{T}$. The subset of $\mathrm{Hilb}^n(\mathbb{C}^2)$ where m_f has nonzero rank is then an open subset coinciding with $\mathrm{Hilb}^n(\mathbb{C}^2)_f$. □

For a partition $\sigma : \sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_m > 0$ of n, consider the ideal

$$I_\sigma(\Lambda) = (Y^{m+1}(Y-\Lambda), Y^{m+1}X) + (X^iY^j : i \geqslant \sigma_j\ ,\ j \leqslant m)$$

inside the polynomial ring $\mathbb{C}[X,Y,\Lambda]$. The quotient $\mathbb{C}[X,Y,\Lambda]/I_\sigma(\Lambda)$ is a free $\mathbb{C}[\Lambda]$-module of rank $n+1$ with basis $\{X^iY^j : i < \sigma_j\} \cup \{Y^{m+1}\}$. In particular, this defines a flat family

$$\psi : \mathrm{Spec}(\mathbb{C}[X,Y,\Lambda]/I_\sigma(\Lambda)) \to \mathrm{Spec}(\mathbb{C}[\Lambda]) \simeq \mathbb{A}^1,$$

of closed subschemes of $\mathbb{C}^2$ of length $n+1$. Notice that the fiber of ψ over $\lambda \neq 0$ is $\mathrm{Spec}(\mathbb{C}[X,Y]/I_\sigma \times \mathbb{C}[X,Y]/(Y-\lambda, X))$, while the fiber over $\lambda = 0$ is $\mathrm{Spec}(\mathbb{C}[X,Y]/I_{\sigma'})$ where $\sigma' : \sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_m \geqslant \sigma_{m+1} > 0$, with $\sigma_{m+1} = 1$, is a partition of $n+1$.

Lemma 14. *The validity of the $n!$-result (rather the $(n+1)!$-result) for the partition σ' implies its validity for the partition σ, where σ' is as above.*

Proof. Since the set of points of $\mathrm{Hilb}^{n+1}(\mathbb{C}^2)$ where the maximal rank condition is satisfied is open (cf. Section 2), we get that it is satisfied in an open subset of $\mathrm{Hilb}^{n+1}(\mathbb{C}^2)$ containing $I_{\sigma'}$. In particular, from the above family, we get that the maximal rank condition is satisfied for $\psi^{-1}(\lambda)$ for some $\lambda \neq 0$. But $\psi^{-1}(\lambda) \simeq \mathrm{Spec}(\mathbb{C}[X,Y]/I_\sigma \times \mathbb{C})$. Finally, observe that the maximal rank condition is satisfied for the product of rings $A \times \mathbb{C}$ if and only if it is satisfied for A. This proves the lemma. □

Theorem 7. *To prove the validity of the $n!$-result for all $n \geqslant 1$ and all the partitions of n, it suffices to prove its validity for only the staircase partitions $\mathfrak{s}_m : m \geqslant m-1 \geqslant \ldots \geqslant 1$ (for all $m \geqslant 1$).*

Proof. For any $I \in \mathrm{Hilb}^n_0(\mathbb{C}^2)$, we first define $d = d(I)$ as the smallest integer such that $\mathbb{C}[X,Y]_d \subseteq I$. (Since $I \in \mathrm{Hilb}^n_0(\mathbb{C}^2)$, d exists.) Then clearly $n \leqslant d(d+1)/2$ with equality if and only if $I = I_{\mathfrak{s}_d}$ for the staircase partition $\mathfrak{s}_d$. Now define $N(I) = d(d+1)/2 - n$. Thus $N(I)$ measures, in some sense, how far I is from $I_{\mathfrak{s}_d}$. The assumption of the theorem means that the maximal rank condition is valid for any I with $N(I) = 0$. We prove the theorem by induction on $N(I)$.

So assume $N(I_\sigma) > 0$ and that the theorem has been proved for I_μ with $N(I_\mu) < N(I_\sigma)$.

Consider the orbit closure $\overline{BI_\sigma}$ inside $\mathrm{Hilb}^n(\mathbb{C}^2)$. By Lemma 11, there exists a B-fixed point $I_\mu \in \overline{BI_\sigma}$ for some partition μ. Since, for any d, $\mathbb{C}[X,Y]_d$ is stable under B (in fact, under $\mathrm{GL}_2(\mathbb{C})$), by Lemma 13, $\mathbb{C}[X,Y]_{d(I_\sigma)}$ is contained in I_μ. In particular, $d(I_\mu) \leqslant d(I_\sigma)$ and thus $N(I_\mu) \leqslant N(I_\sigma)$. Since the set of points in $\mathrm{Hilb}^n(\mathbb{C}^2)$ satisfying the maximal rank condition is open, we may now assume $\sigma = \mu$ or, in other words, we may assume that I_σ is B-stable. Then $\sigma : \sigma_0 > \sigma_1 > \ldots > \sigma_m > 0$ from which we conclude that $d(I_\sigma) \geqslant m+1$ with equality if and only if σ is a staircase partition (use Lemma 12). We therefore assume $d(I_\sigma) > m+1$.

Now, the partition σ' of $n+1$, as defined earlier, also satisfies that $\mathbb{C}[X,Y]_{d(I_\sigma)}$ is contained in $I_{\sigma'}$. But $N(I_{\sigma'}) < N(I_\sigma)$. Thus, by induction, the maximal rank condition follows for $I_{\sigma'}$. Finally, by Lemma 14, the maximal rank condition follows for I_σ. □

Remark 5. Observe that $\mathrm{GL}_2(\mathbb{C})$ has an invariant in $\mathrm{Hilb}^n(\mathbb{C}^2)$ if and only if n is of the form $d(d+1)/2$ and in this case $I_{\mathfrak{s}_d}$, for the staircase partition $\mathfrak{s}_d$, is the only invariant.

For the staircase partition $\mathfrak{s}_m : m \geqslant m-1 \geqslant \ldots \geqslant 1$ of $n := m(m+1)/2$, we have the following (stronger) variant of Theorem 3.

Proposition 5. *Let m be any positive integer and let B_m be a finite-dimensional graded S_n-equivariant quotient of the polynomial ring $R_n := \mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ such that $B_m^d = 0$ for $d > d_{\mathfrak{s}_m}$ and $B_m^{d_{\mathfrak{s}_m}}$ is the one-dimensional sign representation of S_n, where $n := m(m+1)/2$. Assume further that B_m is a Gorenstein algebra. Then*

$$B_m \simeq A_{\mathfrak{s}_m},$$

as graded S_n-algebras.

(*Observe that* $d_{\mathfrak{s}_m} = m(m-1)(m+1)/3$.)

Proof. The proof of this proposition follows by a similar argument as that of Theorem 3 in view of the following Lemma 15. ☐

Definition 3. Let n be a positive integer. Write n in the form

$$n = 1 + 2 + 3 + \ldots + m + s\ ,\ \ 0 \leqslant s < m+1.$$

The *degree* of n is defined as

$$\deg(n) = 1 \cdot 0 + 2 \cdot 1 + \ldots + m \cdot (m-1) + s \cdot m.$$

The integer s is called the *remainder* of n.

Lemma 15. *The lowest degree in R_n in which the sign representation appears is exactly* $\deg(n)$. *If the remainder of n is* 0 *then* (*and only then*) *there is a unique copy of the sign representation in degree* $\deg(n)$ *in R_n.*

Proof. Assume that $L \subseteq R_n^d$ is a copy of the sign representation. Let $l \in L$ be a nonzero element. Write

$$l = \textstyle\sum_{\underline{i},\underline{j}} \alpha_{\underline{i},\underline{j}} X^{\underline{i}} Y^{\underline{j}},$$

where $\underline{i} = (i_1, \ldots, i_n), \underline{j} = (j_1, \ldots, j_n) \in \mathbb{Z}_+^n$, $X^{\underline{i}} := X_1^{i_1} \ldots X_n^{i_n}$ and $Y^{\underline{j}}$ has a similar meaning. Take a nonzero $\alpha_{\underline{i'},\underline{j'}}$. If there exist $s \neq t$ such that $(i'_s, j'_s) = (i'_t, j'_t)$, then the permutation $(s,t) \in \mathrm{S}_n$ acts on l leaving the coefficient of $X^{\underline{i'}} Y^{\underline{j'}}$ invariant. As L is a copy of the sign representation this is impossible. Hence, $(i'_s, j'_s) \neq (i'_t, j'_t)$ whenever $s \neq t$, i.e., $(i'_1, j'_1), (i'_2, j'_2), \ldots, (i'_n, j'_n)$ are distinct elements in $\mathbb{Z}_+^2$.

Consider the function $\phi : \mathbb{Z}_+^2 \to \mathbb{Z}_+$ defined by $\phi(x,y) = x + y$. Call $\phi(x,y)$ the norm of (x,y). Note that there are exactly s elements in $\mathbb{Z}_+^2$ of norm $s-1$. Notice also that the degree of $X^{\underline{i'}} Y^{\underline{j'}}$ is $d = \sum_t \phi(i'_t, j'_t)$. As the pairs (i'_t, j'_t) are distinct, we get that $d \geqslant \deg(n)$.

Now we construct a copy of the sign representation inside $R_n^{\deg(n)}$: Choose n distinct pairs (i_t, j_t) such that the sum of the norms is $\deg(n)$ (start with pairs of norm 0, then take those of norm 1, $\ldots$). Notice that there is a unique way of doing this exactly when the remainder of n is 0. Put

$$L = \mathbb{C}\Big(\sum_{\sigma \in \mathrm{S}_n} \epsilon(\sigma)\sigma(X^{\underline{i}} Y^{\underline{j}})\Big).$$

Then L is a copy of the sign representation sitting in the degree equal to $\deg(n)$. ☐

7. Divided differential operators and the box partition

Let n be a positive integer and let $R_n^{\mathbb{Z}} := \mathbb{Z}[X_1,\ldots,X_n,Y_1,\ldots,Y_n]$ be the polynomial ring over $\mathbb{Z}$. For any $m \geqslant 0$ and any $1 \leqslant s \leqslant n$, the divided differential operators $\partial_{X_s}^{(m)} := \frac{1}{m!}(\frac{\partial}{\partial X_s})^m$ and $\partial_{Y_s}^{(m)} := \frac{1}{m!}(\frac{\partial}{\partial Y_s})^m$ keep $R_n^{\mathbb{Z}}$ stable. In particular, for any $\mathbb{Z}$-module M, we obtain the operators (again denoted by) $\partial_{X_s}^{(m)}$ and $\partial_{Y_s}^{(m)}$ on $R_n^M := R_n^{\mathbb{Z}} \otimes_{\mathbb{Z}} M$.

Now let p be a prime and k a field of char. p. We will particularly be interested in the operators $\partial_{X_s}, \partial_{X_s}^{(p)}$, ∂_{Y_s} and $\partial_{Y_s}^{(p)}$ acting on R_n^k. Similarly, we have the operators $\partial_{X_s}, \partial_{X_s}^{(p)}$ on $P_n^k := k[X_1,\ldots,X_n]$.

Define a k-linear map (in fact, a k-algebra homomorphism) $\phi : R_n^k \to P_n^k$, by $X^{\underline{i}}Y^{\underline{j}} \mapsto X^{\underline{i}+p\underline{j}}$, for any $\underline{i} = (i_1,\ldots,i_n)$ and $\underline{j} = (j_1,\ldots,j_n) \in \mathbb{Z}_+^n$.

Lemma 16. *For any $1 \leqslant s \leqslant n$ and $Q \in R_n^k$,*

(a) $\phi\partial_{X_s} = \partial_{X_s}\phi$,

(b) $\phi(\partial_{Y_s}Q) = \partial_{X_s}^{(p)}\phi(Q)$ *if* $\deg Q$ *in* X_s *is* $< p$.

Proof. (a) Let $\delta_s := (0,\ldots,1,0,\ldots,0) \in \mathbb{Z}_+^n$, where 1 is placed in the s-th place.

$$\phi(\partial_{X_s}X^{\underline{i}}Y^{\underline{j}}) = \phi(i_s(X^{\underline{i}-\delta_s}Y^{\underline{j}})) = i_sX^{\underline{i}-\delta_s+p\underline{j}}.$$

Also,

$$\partial_{X_s}\phi(X^{\underline{i}}Y^{\underline{j}}) = \partial_{X_s}X^{\underline{i}+p\underline{j}} = i_sX^{\underline{i}+p\underline{j}-\delta_s}.$$

This proves (a).

(b) Let $Q = X^{\underline{i}}Y^{\underline{j}}$. Then

$$\phi(\partial_{Y_s}Q) = \phi(j_sX^{\underline{i}}Y^{\underline{j}-\delta_s}) = j_sX^{\underline{i}+p(\underline{j}-\delta_s)}.$$

Also,

$$\partial_{X_s}^{(p)}\phi(Q) = \partial_{X_s}^{(p)}X^{\underline{i}+p\underline{j}} = \binom{i_s+pj_s}{p}X^{\underline{i}+p\underline{j}-p\delta_s} = j_sX^{\underline{i}+p\underline{j}-p\delta_s}, \text{ since } i_s < p.$$

This proves (b). $\square$

Now take $n = p^2$ (where p is a prime) and consider the box partition $\mathfrak{b}_p : p \geqslant p \geqslant \ldots \geqslant p$ of n. Let $V_n^k := k[\partial_{X_1},\ldots,\partial_{X_n},\partial_{Y_1},\ldots,\partial_{Y_n}](\triangle_{\mathfrak{b}_p}) \subset R_n^k$.

Note that $\triangle_{\mathfrak{b}_p}$ has degree $< p$ in each variables X_s and Y_s. The following lemma is clear.

Lemma 17. $\phi(\triangle_{\mathfrak{b}_p}) = \pm\triangle_n$ *where $\triangle_n \in P_n^k$ is the standard Vandermonde determinant* $\prod(X_s - X_t)$ *(product being taken over* $1 \leqslant s < t \leqslant n$*) and* $n := p^2$.

Let $\mathbb{D}_n^k$ be the algebra of all the divided differential operators in n variables, i.e., the algebra over k generated by $\{\partial_{X_s}^{(m)}; 1 \leqslant s \leqslant n, m \geqslant 0\}$ inside the algebra of all the linear operators of P_n^k. Since $\partial_{X_s}^{(p^2)}$ kills $\triangle_n$, by Lemma 16(b), we get the following.

Proposition 6.

$$\phi(V_n^k) = k[\partial_{X_1}, \dots, \partial_{X_n}, \partial_{X_1}^{(p)}, \dots, \partial_{X_n}^{(p)}]\triangle_n = \mathbb{D}_n^k(\triangle_n),$$

where $\mathbb{D}_n^k(\triangle_n)$ *denotes the space* $\{D(\triangle_n) : D \in \mathbb{D}_n^k\}$ *inside* P_n^k.

Corollary 1. $\dim_k V_n^k \geqslant \dim_k \mathbb{D}_n^k(\triangle_n)$.

Conjecture 3. *Let* n *be any positive integer and let* p *be a prime such that* $p^2 \geqslant n$. *Then, for any field* k *of characteristic* p,

$$\dim_k \mathbb{D}_n^k(\triangle_n) = \dim_{\mathbb{C}} \mathrm{Diff}_n(\triangle_n),$$

where $\mathrm{Diff}_n(\Delta_n) := \mathbb{C}[\partial_{X_1}, \dots, \partial_{X_n}] \cdot \Delta_n$.

The latter of course is well-known to have dimension $n!$.

Remark 6. (a) It is easy to see that

$$\dim_k \mathbb{D}_n^k(\triangle_n) \leqslant \dim_{\mathbb{C}} \mathrm{Diff}_n(\triangle_n).$$

(b) It is possible that the above conjecture is true for any prime p (not necessarily under the restriction $p^2 \geqslant n$), as some explicit calculations indicate, e.g., we have verified the conjecture for $n = 5, p = 2$.

(c) The above conjecture is false, in general, for the other polynomials which transform under the action of the symmetric group S_n via the sign character. Take, e.g., $n = 2$ and $f = (X_1 - X_2)(X_1 + X_2)$. Then, for $p = 2$, $\dim_k \mathbb{D}_n^k(f) = 2$, whereas $\dim_{\mathbb{C}} \mathrm{Diff}_n(f) = 4$.

Assuming the above conjecture, we get the following theorem.

Theorem 8. *Assume that* $n = p^2$ *and* k *is a field of char.* p. *If the above Conjecture* 3 *is true, then*

$$\dim_k V_n^k = n!.$$

In particular,

$$\dim_{\mathbb{C}} \mathrm{Diff}_{2n}(\Delta_{\mathfrak{b}_p}) = n!.$$

Thus the validity of Conjecture 3 *will imply the* $n!$ *result for the box partition* $\mathfrak{b}_p$ *of* $n = p^2$.

Proof. By combining Corollary 1 and Conjecture 3, we get that $\dim_k V_n^k \geqslant n!$. But, $\dim_k V_n^k \leqslant \dim_{\mathbb{C}} \mathrm{Diff}_{2n}(\Delta_{\mathfrak{b}_p}) \leqslant n!$. Thus, both the identities of the theorem follow. □

8. Gr description of A_σ (box plus one row case)

Let $n = pq+r$ with any integers $p > r \geqslant 0$ and $p > 1, q \geqslant 1$, and consider the *box plus one row partition* $\sigma = \sigma(p, q, r) : \sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_q > 0$ of n with $\sigma_i = p$ for $0 \leqslant i \leqslant q - 1$ and $\sigma_q = r$. In this section we describe the algebra A_σ as the associated graded ring of a specific filtration, depending only on p, of the cohomology ring $H^*(\mathrm{SL}_n/B, \mathbb{C})$. Here B is the Borel subgroup of SL_n consisting of the upper triangular matrices of determinant one. By a classical result we may identify $H^*(\mathrm{SL}_n/B, \mathbb{C})$ with the ring P_n/J, where $P_n := \mathbb{C}[X_1, \dots, X_n]$ and $J \subset P_n$ denotes the ideal generated by the S_n-invariant elements of positive degree.

8.1. Preliminary results

Recall the definitions of $S_{h,t,k} \in \mathbb{C}[Z_1, \ldots, Z_t]$ and $n_k(\sigma)$ from the appendix. In this section we will use the notation n_k and $n_k^\vee$ to denote $n_k(\sigma)$ and $n_k(\sigma^\vee)$ respectively. Then, for the partition $\sigma = \sigma(p, q, r)$,

$$n_k = \begin{cases} q(p-k) + (r-k) & \text{if } 0 \leqslant k \leqslant r, \\ q(p-k) & \text{if } r \leqslant k < p, \\ 0 & \text{if } k \geqslant p. \end{cases}$$

while

$$n_k^\vee = \begin{cases} p(q-k) + r & \text{if } 0 \leqslant k \leqslant q, \\ 0 & \text{if } k > q. \end{cases}$$

Let J_p denote the ideal in P_n generated by J and the elements $X_1^p, \ldots, X_n^p$. Then

Lemma 18. *Let h, k and $1 \leqslant t \leqslant n$ be nonnegative integers. If $h + t \geqslant n_k + 1$, then*

$$S = S_{h,t,k}(X_{i_1}, X_{i_2}, \ldots, X_{i_t}) \in J_p,$$

whenever $1 \leqslant i_1 < i_2 < \ldots < i_t \leqslant n$. In particular, by Theorem 10, $\hat{J}(\sigma) \subset J_p$, where $\sigma = \sigma(p, q, r)$.

Proof. Observe first of all that the statement is clear when $k \geqslant p$. In the following we hence assume that $k < p$. By symmetry it suffices to consider the case $i_j = j$, $j = 1, \ldots, t$. Assume first that $t \leqslant q$. Let $M = X_1^{s_1} X_2^{s_2} \ldots X_t^{s_t}$ be a monomial of degree h. If $s_j \leqslant p - k - 1$ for every $1 \leqslant j \leqslant t$, then

$$h = \deg(M) \leqslant t(p-k-1) \leqslant q(p-k) - t \leqslant n_k - t \leqslant h - 1,$$

which is a contradiction. Hence, there exists a j such that $s_j \geqslant p - k$. But then each term in S is divisible by some X_j^p and hence $S \in J_p$.

Assume now that $t \geqslant q + 1$ and that the result, by induction, is true for smaller values of t. If $k = 0$ then by Lemma 1.2 in [9] we find $S \in J \subseteq J_p$. So assume that $k \geqslant 1$ and that the statement, by induction, is correct for smaller values of k. As

$$(h+t) + t \geqslant h + t + q + 1 \geqslant n_k + q + 2 \geqslant n_{k-1} + 1\,,$$

we know by induction that

$$S_{h+t,t,k-1}(X_1, \ldots, X_t) \in J_p.$$

Rewriting $S_{h+t,t,k-1}(X_1, \ldots, X_t)$ as

$$\begin{aligned} &\textstyle\sum_{1 \leqslant i_1 < i_2 < \ldots < i_l \leqslant t} (X_1 \ldots X_t)^{k-1} (X_{i_1} X_{i_2} \ldots X_{i_l}) S_{h+t-l}^l (X_{i_1}, \ldots, X_{i_l}) \\ &\quad = \textstyle\sum_{1 \leqslant i_1 < i_2 < \ldots < i_l \leqslant t} \dfrac{(X_1 \ldots X_t)^{k-1}}{(X_{i_1} \ldots X_{i_l})^{k-1}} S_{h+t-l,l,k}(X_{i_1}, \ldots, X_{i_l}), \end{aligned} \tag{3}$$

and noticing, by induction, that all the terms with $l < t$ of the last expression belongs to J_p , we conclude that $S_{h,t,k}(X_1, \ldots, X_t) \in J_p$. $\square$

Let $J_q^\vee$ denote the ideal in P_n generated by the elements of the form

$$(X_{i_1} \ldots X_{i_{r+1}})^q \ , \ 1 \leqslant i_1 < \ldots < i_{r+1} \leqslant n,$$

together with the ideal J_{q+1} (defined similar to J_p).

Lemma 19. *Let h, k and $1 \leqslant t \leqslant n$ be nonnegative integers. If $h+t \geqslant n_k^\vee + 1$, then*

$$S = S_{h,t,k}(X_{i_1}, X_{i_2}, \ldots, X_{i_t}) \in J_q^\vee \,,$$

whenever $1 \leqslant i_1 < i_2 < \ldots < i_t \leqslant n$. In particular, by Theorem 10, *$\hat{J}(\sigma^\vee) \subset J_q^\vee$, where $\sigma = \sigma(p,q,r)$.*

Proof. Notice first of all that the statement is clear when $k > q$. Hence, in the following, we assume that $k \leqslant q$. By symmetry it suffices to consider the case $i_j = j$, $j = 1, \ldots, t$. Assume first that $t \leqslant p$. Assume, if possible, that $S \notin J_q^\vee$. Then, by the definition of $J_q^\vee$, there exists a monomial $M = X_1^{s_1} X_2^{s_2} \ldots X_t^{s_t}$ of degree $h+kt$ in the decomposition of S such that at most r of the indices s_j are $\geqslant q$ and none of the indices are $\geqslant q+1$. Hence

$$h + tk = \deg(M) \leqslant rq + (t-r)(q-1) \leqslant n_k^\vee + tk - t \leqslant h + tk - 1,$$

which is a contradiction.

Thus we are left with the case $t > p$. Assume by induction that the statement is correct for smaller values of t. If $k = 0$ then by Lemma 1.2 in [9] we find $S \in J \subseteq J_q^\vee$. So assume, by another induction, that $k \geqslant 1$ and that the statement is correct for smaller values of k. As

$$(h+t) + t \geqslant h + t + p + 1 \geqslant n_k^\vee + p + 2 \geqslant n_{k-1}^\vee + 1\,,$$

we know by induction that

$$S_{h+t,t,k-1}(X_1, \ldots, X_t) \in J_q^\vee .$$

Using the identity (3) for $S_{h+t,t,k-1}(X_1, \ldots, X_t)$ and noticing, by induction, that all the terms with $l < t$ belong to $J_q^\vee$, we conclude that $S_{h,t,k}(X_1, \ldots, X_t) \in J_q^\vee$. $\square$

Remark 7. By the definition of $\hat{J}(\sigma)$ and $\hat{J}(\sigma^\vee)$ and Lemmas 18, 19, it follows easily that, in fact, $J_p = \hat{J}(\sigma)$ and $J_q^\vee = \hat{J}(\sigma^\vee)$.

8.2. The Gr description

Let x_i denote $X_i + J \in P_n/J$ and consider the filtration of P_n/J:

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \,,$$

where $F_0 = \mathbb{C}$, $F_1 = \mathrm{span}_{\mathbb{C}}\{1, x_1, x_2, \ldots, x_n, x_1^p, x_2^p, \ldots, x_n^p\}$ and F_s is defined to be F_1^s (for any $s \geqslant 1$). Let $\mathrm{Gr}(F)$ denote the associated graded algebra

$$\mathrm{Gr}(F) := \bigoplus_{s \geqslant 0} F_s / F_{s-1},$$

where $F_{-1} := (0)$. Under the standard action of S_n on P_n, each F_s is S_n-stable and hence $\mathrm{Gr}(F)$ is canonically a S_n-module.

Consider now the algebra homomorphism $\pi^\vee : P_n \to \mathrm{Gr}(F)$ defined by $\pi^\vee(X_i) = x_i + F_0 \in F_1/F_0$. By Lemma 18, $\hat{J}(\sigma) \subseteq J_p \subseteq \ker(\pi^\vee)$ and thus, by Theorem 10, we get an induced map of graded algebras

$$\overline{\pi}^\vee : H^*(X_{\sigma^\vee}) \to \mathrm{Gr}(F).$$

Consider also the algebra homomorphism $\pi : P_n \to \mathrm{Gr}(F)$ defined by $\pi(X_i) = x_i^p + F_0 \in F_1/F_0$. As $p(q+1) \geqslant n$, we know that $X_i^{p(q+1)} \in J$ and hence $\pi(X_i^{q+1}) = 0$. Also, by [17], Corollary 3.2.2,

$$(X_{i_1} \ldots X_{i_{r+1}})^{qp} \in J.$$

Thus, from Lemma 19, we conclude that $\hat{J}(\sigma^\vee) \subseteq J_q^\vee \subseteq \ker(\pi)$. This gives us an induced map of graded algebras

$$\overline{\pi} : H^*(X_\sigma) \to \mathrm{Gr}(F).$$

Moreover, both $\overline{\pi}$ and $\overline{\pi}^\vee$ are S_n-equivariant.

Proposition 7. *The map* $\phi = \overline{\pi} \otimes \overline{\pi}^\vee : H^*(X_\sigma) \otimes H^*(X_{\sigma^\vee}) \to \mathrm{Gr}(F)$ *taking* $a \otimes b \mapsto \overline{\pi}(a)\cdot\overline{\pi}^\vee(b)$ *is an* S_n*-equivariant surjective graded algebra homomorphism. Furthermore, the top graded component of* $\mathrm{Gr}(F)$ *is of degree* d_σ *and is* 1*-dimensional supporting the sign representation of* S_n*.*

Proof. As $\mathrm{Im}(\phi)$, by definition, contains F_1/F_0 and F_0 and as $\mathrm{Gr}(F)$ (by definition) is generated by degree one elements, we get that ϕ is surjective. As the top graded component in $H^*(X_\sigma) \otimes H^*(X_{\sigma^\vee})$ is of degree d_σ, the top graded component in $\mathrm{Gr}(F)$ is of degree at most d_σ. As a representation of S_n, the ring $\mathrm{Gr}(F)$ is of course isomorphic to P_n/J and thus $\mathrm{Gr}(F)$ is isomorphic to the regular representation of S_n. In particular, $\mathrm{Gr}(F)$ contains a unique copy of the sign representation of S_n. But $H^*(X_\sigma) \otimes H^*(X_{\sigma^\vee})$ contains a unique copy of the sign representation which occurs in the top degree d_σ (cf. the proof of Proposition 3). Therefore, the copy of the sign representation in $\mathrm{Gr}(F)$ must also be placed in degree d_σ. So the proposition follows from the following. □

Proposition 8. *The top graded component of* $\mathrm{Gr}(F)$ *is* 1*-dimensional.*

Proof. By Remark 3.4.(ii) in [9], we know that the top degrees of $H^*(X_\sigma)$ and $H^*(X_{\sigma^\vee})$ are respectively

$$d = \frac{2qr + p(q-1)q}{2},$$
$$d^\vee = \frac{qp(p-1) + r(r-1)}{2}.$$

The image of the top degree in $H^*(X_\sigma) \otimes H^*(X_{\sigma^\vee})$ under ϕ, can therefore be represented by elements in P_n of degree

$$pd + d^\vee = \frac{n(n-1)}{2}.$$

But modulo J there is only one element (up to a constant) in P_n of this degree. □

Corollary 2. *Let* $n = pq + r$ *be as in the beginning of this section. Then there exists a* S_n*-equivariant isomorphism of graded algebras:*

$$\mathrm{Gr}(F) \simeq A_\sigma,$$

where $\sigma = \sigma(p, q, r)$*.*

Proof. Define the ideal $I \subset \mathrm{Gr}(F)$ by

$$I = \{a \in \mathrm{Gr}(F) : \text{the top degree component of}\, ab\, \text{equals}\, 0\, \text{for all}\, b \in \mathrm{Gr}(F)\}.$$

Clearly I is a S_n-stable graded ideal of $\mathrm{Gr}(F)$. Then $\mathrm{Gr}(F)/I$ satisfies all the characterizing properties of Theorem 3 (in view of Proposition 7). Thus $\mathrm{Gr}(F)/I \simeq T_\sigma$. But, by Theorem 4 and the $n!$ result, we know that T_σ has complex dimension $n! = \dim_{\mathbb{C}} \mathrm{Gr}(F)$. This forces $I = 0$ and hence $\mathrm{Gr}(F) \simeq T_\sigma \simeq A_\sigma$. □

Remark 8. (a) In the above proof we have used the $n!$ result for the partition $\sigma = \sigma(p, q, r)$. Conversely, the validity of the above corollary clearly implies the $n!$ result for σ.

(b) In [12], Section 6, a statement equivalent to Corollary 2 for 2-row shapes (i.e., $q = 1$) appears without proof. In the same paper it is noted that computational experiments suggest that Corollary 2 should be true. Recently, M. Haiman informed us that he had an unpublished proof of Corollary 2 (assuming the validity of the $n!$-conjecture for $\sigma(p, q, r)$) prior to our work.

9. Appendix: Cohomology of Springer fibers

We recall some well-known results on the cohomology of Springer fibers in the case of $G = \mathrm{SL}_n(\mathbb{C})$ (which we have used in the paper).

Let $\sigma : \sigma_0 \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_m > 0$ be a partition of n and let $\sigma^\vee : \sigma'_0 \geqslant \ldots \geqslant \sigma'_{m'} > 0$ be the dual partition. For any $m' < i \leqslant n$, set $\sigma'_i = 0$. For any $1 \leqslant k \leqslant n$, recall the definition of the integer $d_k(\sigma)$ from Section 3, equation (1). For any $k \geqslant 0$, let $n_k = n_k(\sigma)$ denote the integer

$$n_k := \sigma'_k + \sigma'_{k+1} + \ldots.$$

Let $G = \mathrm{SL}_n(\mathbb{C})$ and let X_σ be the Springer fiber (over complex numbers) corresponding to a fixed nilpotent matrix M_σ (of size $n \times n$) with Jordan blocks of sizes $\sigma_0, \sigma_1, \ldots, \sigma_m$. Let $H^*(X_\sigma)$ denote the cohomology ring of X_σ with coefficients in $\mathbb{C}$.

The following description of $H^*(X_\sigma)$ is due to Tanisaki [22], though in its present form we have taken it from [13], p. 84–85. The notation e_r denotes the r-th elementary symmetric polynomial. As earlier, $P_n := \mathbb{C}[X_1, \ldots, X_n]$.

Theorem 9. *As a graded algebra, we have an* S_n*-equivariant isomorphism*

$$H^*(X_\sigma) \simeq P_n / J(\sigma),$$

where $J(\sigma)$ *is the ideal generated by all polynomials of the form*

$$e_r(X_{s_1}, X_{s_2}, \ldots, X_{s_k})$$

subject to the conditions $1 \leqslant s_1 < \ldots < s_k \leqslant n$ *and all* $k, r \geqslant 1$ *satisfying* $k - d_k(\sigma) < r \leqslant k$.

Definition 4. Let $t \geqslant 1$ and $h, k \geqslant 0$ be integers. Let $S_h^t \in \mathbb{C}[Z_1, \dots, Z_t]$ denote the sum of all monomials of degree h in the variables $Z_1, \dots, Z_t$. We then define

$$S_{h,t,k} = (Z_1 \dots Z_t)^k S_h^t \in \mathbb{C}[Z_1, \dots, Z_t].$$

The following description of the cohomology of X_σ is taken from [9], Theorem 2.2.

Theorem 10. *Let $\hat{J}(\sigma)$ denote the ideal in P_n generated by the polynomials*

$$S_{h,t,k}(X_{i_1}, X_{i_2}, \dots, X_{i_t})$$

for nonnegative integers h, k and $1 \leqslant t \leqslant n$ subject to the condition $h + t = n_k + 1$, and $1 \leqslant i_1 < i_2 < \dots < i_t \leqslant n$. Then there exists an S_n*-equivariant isomorphism*

$$H^*(X_{\sigma^\vee}) \simeq P_n / \hat{J}(\sigma).$$

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