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Algebraization of Frobenius splitting via quantum groups

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Abstract

An important breakthrough in understanding the geometry of Schubert varieties was the introduction of the notion of Frobenius split varieties and the result that the flag varieties G/P are Frobenius split. The aim of this article is to give in this case a complete and self contained representation theoretic approach to this method. The geometric Frobenius method (in char $k = p > 0$) will here be replaced by Lusztig's Frobenius maps for quantum groups at roots of unity (which exist not only for primes but any odd integer $\ell > 1$).

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0. Introduction

The passage from representations of quantum groups at roots of unity in characteristic zero to representations of algebraic groups in characteristic p is extremely important in view of Lusztig's conjectures. (Recall that Andersen-Jantzen-Soergel confirmed the conjectured link in Lusztig's program but only asymptotically.) The aim of the present article is to establish that Lusztig's two Frobenius maps in characteristic zero lead naturally to two familiar objects in characteristic p. One Frobenius map leads simply to the Frobenius map in characteristic p, while the other leads to the so-called canonical Frobenius splitting on G/B and related varieties.

Let k be an algebraically closed field of arbitrary characteristic and let G be a semisimple simply-connected algebraic group over k with a fixed Borel subgroup B and the associated Weyl group W. Let $X = G/B$ be the flag variety and let (for any $w \in W$) $X(w) \subset X$ be the Schubert subvariety, which is the closure of the B-orbit BwB in G/B. For a homogeneous line bundle $\mathcal{L}$ on X, the cohomology groups $H^i(X, \mathcal{L})$ are G-modules and the corresponding groups $H^i(X(w), \mathcal{L}|_{X(w)})$ inherit naturally the structure of B-modules. These modules have been extensively studied from algebro-geometric as well as representation theoretic points of view.

An important breakthrough in understanding the geometry of Schubert varieties was the introduction, by Mehta-Ramanathan and Ramanan-Ramanathan, of the notion of a Frobenius $\mathcal{D}$–split variety X (defined over k of char $p > 0$) and compatibly split subvarieties, where $\mathcal{D}$ is a line bundle together with a nonzero section (cf. Definition 6.1). 'Very few' projective varieties turn out to be split but those which do have rather remarkable geometric and cohomological properties. The most important class of examples of varieties which are Frobenius split arise in group theory. In particular, the flag varieties $X = G/B$ are Frobenius split (in fact are $\mathcal{D}$-split for the homogeneous line bundle $\mathcal{D}$ corresponding to the character $-2(p-1)\rho$, where ρ is half the sum of positive roots) compatibly splitting all the Schubert subvarieties and so are the product varieties $X \times X$ compatibly splitting all the G-Schubert subvarieties. This leads to various important geometric facts about them (normality, projective normality, Cohen-Macaulay, projective Cohen-Macaulay, rational singularity etc.) and various representation theoretic results (vanishing theorems, Demazure character formula, good filtrations, etc.) (see the papers [MR1], [RR], [R1], [R2], [MR2], [M] etc.). However, this geometric method does not provide explicit representation theoretic information directly.

The aim of this article is to give a complete and self contained representation theoretic approach to these methods. The algebro–geometric "Frobenius methods" will here be replaced by Lusztig's two Frobenius maps for quantized enveloping algebras at roots of unity.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and denote by $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ the corresponding Borel subalgebra. Assume that $(\mathfrak{g}, \mathfrak{b})$ corresponds to the pair (G, B). Fix an odd integer $\ell > 1$ which is assumed to be coprime to 3 if $\mathfrak{g}$ has a component of type G_2. Choose a primitive ℓ^{th} root of unity ξ and let $\mathbb{Z}_\xi$ be the corresponding ring of cyclotomic integers. Let $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$ denote the corresponding quantized enveloping algebra over $\mathbb{Z}_\xi$ obtained by the base change $\mathbb{Z}[v, v^{-1}] \to \mathbb{Z}_\xi, v \mapsto \xi$, from Lusztig's $\mathbb{Z}[v, v^{-1}]$-form of the quantized enveloping algebra $U_{\mathbb{Q}(v)}(\mathfrak{g})$ (divided by the ideal generated by the central elements $K_i^\ell - 1$) and let $\bar{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$ be the corresponding classical enveloping algebra over $\mathbb{Z}_\xi$ (obtained via the base change $\mathbb{Z} \hookrightarrow \mathbb{Z}_\xi$ from Kostant's $\mathbb{Z}$-form of the classical enveloping algebra $\bar{U}(\mathfrak{g})$ over $\mathbb{C}$). The subalgebras $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}), \mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{n}), \mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{n}^-)$ (resp. $\bar{U}_{\mathbb{Z}_\xi}(\mathfrak{b}), \bar{U}_{\mathbb{Z}_\xi}(\mathfrak{n}), \bar{U}_{\mathbb{Z}_\xi}(\mathfrak{n}^-)$) are defined similarly.

Lusztig [Lu2] has defined Frobenius homomorphisms $\mathrm{Fr} : \mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g}) \to \bar{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$ which maps the generators by dividing the exponents by ℓ when possible (cf. Theorem 1.1), and $\mathrm{Fr}' : \bar{U}_{\mathbb{Z}_\xi}(\mathfrak{n}) \to \mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{n})$ which maps the generators by multiplying the exponents by ℓ. In fact, Fr' extends to a homomorphism (still denoted by) $\mathrm{Fr}' : \bar{U}_{\mathbb{Z}_\xi}(\mathfrak{b}) \to \mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})$ (cf. Theorem 1.2). We make crucial use of these maps Fr and Fr' together with the homological machinery developed by Andersen-Polo-Wen [APW] to define natural functors (for any $\bar{U}_{\mathbb{Z}_\xi}(\mathfrak{b})$-module $\bar{M}$):

$$\mathrm{Fr}^* : H^i\Big(\bar{U}(\mathfrak{g})/\bar{U}(\mathfrak{b}), \bar{M}\Big)^{\mathrm{Fr}} \longrightarrow H^i\Big(\mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{b}), \bar{M}^{\mathrm{Fr}}\Big)$$

of $\mathfrak{U}(\mathfrak{g})$-modules (cf. Theorem 2.3), and (for any $\mathfrak{U}(\mathfrak{b})$-module M),

$$\mathrm{Fr}'^* : H^i\Big(\mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{b}), M\Big)^{\mathrm{Fr}'} \to H^i\Big(\bar{U}(\mathfrak{g})/\bar{U}(\mathfrak{b}), M^{\mathrm{Fr}'}\Big)$$

of $\bar{U}_{\mathbb{Z}_\xi}(\mathfrak{b})$-modules (cf. Theorem 3.8), where we have abbreviated $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$ by $\mathfrak{U}(\mathfrak{g})$ etc. and $\bar{M}^{\mathrm{Fr}}$ is a $\mathfrak{U}(\mathfrak{b})$-module under Fr and the superscript Fr' has a similar meaning. Moreover, the composition of these two maps $\mathrm{Fr}'^* \circ \mathrm{Fr}^*$ is the identity map (cf. Corollary 3.9). The first map is our representation theoretic replacement of the Frobenius morphism F (which corresponds to the p^{th} power map on functions) and the second corresponds to a splitting map. (For one dimensional representations, these maps are given more explicitly in [KL] for $i = 0$.)

To define a representation theoretic analogue of the $\mathcal{D}$-splitting in our setting, consider the element $F_o \in \mathfrak{U}(\mathfrak{n}^-)$ defined as the product of all divided $(\ell - 1)$-powers $F_\beta^{(\ell-1)}$ of Lusztig's root vectors, where the ordering has been chosen relative to a reduced decomposition of the longest element in the Weyl group. The idea is then to "twist" the splitting Fr'^* by F_o to get, for any $\bar{U}(\mathfrak{b})$-module $\bar{M}$, a functorial $\bar{U}(\mathfrak{b})$-module map

$$\mathrm{Fr}'^*_\gamma : H^i\Big(\mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{b}), \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} \to H^i(\bar{U}(\mathfrak{g})/\bar{U}(\mathfrak{b}), \bar{M}),$$

where χ^ξ_γ stands for the one-dimensional $\mathfrak{U}(\mathfrak{b})$-representation of weight $\gamma = -2(\ell-1)\rho$ (cf. Theorem 4.7). Moreover, both the maps Fr'^* and Fr'^*_γ also commute with the action (induced by Fr') of $\bar{U}(\mathfrak{n}^-)$. Further, all the above three maps are compatible with any base change. (Note that though $\bar{U}(\mathfrak{b})$ and $\bar{U}(\mathfrak{n}^-)$ both act on $H^i\Big(\mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{b}), M\Big)$ and similarly on $H^i\Big(\mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{b}), \chi^\xi_\gamma \otimes \bar{M}^{\mathrm{Fr}}\Big)$ via Fr', these actions do not in general glue together to provide a $\bar{U}(\mathfrak{g})$-action.)

Now assume that $\ell = p$ is a prime and k is an algebraically closed field of characteristic p. Since the constructions of $\mathrm{Fr}^*, \mathrm{Fr}'^*$ and Fr'^*_γ are compatible with any base change, we consider them under the base change $\mathbb{Z}_\xi \to k$ taking $\xi \mapsto 1$. Recall that, for any $\mathfrak{U}(\mathfrak{b})$-module M, there is a canonical isomorphism (cf. Proposition 5.1),

$$H^i(\mathfrak{U}_k(\mathfrak{g})/\mathfrak{U}_k(\mathfrak{b}), M_k) \simeq H^i(G/B, \mathcal{L}(M)),$$

and similarly, for a $\bar{U}(\mathfrak{b})$-module $\bar{M}$, there is a canonical isomorphism

$$H^i(\bar{U}_k(\mathfrak{g})/\bar{U}_k(\mathfrak{b}), \bar{M}_k) \simeq H^i(G/B, \mathcal{L}(\bar{M})),$$

where $M_k := M \otimes_{\mathbb{Z}_\xi} k$ etc. and $\mathcal{L}(M)$ denotes the homogeneous vector bundle on the flag variety G/B associated to the $\mathfrak{U}_k(\mathfrak{b})$ (and hence B) module M_k. Using these identifications and the usual Serre vanishing theorem, one readily deduces from the above functors Fr^* and Fr'^*_γ the standard Kempf vanishing theorem asserting that for any weight λ such that $\lambda + \rho$ is dominant, $H^i(G/B, \mathcal{L}(-\lambda)) = 0$ for all $i > 0$ (cf. Theorem 5.2).

In Section 6 we establish a precise connection between our representation theoretic approach with the algebro-geometric Frobenius splitting mentioned earlier. Actually, by an appropriate 'sheafification' we obtain from the functors Fr'^* and Fr'^*_γ an entirely new proof (purely from the representation theory of quantum groups) of the Frobenius splitting and the stronger Frobenius $\mathcal{D} = \mathcal{L}(\gamma)$-splitting respectively of the flag variety G/B, and these compatibly split all the Schubert subvarieties X_w (cf. Theorems 6.4, 6.5 and 6.7). In fact, from our constructions, it is immediately clear that the splitting thus obtained is canonical (in the sense of Mathieu), a property which is not so transparent (though true) from the original (geometric) proof of the splitting of G/B given by Mehta-Ramanathan. In particular, from the uniqueness (noted by Mathieu) of the canonical splitting on G/B, we deduce that our splitting coincides with the original splitting given in [MR1].

Since our constructions live at the level of algebras over $\mathbb{Z}_\xi$, we can view them as 'characteristic zero lift' of the (characteristic p) Frobenius splitting and Frobenius $\mathcal{D}$-splitting of G/B. Also, they are defined for any odd integer $\ell > 1$ which is coprime to 3 if $\mathfrak{g}$ has a component of type G_2 (not only for primes). It is possible that the above restriction on ℓ can be removed by using certain results of Kaneda and Andersen-Paradowski (cf. Remark A.8).

We extend the above constructions and results to cover the case of the product flag variety $G/B \times G/B$ and deduce, by methods as above, that $G/B \times G/B$ is Frobenius split such that all the G-Schubert subvarieties (in particular the diagonal) are compatibly split (cf. Theorem 7.5).

We also deduce the splitting of the Bott-Samelson desingularization from an analogous quantum setup (cf. Section 8).

Even though in a large part of the paper, for notational convenience, we considered the case of the (full) flag variety G/B, most of the results can easily be generalised to cover the case of G/P for any parabolic subgroup P. We formulate the extensions in Section 9 but omit the proofs as they are similar to that of G/B.

For completeness and convenience of the reader, we collect various important (and by now standard) consequences of the above Frobenius splitting results in the appendix. This includes normality, the Demazure character formula, projective normality, Cohen-Macaulay, projective Cohen-Macaulay, and rational singularity of Schubert varieties in G/P. In particular, with our setup, these results follow from the representation theory of quantum groups and the Serre vanishing theorem. There are other algebraic proofs of the Demazure character formula using quantum groups: by Kashiwara using his crystal base [Kas] and by Littelmann using his LS path model [Li].

We believe that many other results (concerning the Frobenius splitting property of varieties arising in group theory) are amenable to the methods of this paper: e.g., we believe that one can deduce the 'good filtration property' originally due to Donkin in most cases (and proved by Mathieu in general).

This paper is a sequel to our paper [KL], where a weaker form of some of the results of this paper are proved. However, we have kept the exposition of this paper almost self-contained (with the exception of [KL, Lemma 3 and Th. 1], which we use here without including a proof).

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1. Notation, preliminaries and review of certain results of Lusztig and Andersen-Polo-Wen

Let us fix a Cartan matrix of finite type $A = (a_{i,j})_{1 \le i,j \le n}$. Then there is a diagonal matrix D with positive integral diagonal entries $(d_1, \cdots, d_n)$ with $d_i \in \{1, 2, 3\}$ such that g.c.d. $(d_1, \cdots, d_n) = 1$ and DA is symmetric and positive definite.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the semisimple Lie algebra over $\mathbb{C}$ associated to the Cartan matrix A. Recall that $\mathfrak{g}$ is generated by its Cartan subalgebra $\mathfrak{h}$ and positive root vectors $\{\bar{E}_1, \cdots, \bar{E}_n\}$ and negative root vectors $\{\bar{F}_1, \cdots, \bar{F}_n\}$ subject to certain relations. Let $\mathfrak{b}; \mathfrak{n}; \mathfrak{b}^-; \mathfrak{n}^-$ be the Lie subalgebras of $\mathfrak{g}$ generated respectively by $\{\mathfrak{h}, \bar{E}_1, \cdots, \bar{E}_n\}$; $\{\bar{E}_1, \cdots, \bar{E}_n\}$; $\{\mathfrak{h}, \bar{F}_1, \cdots, \bar{F}_n\}$; $\{\bar{F}_1, \cdots, \bar{F}_n\}$. Let $\bar{U}_{\mathbb{Z}}(\mathfrak{g})$ be the Kostant $\mathbb{Z}$-form of the enveloping algebra $\bar{U}(\mathfrak{g})$, i.e., the $\mathbb{Z}$-subalgebra of $\bar{U}(\mathfrak{g})$ generated by $\{\bar{E}_i^{(m)}, \bar{F}_i^{(m)}, \binom{h_i}{m}; 1 \le i \le n, m \in \mathbb{Z}_+\}$, where $\bar{E}_i^{(m)} := \frac{\bar{E}_i^m}{m!}$, $h_i := [\bar{E}_i, \bar{F}_i]$ and $\binom{h_i}{m} := \frac{h_i(h_i-1)\cdots(h_i-m+1)}{m!}$. Let $\bar{U}_{\mathbb{Z}}(\mathfrak{b})$, $\bar{U}_{\mathbb{Z}}(\mathfrak{n})$ be the subalgebras of $\bar{U}_{\mathbb{Z}}(\mathfrak{g})$ generated by $\left\{\bar{E}_i^{(m)}, \binom{h_i}{m}; 1 \le i \le n, m \in \mathbb{Z}_+\right\}$ and $\left\{\bar{E}_i^{(m)}; 1 \le i \le n, m \in \mathbb{Z}_+\right\}$ respectively. Similarly, let $\bar{U}_{\mathbb{Z}}^0$ be the subalgebra of $\bar{U}_{\mathbb{Z}}(\mathfrak{g})$ generated by $\left\{\binom{h_i}{m}; 1 \le i \le n, m \in \mathbb{Z}_+\right\}$. For any $\mathbb{Z}$-algebra $\mathcal{B}$, by $\bar{U}_{\mathcal{B}}(\mathfrak{g})$ we mean

$$\bar{U}_{\mathcal{B}}(\mathfrak{g}) = \bar{U}_{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathcal{B},$$

with a similar meaning for $\bar{U}_{\mathcal{B}}(\mathfrak{b})$ etc.

Now we come to the corresponding quantized algebras. Let $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$, where v is an indeterminate and let $\mathbb{Q}(v)$ be the quotient field of $\mathcal{A}$. Let $U_{\mathbb{Q}(v)}(\mathfrak{g})$ be the quantized enveloping algebra over the field $\mathbb{Q}(v)$ generated by $\left\{E_i, F_i, K_i^{\pm 1}; 1 \le i \le n\right\}$ and subject to certain relations (cf. [Lu2, §1.1]). Let $U_{\mathcal{A}}(\mathfrak{g})$ be the $\mathcal{A}$-form of $U_{\mathbb{Q}(v)}(\mathfrak{g})$ defined by Lusztig, i.e., $U_{\mathcal{A}}(\mathfrak{g})$ is the $\mathcal{A}$-subalgebra of $U_{\mathbb{Q}(v)}(\mathfrak{g})$ generated by $\left\{E_i^{(m)}, F_i^{(m)}, K_i^{\pm 1}; 1 \le i \le n, m \in \mathbb{Z}_+\right\}$, where

$$E_i^{(m)} := \frac{E_i^m}{[m]!_{d_i}},\ [m]!_{d_i} := \prod_{h=1}^{m} \frac{v^{d_i h} - v^{-d_i h}}{v^{d_i} - v^{-d_i}} \in \mathcal{A}.$$

Let $U_{\mathcal{A}}(\mathfrak{n})$, $U_{\mathcal{A}}^0$ be the $\mathcal{A}$-subalgebras of $U_{\mathcal{A}}(\mathfrak{g})$ generated by $\left\{E_i^{(m)}; 1 \le i \le n, m \in \mathbb{Z}_+\right\}$ and $\left\{K_i^{\pm 1}, \begin{bmatrix} K_i; c \\ m \end{bmatrix}; 1 \le i \le n, m \in \mathbb{Z}_+, c \in \mathbb{Z}\right\}$ respectively, where

$$\begin{bmatrix} K_i; c \\ m \end{bmatrix} := \prod_{h=1}^{m} \frac{K_i v^{d_i(c-h+1)} - K_i^{-1} v^{-d_i(c-h+1)}}{v^{d_i h} - v^{-d_i h}}.$$

Also let $U_{\mathcal{A}}(\mathfrak{b})$ be the subalgebra of $U_{\mathcal{A}}(\mathfrak{g})$ generated by $U_{\mathcal{A}}^0$ and $U_{\mathcal{A}}(\mathfrak{n})$. We similarly define $U_{\mathcal{A}}(\mathfrak{n}^-)$ and $U_{\mathcal{A}}(\mathfrak{b}^-)$. Then $U_{\mathbb{Q}(v)}(\mathfrak{g})$ is a Hopf algebra with comultiplication Δ, antipode (an anti-automorphism) S and co-unit ϵ given by:

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i,\ \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i,\ \Delta K_i = K_i \otimes K_i\,,$$

$$SE_i = -K_i^{-1} E_i,\ SF_i = -F_i K_i,\ SK_i = K_i^{-1}\,,$$

$$\epsilon E_i = \epsilon F_i = 0,\ \epsilon K_i = 1\,.$$

Then Δ and S keep $U_{\mathcal{A}}(\mathfrak{g}), U_{\mathcal{A}}(\mathfrak{b}), U_{\mathcal{A}}(\mathfrak{b}^-)$ and $U_{\mathcal{A}}^0$ stable.

We now fix an odd integer $\ell > 1$ which is, in addition, assumed to be coprime to 3 if G_2 is a component of $\mathfrak{g} = \mathfrak{g}(A)$. This will be our tacit restriction on ℓ. Now choose a primitive ℓ^{th} root of unity ξ and let $\mathbb{Z}_\xi$ be the corresponding ring of cyclotomic integers with quotient field $\mathbb{Q}_\xi$; i.e., $\mathbb{Q}_\xi$ (resp. $\mathbb{Z}_\xi$) is obtained from $\mathbb{Q}$ (resp. $\mathbb{Z}$) by attaching ξ.

For any $\mathcal{A}$-algebra $\mathcal{B}$, by $U_\mathcal{B}(\mathfrak{g})$ we mean

$$U_\mathcal{B}(\mathfrak{g}) = U_\mathcal{A}(\mathfrak{g}) \otimes_\mathcal{A} \mathcal{B},$$

and a similar meaning for $U_\mathcal{B}(\mathfrak{b})$ etc. In particular, taking $\mathcal{B} = \mathbb{Z}_\xi$ with the homomorphism $\mathcal{A} \to \mathcal{B}$, $v \mapsto \xi$, we get $U_{\mathbb{Z}_\xi}(\mathfrak{g})$ etc.

We recall the following result due to Lusztig [Lu2, Cor. 8.14].

THEOREM 1.1. *There is a unique $\mathbb{Z}_\xi$-algebra homomorphism*

$$\mathrm{Fr} : U_{\mathbb{Z}_\xi}(\mathfrak{g}) \to \bar{U}_{\mathbb{Z}_\xi}(\mathfrak{g}),$$

taking $E_i^{(m)} \mapsto \bar{E}_i^{(m/\ell)}$, $F_i^{(m)} \mapsto \bar{F}_i^{(m/\ell)}$, $K_i^{\pm 1} \mapsto 1$; where $\bar{E}_i^{(h)}$ and $\bar{F}_i^{(h)}$ mean 0 if $h \notin \mathbb{Z}_+$ and, moreover, $\mathbb{Z} \hookrightarrow \mathbb{Z}_\xi$ is the canonical inclusion.

Then Fr *takes $\binom{K_i;0}{m} \mapsto \binom{h_i}{m/\ell}$ if ℓ divides m, and 0 otherwise.*

We also have the following theorem [KL, Lemma 3]. The corresponding result with $\mathfrak{b}$ replaced by $\mathfrak{n}$ was proved by Lusztig [Lu2, Lemma 8.6].

THEOREM 1.2. *There is a unique $\mathbb{Z}_\xi$-algebra homomorphism*

$$\mathrm{Fr}' = \mathrm{Fr}'_\mathfrak{b} : \bar{U}_{\mathbb{Z}_\xi}(\mathfrak{b}) \to U_{\mathbb{Z}_\xi}(\mathfrak{b}) / \langle K_i^\ell - 1; 1 \le i \le n \rangle\,,$$

taking $\bar{E}_i^{(m)} \mapsto E_i^{(m\ell)}$, $\binom{h_i}{m} \mapsto \binom{K_i;0}{\ell m}$; where $\langle\ \rangle$ denotes the (two sided) ideal of $U_{\mathbb{Z}_\xi}(\mathfrak{b})$ generated by the central elements $\{K_i^\ell - 1\}$.

Similarly, we have the $\mathbb{Z}_\xi$-algebra homomorphism

$$\mathrm{Fr}'_{\mathfrak{b}^-} : \bar{U}_{\mathbb{Z}_\xi}(\mathfrak{b}^-) \to U_{\mathbb{Z}_\xi}(\mathfrak{b}^-) / \langle K_i^\ell - 1; 1 \le i \le n \rangle\,.$$

Let $X := \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{Z} \text{ for all } 1 \le i \le n\}$ be the set of weights and $X^+ := \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{Z}_+ \text{ for all } 1 \le i \le n\}$ be the set of dominant weights. For any $\lambda \in X$, define the character $\chi_\lambda : U^0_\mathcal{A} \to \mathcal{A}$ by

$$\chi_\lambda\Big(K_i^{\pm 1}\Big) = v^{\pm d_i \lambda(h_i)}, \qquad \text{and}$$

$$\chi_\lambda \begin{bmatrix} K_i; c \\ m \end{bmatrix} = \begin{bmatrix} \lambda(h_i) + c \\ m \end{bmatrix}_{d_i},$$

where, for $m \in \mathbb{Z}_+$ and $c \in \mathbb{Z}$,

$$\begin{bmatrix} c \\ m \end{bmatrix}_{d_i} := \prod_{h=1}^{m} \frac{v^{d_i(c-h+1)} - v^{-d_i(c-h+1)}}{v^{d_i h} - v^{-d_i h}} \in \mathcal{A}.$$

In particular, χ_λ gives rise to a homomorphism by extending the scalars (denoted by)

$$\chi_\lambda^\xi : U^0_{\mathbb{Z}_\xi} \to \mathbb{Z}_\xi.$$

Moreover, χ_λ^ξ descends to give a homomorphism (again denoted by)

$$\chi_\lambda^\xi : U^0_{\mathbb{Z}_\xi}/\langle K_i^\ell - 1; 1 \le i \le n\rangle \to \mathbb{Z}_\xi.$$

Similarly, there is a homomorphism

$$\begin{array}{rcl} \bar{\chi}_\lambda : \bar{U}^0_{\mathbb{Z}} & \to \mathbb{Z}, & \text{taking} \\ \begin{pmatrix} h_i \\ m \end{pmatrix} & \mapsto & \begin{pmatrix} \lambda(h_i) \\ m \end{pmatrix}, \end{array}$$

where for $h \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, $\binom{h}{m}$ is the ordinary binomial coefficient

$$\begin{pmatrix} h \\ m \end{pmatrix} := \frac{h(h-1)\cdots(h-m+1)}{m!}.$$

By extending the scalars $\mathbb{Z} \hookrightarrow \mathbb{Z}_\xi$, we get a $\mathbb{Z}_\xi$-algebra homomorphism (still denoted by)

$$\bar{\chi}_\lambda : \bar{U}^0_{\mathbb{Z}_\xi} \to \mathbb{Z}_\xi.$$

Let us denote, by the corresponding Gothic letter,

$$\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g}) := U_{\mathbb{Z}_\xi}(\mathfrak{g})/\langle K_i^\ell - 1; 1 \le i \le n\rangle$$

with a similar meaning for $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})$ and $\mathfrak{U}^0_{\mathbb{Z}_\xi}$. By its definition, Fr descends to a homomorphism (again denoted by)

$$\mathrm{Fr} : \mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g}) \to \bar{U}_{\mathbb{Z}_\xi}(\mathfrak{g}).$$

Definition 1.3. A $\mathfrak{U}^0_{\mathbb{Z}_\xi}$-module M is called a *weight module* if

$$\begin{array}{rcl} M & = & \oplus_{\lambda\in X} M_\lambda, \qquad \text{where} \\ M_\lambda & := & \{v \in M : av = \chi_\lambda^\xi(a)v, \text{ for all } a \in \mathfrak{U}^0_{\mathbb{Z}_\xi}\}. \end{array}$$

Let $\mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$ (resp. $\mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{g})$) be the category of those $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})$ (resp. $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$) -modules M such that $M = F_\mathfrak{b}(M)$ (resp. $M = F_\mathfrak{g}(M)$), where

$$\begin{array}{rcl} F_\mathfrak{b}(M) & := & \Big\{v \in \oplus_{\lambda\in X} M_\lambda : E_i^{(m)}v = 0, \text{ for } m \ge m(v), \text{for some } m(v) \in \mathbb{Z}_+\Big\}, \\ F_\mathfrak{g}(M) & := & \Big\{v \in \oplus_{\lambda\in X} M_\lambda : E_i^{(m)}v = F_i^{(m)}v = 0, \text{ for } m \ge m(v)\Big\}. \end{array}$$

In particular, any $M \in \mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$ is a weight module under $\mathfrak{U}^0_{\mathbb{Z}_\xi}$.

We similarly define the notion of weight modules for $\bar{U}^0_{\mathbb{Z}}$ (resp. $\bar{U}^0_{\mathbb{Z}_\xi}$) and define the categories $\bar{\mathcal{C}}_{\mathbb{Z}}(\mathfrak{b})$ and $\bar{\mathcal{C}}_{\mathbb{Z}}(\mathfrak{g})$ (resp. $\bar{\mathcal{C}}_{\mathbb{Z}_\xi}(\mathfrak{b})$ and $\bar{\mathcal{C}}_{\mathbb{Z}_\xi}(\mathfrak{g})$). Then these are abelian categories (see [APW, §2.2]).

(1.4) *The induction functor.* Following [APW, §2], define the induction functor $H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})/\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}), -\Big) : \mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b}) \to \mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{g})$ as follows:

Take $M \in \mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$. Consider the space $\tilde{M} := \mathrm{Hom}_{\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})}\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g}), M\Big)$ of $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})$-module maps from $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$ to M, where $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})$ acts on $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$ via left multiplication. Then $\tilde{M}$ is a $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$-module under $(a \cdot f)(b) = f(ba)$, for $a, b \in \mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$. Now set

$$H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})/\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}), M\Big) := F_{\mathfrak{g}}(\tilde{M}).$$

Then this is a left exact covariant functor.

There is a natural $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})$-module homomorphism

$$ev : H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})/\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}), M\Big) \to M,$$

defined by

$$ev(f) = f(1).$$

For $M \in \mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{g})$ we have a $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$-module isomorphism:

$$\theta : M \to H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})/\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}), M\Big)$$

given by $\theta(m)a = a.m$, for $m \in M$ and $a \in \mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$.

For a $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$-module M, the dual space $M^* := \mathrm{Hom}_{\mathbb{Z}_\xi}(M, \mathbb{Z}_\xi)$ is a $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$-module under

$$(af)m = f(S(a)m)\,,\ \text{for}\, a \in \mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})\,,\, f \in M^* \ \text{and}\, m \in M\,.$$

For $\lambda \in X^+$, there is a $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$-module isomorphism

$$\beta : V_\xi(\lambda)^* \to H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})/\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}), \chi^\xi_{-\lambda}\Big) \tag{1}$$

given by $\beta(f)(a) = f(S(a)v_\lambda)\, v^*_\lambda$, for $f \in V_\xi(\lambda)^*$ and $a \in \mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})$, where $V_\xi(\lambda)$ is the Weyl module over $\mathbb{Z}_\xi$ with highest weight λ (cf. [APW, Prop. 1.20 (ii)] or [KL, §1]), v_λ is a highest weight primitive vector of $V_\xi(\lambda)$ and $v^*_\lambda \in \mathrm{Hom}(V_\xi(\lambda)_\lambda, \mathbb{Z}_\xi)$ is given by $v^*_\lambda(v_\lambda) = 1$. (Observe that β does not depend upon the choice of v_λ.)

Exactly the same way we define the functor

$$H^0\Big(\bar{U}_{\mathbb{Z}}(\mathfrak{g})/\bar{U}_{\mathbb{Z}}(\mathfrak{b}), -\Big) : \bar{\mathcal{C}}_{\mathbb{Z}}(\mathfrak{b}) \to \bar{\mathcal{C}}_{\mathbb{Z}}(\mathfrak{g}).$$

We also need to consider the induction functor

$$H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})/\mathfrak{U}^0_{\mathbb{Z}_\xi}, -\Big) : \mathcal{C}^0_{\mathbb{Z}_\xi} \to \mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$$

defined by

$$H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})/\mathfrak{U}^0_{\mathbb{Z}_\xi}, M\Big) = F_{\mathfrak{b}}\bigg(\mathrm{Hom}_{\mathfrak{U}^0_{\mathbb{Z}_\xi}}\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}), M\Big)\bigg)\,,$$

where $\mathcal{C}^0_{\mathbb{Z}_\xi}$ is the category of weight modules of $\mathfrak{U}^0_{\mathbb{Z}_\xi}$.

Similarly, one defines the functor

$$H^0\Big(\bar U_{\mathbb{Z}}(\mathfrak{b})/\bar U^0_{\mathbb{Z}},-\Big):\bar{\mathcal{C}}^0_{\mathbb{Z}}\to\bar{\mathcal{C}}_{\mathbb{Z}}(\mathfrak{b}).$$

PROPOSITION 1.5 ([APW, Prop. 2.11 and Cor. 2.13]).

a) *All the abelian categories* $\mathcal{C}^0_{\mathbb{Z}_\xi}$, $\mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$, $\mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{g})$ *have enough injective objects.*

b) *The induction functors* $H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})/\mathfrak{U}^0_{\mathbb{Z}_\xi},-\Big)$ *and* $H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})/\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}),-\Big)$ *take injective objects to injective objects.*

c) *The induction functor* $H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})/\mathfrak{U}^0_{\mathbb{Z}_\xi},-\Big)$ *is an exact functor, which takes* $\mathbb{Z}_\xi$*-free modules to* $\mathbb{Z}_\xi$*-free modules.*

An analogous result is true for the categories $\bar{\mathcal{C}}^0_{\mathbb{Z}_\xi},\bar{\mathcal{C}}_{\mathbb{Z}_\xi}(\mathfrak{b}),\bar{\mathcal{C}}_{\mathbb{Z}_\xi}(\mathfrak{g})$ *(and* $\bar{\mathcal{C}}^0_{\mathbb{Z}}$, $\bar{\mathcal{C}}_{\mathbb{Z}}(\mathfrak{b}),\bar{\mathcal{C}}_{\mathbb{Z}}(\mathfrak{g})$*) as well.*

(Actually the setting of the above proposition in [APW] is slightly different, but the same proof works.)

Fix $M\in\mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$. We need a certain specific resolution of M in the category $\mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$. By the Frobenius reciprocity [APW, Prop. 2.12], M is a $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})$-submodule of $Q_0 := H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})/\mathfrak{U}^0_{\mathbb{Z}_\xi},M\Big)$ under $m\mapsto i_m$ (where $i_m(X)=X.m$) and, moreover, M is a $\mathfrak{U}^0_{\mathbb{Z}_\xi}$ direct summand of Q_0. Apply the same to Q_0/M and set $Q_1 := H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})/\mathfrak{U}^0_{\mathbb{Z}_\xi},Q_0/M\Big)$, etc. This gives a resolution of M in $\mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$:

$$(*)\qquad 0\to M\to Q_0\to Q_1\to\cdots.$$

If M is $\mathbb{Z}_\xi$-free then so are each of the Q_i. We refer to $(*)$ as the *standard resolution* of M in the category $\mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$.

Definition 1.6. Since the category $\mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$ has enough injectives (by Proposition 1.5), the right derived functors of $H^0\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})/\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}),-\Big)$ are defined. Denote them by $H^i\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})/\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}),-\Big)$.

Similarly the $H^i\Big(\bar U_{\mathbb{Z}}(\mathfrak{g})/\bar U_{\mathbb{Z}}(\mathfrak{b}),-\Big)$ are defined.

We will abbreviate $H^i\Big(\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{g})/\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b}),-\Big)$ (resp. $H^i\Big(\bar U_{\mathbb{Z}_\xi}(\mathfrak{g})/\bar U_{\mathbb{Z}_\xi}(\mathfrak{b}),-\Big)$) by $H^i\Big(\mathfrak{X},-\Big)$ (resp. $H^i\Big(\bar X,-\Big)$).

PROPOSITION 1.7 ([APW, Prop. 2.19]). *For any* $M\in\mathcal{C}_{\mathbb{Z}_\xi}(\mathfrak{b})$, *the modules* Q_j *in the standard resolution* $(*)$ *of* M *have*

$$H^i\Big(\mathfrak{X},Q_j\Big)=0\quad\textit{for all } i>0, j\ge 0.$$

A similar result is true for $H^i\Big(\bar X,\bar Q_j\Big)$.

2. Definition of quantized Frobenius homomorphism

From now on we drop the subscript $\mathbb{Z}_\xi$ from $\mathfrak{U}^0_{\mathbb{Z}_\xi}$, $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{b})$ etc.; i.e., $\mathfrak{U}^0$ means $\mathfrak{U}^0_{\mathbb{Z}_\xi}$ etc. Similarly, the category $\bar{\mathcal{C}}_{\mathbb{Z}_\xi}(\mathfrak{b})$ is abbreviated as $\bar{\mathcal{C}}(\mathfrak{b})$ etc.

Definition 2.1. For any $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{b})$, let $\bar{M}^{\mathrm{Fr}} \in \mathcal{C}(\mathfrak{b})$ be defined by taking $\bar{M}^{\mathrm{Fr}} = \bar{M}$ as a $\mathbb{Z}_\xi$-module and the action of $a \in \mathfrak{U}(\mathfrak{b})$ on $\bar{M}^{\mathrm{Fr}}$ is defined as

(1) $$a \odot m = \mathrm{Fr}(a) \cdot m.$$

Observe that, for $\lambda \in X$,

(2) $$\bar{\chi}_\lambda \circ \mathrm{Fr}_{|\mathfrak{U}^0} = \chi^\xi_{\ell\lambda}.$$

To prove (2), use [Lu3, Lemma 34.1.2(c)].

From (2) it is easy to see that $\bar{M}^{\mathrm{Fr}}$ is a weight module and hence $\bar{M}^{\mathrm{Fr}} \in \mathcal{C}(\mathfrak{b})$.

Clearly, for any $\bar{U}(\mathfrak{b})$-module morphism $f : \bar{M} \to \bar{N}$ $(\bar{M}, \bar{N} \in \bar{\mathcal{C}}(\mathfrak{b}))$, the same map $f : \bar{M}^{\mathrm{Fr}} \to \bar{N}^{\mathrm{Fr}}$ is a $\mathfrak{U}(\mathfrak{b})$-module morphism.

Exactly the same way we can define $\bar{M}^{\mathrm{Fr}} \in \mathcal{C}(\mathfrak{g})$ (resp. $\in \mathcal{C}^0$) for $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{g})$ (resp. $\in \bar{\mathcal{C}}^0$).

LEMMA 2.2. a) *For any $\bar{M} \in \bar{\mathcal{C}}^0$, there is a functorial $\mathfrak{U}(\mathfrak{b})$-module map*

$$\mathrm{Fr}^*_{\mathfrak{b}} : \Big(H^0(\bar{U}(\mathfrak{b})/\bar{U}^0, \bar{M})\Big)^{\mathrm{Fr}} \longrightarrow H^0\Big(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, \bar{M}^{\mathrm{Fr}}\Big),$$

defined by

(1) $$(\mathrm{Fr}^*_{\mathfrak{b}} f)(a) = f(\mathrm{Fr}(a)),$$

for $f \in \Big(H^0(\bar{U}(\mathfrak{b})/\bar{U}^0, \bar{M})\Big)^{\mathrm{Fr}} := \Big(F_{\mathfrak{b}}\Big(\mathrm{Hom}_{\bar{U}^0}(\bar{U}(\mathfrak{b}), \bar{M})\Big)\Big)^{\mathrm{Fr}}$ *and* $a \in \mathfrak{U}(\mathfrak{b})$.

Similarly,

b) *For any $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{b})$, there is a functorial $\mathfrak{U}(\mathfrak{g})$-module map*

$$\mathrm{Fr}^* : H^0(\bar{X}, \bar{M})^{\mathrm{Fr}} \longrightarrow H^0\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}}\Big),$$

defined by

(2) $$(\mathrm{Fr}^* f)(a) = f(\mathrm{Fr}(a)),\ \textit{for}\ f \in H^0(\bar{X}, \bar{M})^{\mathrm{Fr}}\ \textit{and}\ a \in \mathfrak{U}(\mathfrak{g})\,.$$

Proof. We prove (a); the proof of (b) is identical.

Clearly, $\mathrm{Fr}^*_{\mathfrak{b}} f$ is a $\mathfrak{U}^0$-module map. Moreover, for $a, b \in \mathfrak{U}(\mathfrak{b})$,

$$\begin{aligned}\Big(a \cdot (\mathrm{Fr}^*_{\mathfrak{b}} f)\Big)\, b = (\mathrm{Fr}^*_{\mathfrak{b}} f)\,(ba) &= f\,(\mathrm{Fr}\, b\, \mathrm{Fr}\, a) \\ &= \Big((\mathrm{Fr}\, a) \cdot f\Big)(\mathrm{Fr}\, b).\end{aligned}$$

This implies that

$$a \cdot (\mathrm{Fr}^*_{\mathfrak{b}} f) = \mathrm{Fr}^*_{\mathfrak{b}}\Big((\mathrm{Fr}\, a) \cdot f\Big). \tag{3}$$

By (3) it is easy to see that $\mathrm{Fr}^*_{\mathfrak{b}} f \in H^0\Big(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, \bar{M}^{\mathrm{Fr}}\Big)$ and, moreover, $\mathrm{Fr}^*_{\mathfrak{b}}$ is a $\mathfrak{U}(\mathfrak{b})$-module map. □

Now we extend the above $\mathfrak{U}(\mathfrak{g})$-module map Fr^* to an arbitrary H^i, still keeping the same notation.

THEOREM 2.3. *For any $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{b})$, there exists a functorial $\mathfrak{U}(\mathfrak{g})$-module map*

$$\mathrm{Fr}^* : H^i(\bar{X}, \bar{M})^{\mathrm{Fr}} \longrightarrow H^i\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}}\Big) \tag{1}$$

in the sense that for any $\bar{U}(\mathfrak{b})$-module morphism $\bar{M} \to \bar{N}$ $(\bar{M}, \bar{N} \in \bar{\mathcal{C}}(\mathfrak{b}))$, the following diagram is commutative

$$\begin{array}{ccc} H^i\Big(\bar{X}, \bar{M}\Big)^{\mathrm{Fr}} & \xrightarrow{\mathrm{Fr}^*} & H^i\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}}\Big) \\ \big\downarrow & & \big\downarrow \\ H^i\Big(\bar{X}, \bar{N}\Big)^{\mathrm{Fr}} & \xrightarrow{\mathrm{Fr}^*} & H^i\Big(\mathfrak{X}, \bar{N}^{\mathrm{Fr}}\Big), \end{array} \tag{D}$$

where the vertical maps are induced maps in cohomology.

Proof. Let

$$0 \longrightarrow \bar{M} \longrightarrow \bar{Q}_0 \xrightarrow{\bar{\varepsilon}_0} \bar{Q}_1 \xrightarrow{\bar{\varepsilon}_1} \cdots, \tag{2}$$

$$0 \longrightarrow \bar{M}^{\mathrm{Fr}} \longrightarrow Q_0 \xrightarrow{\varepsilon_0} Q_1 \xrightarrow{\varepsilon_1} \cdots, \tag{3}$$

be the standard resolutions in the categories $\bar{\mathcal{C}}(\mathfrak{b})$ and $\mathcal{C}(\mathfrak{b})$ respectively (cf. $(*)$ of §1.5). We construct by induction on i, using Lemma 2.2, a $\mathfrak{U}(\mathfrak{b})$-module morphism

$$\theta_i : \bar{Q}_i^{\mathrm{Fr}} \to Q_i \qquad \text{for each } i \geq 0,$$

making the following squares commutative:

$$\begin{array}{ccccccc} \bar{M}^{\mathrm{Fr}} & \longrightarrow & \bar{Q}_0^{\mathrm{Fr}} & \longrightarrow & \bar{Q}_1^{\mathrm{Fr}} & \longrightarrow & \cdots \\ \big\downarrow{\scriptstyle I} & & \big\downarrow{\scriptstyle \theta_0} & & \big\downarrow{\scriptstyle \theta_1} & & \\ \bar{M}^{\mathrm{Fr}} & \longrightarrow & Q_0 & \longrightarrow & Q_1 & \longrightarrow & \cdots. \end{array} \tag{D'}$$

We first take $i = 0$. Then, by definition,

$$\begin{aligned} \bar{Q}_0 &= H^0(\bar{U}(\mathfrak{b})/\bar{U}^0, \bar{M}), \qquad \text{and} \\ Q_0 &= H^0(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, \bar{M}^{\mathrm{Fr}}). \end{aligned}$$

Then θ_0 is the map $\mathrm{Fr}^*_{\mathfrak{b}}$ of Lemma 2.2(a). Assume now that we have constructed θ_j ($j \leq i$) making all the squares in (D′) commutative up to θ_i. Now we construct θ_{i+1} as follows: By definition

$$\begin{aligned} \bar{Q}_{i+1} &= H^0(\bar{U}(\mathfrak{b})/\bar{U}^0, \bar{Q}_i/\mathrm{Image}\ \bar{\varepsilon}_{i-1}), \quad \text{and} \\ Q_{i+1} &= H^0(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, Q_i/\mathrm{Image}\ \varepsilon_{i-1}). \end{aligned}$$

By Lemma 2.2(a), we have the $\mathfrak{U}(\mathfrak{b})$-module map

$$\mathrm{Fr}^*_{\mathfrak{b}} : (\bar{Q}_{i+1})^{\mathrm{Fr}} \longrightarrow H^0\Big(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, (\bar{Q}_i/\mathrm{Image}\ \bar{\varepsilon}_{i-1})^{\mathrm{Fr}}\Big).$$

From the commutativity of (D′) for the square containing θ_{i-1} and θ_i, we get a $\mathfrak{U}(\mathfrak{b})$ (and hence $\mathfrak{U}^0$)-module map $\bar{\theta}_i : (\bar{Q}_i/\mathrm{Image}\ \bar{\varepsilon}_{i-1})^{\mathrm{Fr}} \to Q_i/\mathrm{Image}\ \varepsilon_{i-1}$ induced by θ_i. Inducing the map $\bar{\theta}_i$ via the functor $H^0(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, -)$ and composing this with $\mathrm{Fr}^*_{\mathfrak{b}}$ we get the desired $\mathfrak{U}(\mathfrak{b})$-module map θ_{i+1}. This completes the induction.

The resolution (2) gives rise to a complex by taking $H^0(\bar{X}, -)^{\mathrm{Fr}}$:

$$(4) \qquad 0 \to H^0(\bar{X}, \bar{Q}_0)^{\mathrm{Fr}} \to H^0(\bar{X}, \bar{Q}_1)^{\mathrm{Fr}} \to \cdots.$$

Similarly, the resolution (3) gives rise to the complex:

$$(5) \qquad 0 \to H^0(\mathfrak{X}, Q_0) \to H^0(\mathfrak{X}, Q_1) \to \cdots.$$

Define the $\mathfrak{U}(\mathfrak{g})$-module map (for any $i \geq 0$)

$$\beta_i : H^0(\bar{X}, \bar{Q}_i)^{\mathrm{Fr}} \to H^0\Big(\mathfrak{X}, Q_i\Big)$$

as the composite of $\mathrm{Fr}^* : H^0(\bar{X}, \bar{Q}_i)^{\mathrm{Fr}} \to H^0\Big(\mathfrak{X}, \bar{Q}_i^{\mathrm{Fr}}\Big)$ (guaranteed by Lemma 2.2(b)) and the map $\theta_i^* : H^0\Big(\mathfrak{X}, \bar{Q}_i^{\mathrm{Fr}}\Big) \to H^0\Big(\mathfrak{X}, Q_i\Big)$ induced by the $\mathfrak{U}(\mathfrak{b})$-module map $\theta_i : \bar{Q}_i^{\mathrm{Fr}} \to Q_i$.

The $\mathfrak{U}(\mathfrak{g})$-module maps β_i give rise to a cochain map from the cochain complex (4) to the cochain complex (5). Taking cohomology, we get the existence of the map (1). (Observe that Fr^* being an exact functor, the i^{th} cohomology of the complex (4) is the same as $H^i(\bar{X}, \bar{M})^{\mathrm{Fr}}$.) The functoriality of Fr^* follows from the functoriality of all the constructions involved. □

Remark 2.4. (a) As we will see in a subsequent section, Fr^* is a quantization of the map induced on the cohomology of homogeneous vector bundles from the Frobenius morphism of the flag varieties G/B.

(b) As informed by H. H. Andersen, for an extension $\mathbb{Z}_\xi \to k$ where k is a field, Theorem 2.3 can also be deduced from [AW, Prop. 2.4].

3. Definition of quantized Frobenius splitting

We continue to use the same abbreviation $\mathfrak{U}^0$ for $\mathfrak{U}^0_{\mathbb{Z}_\xi}$ etc. as given in the beginning of Section 2.

Recall the definition of the algebra homomorphism Fr' from Theorem 1.2.

Analogous to the Definition 2.1, we make the following.

Definition 3.1. For any $M \in \mathcal{C}(\mathfrak{b})$, let $M^{\mathrm{Fr}'} \in \bar{\mathcal{C}}(\mathfrak{b})$ be defined by taking $M^{\mathrm{Fr}'} = M$ as a $\mathbb{Z}_\xi$-module and the action of $a \in \bar{U}(\mathfrak{b})$ on $M^{\mathrm{Fr}'}$ is defined by

$$a \odot m = \mathrm{Fr}' a \cdot m. \tag{1}$$

Observe that for $\lambda \in X$, with $\lambda = \lambda_0 + \ell\lambda_1$, where $0 \leq \lambda_0(h_i) \leq \ell - 1$ for all $1 \leq i \leq n$, and $\lambda_1 \in X$,

$$\chi^\xi_\lambda \circ \mathrm{Fr}'_{|\bar{U}^0} = \bar{\chi}_{\lambda_1}. \tag{2}$$

To prove (2), again use [Lu3, Lemma 34.1.2(c)] or [KL, Lemma 3].

Similarly, for any $M \in \mathcal{C}^0$, we define $M^{\mathrm{Fr}'} \in \bar{\mathcal{C}}^0$.

Clearly for any $\mathfrak{U}(\mathfrak{b})$-module morphism $f : M \to N$ $(M, N \in \mathcal{C}(\mathfrak{b}))$, the same map $f : M^{\mathrm{Fr}'} \to N^{\mathrm{Fr}'}$ is a $\bar{U}(\mathfrak{b})$-module morphism.

Exactly by the same proof as that of Lemma 2.2(a), we get the following:

LEMMA 3.2. *For any $M \in \mathcal{C}^0$, there is a functorial $\bar{U}(\mathfrak{b})$-module map*

$$\mathrm{Fr}'^*_{\mathfrak{b}} : H^0(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, M)^{\mathrm{Fr}'} \to H^0\Big(\bar{U}(\mathfrak{b})/\bar{U}^0, M^{\mathrm{Fr}'}\Big),$$

defined by

$$(\mathrm{Fr}'^*_{\mathfrak{b}} f)(a) = f(\mathrm{Fr}'(a)), \tag{1}$$

for $f \in H^0(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, M)^{\mathrm{Fr}'}$, and $a \in \bar{U}(\mathfrak{b})$. □

For a $\mathfrak{U}^0$-module V, $V^{\frac{1}{\ell}}$ denotes the sum of weight spaces corresponding to the weights $\lambda \in \ell X$.

PROPOSITION 3.3. *For any $M \in \mathcal{C}(\mathfrak{b})$, there is a functorial $\bar{U}(\mathfrak{b}^-)$-module map*

$$\mathrm{Fr}'^* : \Big(H^0\Big(\mathfrak{X}, M\Big)^{\frac{1}{\ell}}\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \to H^0\Big(\bar{X}, M^{\mathrm{Fr}'}\Big),$$

defined by

$$(\mathrm{Fr}'^* f)(a) = f(\mathrm{Fr}'(a)), \textit{ for } f \in \Big(H^0\Big(\mathfrak{X}, M\Big)^{\frac{1}{\ell}}\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \textit{ and } a \in \bar{U}(\mathfrak{b}^-). \tag{1}$$

Moreover, for any $m \geq 0$ and $f \in H^0(\mathfrak{X}, M)^{\frac{1}{\ell}}$,

$$\bar{E}_i^{(m)} \cdot (\mathrm{Fr}'^* f) = \mathrm{Fr}'^*(E_i^{(m\ell)} \cdot f). \tag{2}$$

Consider the projection

$$\pi : H^0(\mathfrak{X}, M) \to H^0(\mathfrak{X}, M)^{\frac{1}{\ell}}$$

obtained by decomposing

$$H^0(\mathfrak{X}, M) = H^0(\mathfrak{X}, M)^{\frac{1}{\ell}} \oplus \Big(H^0(\mathfrak{X}, M)^{\frac{1}{\ell}}\Big)^{\perp},$$

where $\perp$ *is the sum of weight spaces corresponding to the weights* $\lambda \notin \ell X$. (*Observe that* π *is a* $\bar{U}(\mathfrak{b}^-)$*-module map if the module structures of both the domain and range are twisted by* $\mathrm{Fr}'_{\mathfrak{b}^-}$ *and similarly* π *is a* $\bar{U}(\mathfrak{b})$*-module map.*) *Composing* Fr'^* *with* π, *we get a* $\bar{U}(\mathfrak{b}^-)$*-module map* (*again denoted by*)

$$\mathrm{Fr}'^* : H^0\Big(\mathfrak{X}, M\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \to H^0\Big(\bar{X}, M^{\mathrm{Fr}'}\Big).$$

Proof. We have by the triangular decomposition [Lu2, Th. 6.7(d)],

$$\text{(D)}\quad \begin{array}{ccccc} H^0\Big(\mathfrak{X}, M\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} & \hookrightarrow & \mathrm{Hom}_{\mathfrak{U}(\mathfrak{b})}\Big(\mathfrak{U}(\mathfrak{g}), M\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} & \overset{\delta}{\simeq} & \mathrm{Hom}_{\mathfrak{U}^0}\Big(\mathfrak{U}(\mathfrak{b}^-), M\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \\ & & & & \psi \downarrow \\ H^0\Big(\bar{X}, M^{\mathrm{Fr}'}\Big) & \hookrightarrow & \mathrm{Hom}_{\bar{U}(\mathfrak{b})}\Big(\bar{U}(\mathfrak{g}), M^{\mathrm{Fr}'}\Big) & \overset{\bar{\delta}}{\simeq} & \mathrm{Hom}_{\bar{U}^0}\Big(\bar{U}(\mathfrak{b}^-), M^{\mathrm{Fr}'}\Big), \end{array}$$

where $\psi = \mathrm{Fr}'^*_{\mathfrak{b}^-}$ is the $\bar{U}(\mathfrak{b}^-)$-module map of Lemma 3.2 with $\mathfrak{b}$ replaced by $\mathfrak{b}^-$, and $\delta, \bar{\delta}$ are the restriction maps.

We now show that

$$(*)\qquad \psi\bigg(\Big(H^0\Big(\mathfrak{X}, M\Big)^{\frac{1}{\ell}}\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}}\bigg) \subset H^0\Big(\bar{X}, M^{\mathrm{Fr}'}\Big).$$

Take a weight vector $f \in H^0(\mathfrak{X}, M)$ with respect to the $\mathfrak{U}^0$-action. Then since ψ is a $\bar{U}(\mathfrak{b}^-)$-module map, by (2) of §(3.1), $\psi(f)$ is a weight vector with respect to the $\bar{U}^0$-action. Next,

$$\psi\Big(F_i^{(m\ell)} \cdot f\Big) = \psi\Big(\bar{F}_i^{(m)} \odot f\Big) = \bar{F}_i^{(m)} \cdot \psi(f), \qquad \text{for any } m \geq 0.$$

In particular, $\bar{F}_i^{(m)} \cdot \psi(f) = 0$ for all large enough m. So, to prove the assertion $(*)$, it suffices to show that for $f \in \mathrm{Hom}_{\mathfrak{U}(\mathfrak{b})}\Big(\mathfrak{U}(\mathfrak{g}), M\Big)^{\frac{1}{\ell}}$,

$$(3)\qquad \psi\Big(E_i^{(m\ell)} \cdot f\Big) = \bar{E}_i^{(m)} \cdot \psi(f), \text{ for any } m \geq 0.$$

This will also prove (2).

As a preparation, we prove the following lemmas.

LEMMA 3.4. *For any nonnegative integers* $m, m_1, \cdots, m_r$ *and* $1 \leq i, i_1, \cdots, i_r \leq n$, *let* $1 \leq j_1 < \cdots < j_s \leq r$ *be precisely the indices such that* $i_{j_p} = i$.

Then in the quantized enveloping algebra $U_{\mathcal{A}}(\mathfrak{g})$:

$$(1)\qquad F_{i_r}^{(m_r)}\cdots F_{i_1}^{(m_1)}E_i^{(m)} = \sum_{t=(t_1,\cdots,t_s)\in\mathbb{Z}_+^s} E_i^{(m-\sum_{k=1}^s t_k)}F_{i_r}^{(m_r)} \cdots F_{i_{j_s}}^{(m_{j_s}-t_s)}\cdots F_{i_{j_1}}^{(m_{j_1}-t_1)}\cdots F_{i_1}^{(m_1)}A_t\ ,$$

where $E_i^{(m')}$ and $F_i^{(m')}$ are interpreted as 0 if $m'<0$,

$$A_t := \begin{bmatrix} K_i^{-1}; m_{j_1}-m-\sum_{j<j_1}a_jm_j \\ t_1\end{bmatrix}\begin{bmatrix} K_i^{-1}; m_{j_2}-m-\sum_{j<j_2}a_jm_j - t_1 \\ t_2\end{bmatrix} \cdots \begin{bmatrix} K_i^{-1}; m_{j_s}-m-\sum_{j<j_s}a_jm_j-(t_1+\cdots+t_{s-1}) \\ t_s\end{bmatrix},$$

$$a_j := -\alpha_{i_j}(h_i),\quad \text{and}\quad \begin{bmatrix} K_i^{-1}; c \\ t\end{bmatrix} := \prod_{s=1}^{t}\frac{K_i^{-1}v^{d_i(c-s+1)}-K_iv^{-d_i(c-s+1)}}{v^{d_is}-v^{-d_is}}.$$

Proof. Prove the lemma by induction on r, using the commutation relations [Lu2, §6.5], the following lemma and the $\mathcal{A}$-algebra automorphism ω of $\mathfrak{U}_{\mathcal{A}}(\mathfrak{g})$ as in [Lu3, §3.1.3]. □

LEMMA 3.5. *For $t\in\mathbb{N}:=\{1,2,\cdots\}$ and $c\in\mathbb{Z}$, as elements of $\mathfrak{U}_{\mathcal{A}}(\mathfrak{J})$,*

$$\begin{bmatrix} K_i^{-1}; c \\ t\end{bmatrix} E_j^{(m)} = E_j^{(m)}\begin{bmatrix} K_i^{-1}; c-a_{ij}m \\ t\end{bmatrix}$$

and

$$\begin{bmatrix} K_i^{-1}; c \\ t\end{bmatrix} F_j^{(m)} = F_j^{(m)}\begin{bmatrix} K_i^{-1}; c+a_{ij}m \\ t\end{bmatrix},$$

where $a_{ij}:=\alpha_j(h_i)$.

Proof. Apply the automorphism ω of [Lu3, §3.1.3] to the identities [Lu2, §6.5]. □

LEMMA 3.6. *For any $m\in\mathbb{Z}$ and $t\geq 0$*

$$\mathrm{Fr}\begin{bmatrix} K_i; \ell m \\ t\end{bmatrix} = 0$$

if t is not divisible by ℓ, where $\begin{bmatrix} K_i;\ell m \\ t\end{bmatrix}$ is interpreted as an element of $\mathfrak{U}_{\mathbb{Z}_\xi}(\mathfrak{J})$.

Proof. First assume that $m<0$. Then the lemma follows from [Lu2, §6.4-b3]. By [Lu2, §6.5–a6], we have $\begin{bmatrix} K_i;\ell m \\ t\end{bmatrix}F_i^{(\ell m)} = F_i^{(\ell m)}\begin{bmatrix} K_i;-\ell m \\ t\end{bmatrix}$. From this the case $m>0$ also follows. □

Lemma 3.7. *For $t \in \mathbb{Z}_+$, $c, a \in \mathbb{Z}$,*

$$(1) \quad \begin{bmatrix} K_i^{-1}; c \\ t \end{bmatrix} = (-1)^t \begin{bmatrix} K_i; t-1-c \\ t \end{bmatrix} \text{ as elements of } \mathfrak{U}_{\mathcal{A}}(\mathfrak{J}), \text{ and}$$

$$(2) \quad \begin{bmatrix} a \\ t \end{bmatrix}_{d_i}^{\xi} = (-1)^t \begin{bmatrix} -a+t-1 \\ t \end{bmatrix}_{d_i}^{\xi},$$

where $\left[\begin{smallmatrix} a \\ t \end{smallmatrix}\right]_{d_i}^{\xi}$ *denotes* $\left[\begin{smallmatrix} a \\ t \end{smallmatrix}\right]_{d_i} \in \mathcal{A}$ *evaluated at* $v = \xi$.

We also recall the q-binomial identity (for $0 \leq a_0, t_0 \leq \ell - 1, a_1 \in \mathbb{Z}$, $t_1 \in \mathbb{Z}_+$*) from* [Lu3, *Lemma* 34.1.2]:

$$(3) \quad \begin{bmatrix} a_0 + \ell a_1 \\ t_0 + \ell t_1 \end{bmatrix}_{d_i}^{\xi} = \begin{pmatrix} a_1 \\ t_1 \end{pmatrix} \begin{bmatrix} a_0 \\ t_0 \end{bmatrix}_{d_i}^{\xi},$$

where $\binom{a_1}{t_1} \in \mathbb{Z}_+$ *is the ordinary binomial coefficient.*

Proof. The first identity follows from the definition. For the second identity see [Lu3, p. 266]. □

Proof of (3) *of* §3.3 *continued.* First take $f \in \mathrm{Hom}_{\mathfrak{U}(\mathfrak{b})}(\mathfrak{U}(\mathfrak{g}), M)$. Then, by Lemma 3.4 (following the same notation),

$$(1) \quad \psi\Big(E_i^{(m\ell)} \cdot f\Big)\Big(\bar{F}_{i_r}^{(m_r)} \cdots \bar{F}_{i_1}^{(m_1)}\Big) = f\Big(F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)} E_i^{(m\ell)}\Big) = \sum_t E_i^{(\ell m - \sum_{k=1}^s t_k)} \cdot \Big((A_t \cdot f)(F_t)\Big),$$

where $F_t := F_{i_r}^{(\ell m_r)} \cdots F_{i_{j_s}}^{(\ell m_{j_s} - t_s)} \cdots F_{i_{j_1}}^{(\ell m_{j_1} - t_1)} \cdots F_{i_1}^{(\ell m_1)}$.

Now assume that $f \in \mathrm{Hom}_{\mathfrak{U}(\mathfrak{b})}(\mathfrak{U}(\mathfrak{g}), M)^{\frac{1}{\ell}}$, and f is of weight $\ell\lambda$, for $\lambda \in X$. Then (by Lemma 3.7) the above sum reduces to

$$(*) \quad \sum_t E_i^{(\ell m - \sum_{k=1}^s t_k)} \cdot \Big(\begin{bmatrix} -\ell\lambda(h_i) + \ell p'_1 \\ t_1 \end{bmatrix}_{d_i}^{\xi} \cdots \begin{bmatrix} -\ell\lambda(h_i) + \ell p'_s - (t_1 + \cdots + t_{s-1}) \\ t_s \end{bmatrix}_{d_i}^{\xi} f(F_t) \Big),$$

for some $p'_1, \cdots, p'_s \in \mathbb{Z}$. If at least one of $t_1, \cdots, t_s$ is not divisible by ℓ, say t_j, and t_j is the first one with this property, then

$$\begin{bmatrix} -\ell\lambda(h_i) + \ell p'_j - (t_1 + \cdots + t_{j-1}) \\ t_j \end{bmatrix}_{d_i}^{\xi} = 0,$$

by (3) of Lemma 3.7.

So the sum $(*)$ reduces to $t = (t_1, \cdots, t_s)$ such that $\ell | t$, i.e., each t_k is divisible by ℓ giving

$$\psi\Big(E_i^{(m\ell)} \cdot f\Big)\Big(\bar{F}_{i_r}^{(m_r)} \cdots \bar{F}_{i_1}^{(m_1)}\Big) = \sum_{\ell | t} E_i^{(\ell m - \sum_{k=1}^s t_k)} \cdot \Big((A_t \cdot f)(F_t)\Big). \tag{2}$$

On the other hand, applying Fr to the commutation relation as in Lemma 3.4, we get

$$\begin{aligned}\Big(\bar{E}_i^{(m)} \cdot (\psi f)\Big)\Big(\bar{F}_{i_r}^{(m_r)} \cdots \bar{F}_{i_1}^{(m_1)}\Big) \\ &= \sum_{\ell|t} \Big(\mathrm{Fr}'\Big(\mathrm{Fr}\, E_i^{(\ell m - \sum_{k=1}^s t_k)}\Big)\Big) \cdot \Big(((\mathrm{Fr}'\mathrm{Fr}\, A_t) \cdot f)(F_t)\Big) \\ &= \sum_{\ell|t} E_i^{(\ell m - \sum_{k=1}^s t_k)} \cdot \Big((A_t \cdot f)(F_t)\Big).\end{aligned} \tag{3}$$

Comparing (2) and (3), we get

$$\psi\Big(E_i^{(m\ell)} \cdot f\Big) = \bar{E}_i^{(m)} \cdot (\psi f), \text{ for all } f \in \mathrm{Hom}_{\mathfrak{U}(\mathfrak{b})}(\mathfrak{U}(\mathfrak{g}), M)^{\frac{1}{\ell}}.$$

This proves (3) of Proposition 3.3 and hence Proposition 3.3 itself. □

Now we extend the $\bar{U}(\mathfrak{b}^-)$-module map Fr'^* of Proposition 3.3 to an arbitrary cohomology H^i.

THEOREM 3.8. *For any $M \in \mathcal{C}(\mathfrak{b})$, there exists a functorial $\bar{U}(\mathfrak{b}^-)$-module map for all $i \geq 0$:*

$$\mathrm{Fr}'^* : H^i\Big(\mathfrak{X}, M\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \to H^i\Big(\bar{X}, M^{\mathrm{Fr}'}\Big).$$

Moreover, for any $m \geq 0$ and $f \in H^i(\mathfrak{X}, M)$,

$$\bar{E}_i^{(m)} \cdot (\mathrm{Fr}'^* f) = \mathrm{Fr}'^*(E_i^{(m\ell)} \cdot f), \tag{1}$$

i.e., $\mathrm{Fr}'^* : H^i\Big(\mathfrak{X}, M\Big)^{\mathrm{Fr}'_{\mathfrak{b}}} \to H^i\Big(\bar{X}, M^{\mathrm{Fr}'}\Big)$ is a $\bar{U}(\mathfrak{b})$-module map as well, for all $i \geq 0$.

Proof. The proof is parallel to the proof of Theorem 2.3. Let

$$0 \longrightarrow M \longrightarrow Q_0 \xrightarrow{\varepsilon_0} Q_1 \xrightarrow{\varepsilon_1} \cdots,$$

$$0 \longrightarrow M^{\mathrm{Fr}'} \longrightarrow \bar{Q}_0 \xrightarrow{\bar{\varepsilon}_0} \bar{Q}_1 \xrightarrow{\bar{\varepsilon}_1} \cdots,$$

be the standard resolutions in categories $\mathcal{C}(\mathfrak{b})$ and $\bar{\mathcal{C}}(\mathfrak{b})$ respectively. By induction, we construct $\bar{U}(\mathfrak{b})$-module morphisms $\theta_j : Q_j^{\mathrm{Fr}'} \to \bar{Q}_j$ making the squares commutative up to θ_j.

First define

$$\theta_0 : Q_0^{\mathrm{Fr}'} := H^0(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, M)^{\mathrm{Fr}'} \to \bar{Q}_0 := H^0(\bar{U}(\mathfrak{b})/\bar{U}^0, M^{\mathrm{Fr}'})$$

as the map $\mathrm{Fr}'^*_{\mathfrak{b}}$ of Lemma 3.2. Having defined θ_j, define (abbreviating Image by Im)

$$\theta_{j+1} : Q_{j+1}^{\mathrm{Fr}'} := H^0(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, Q_j/\mathrm{Im}\,\varepsilon_{j-1})^{\mathrm{Fr}'} \to \bar{Q}_{j+1} := H^0(\bar{U}(\mathfrak{b})/\bar{U}^0, \bar{Q}_j/\mathrm{Im}\ \bar{\varepsilon}_{j-1})$$

as the composite

$$\begin{aligned} H^0(\mathfrak{U}(\mathfrak{b})/\mathfrak{U}^0, Q_j/\mathrm{Im}\,\varepsilon_{j-1})^{\mathrm{Fr}'} &\xrightarrow{\mathrm{Fr}'^*_{\mathfrak{b}}} H^0\Big(\bar{U}(\mathfrak{b})/\bar{U}^0, (Q_j/\mathrm{Im}\,\varepsilon_{j-1})^{\mathrm{Fr}'}\Big) \\ &\longrightarrow H^0(\bar{U}(\mathfrak{b})/\bar{U}^0, \bar{Q}_j/\mathrm{Im}\ \bar{\varepsilon}_{j-1})\,, \end{aligned}$$

where the second map is induced from the $\bar{U}(\mathfrak{b})$-module map θ_j. Finally, define a cochain map $H^0(\mathfrak{X}, Q_\bullet)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \to H^0(\bar{X}, \bar{Q}_\bullet)$ as the composite map

$$H^0(\mathfrak{X}, Q_\bullet)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \xrightarrow{\mathrm{Fr}'^*} H^0(\bar{X}, Q_\bullet^{\mathrm{Fr}'}) \to H^0(\bar{X}, \bar{Q}_\bullet)\,,$$

where the second map is induced from the $\bar{U}(\mathfrak{b})$-module maps $\theta_\bullet$. This proves the theorem. □

Corollary 3.9. *For any $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{b})$, the composite map*

$$\mathrm{Fr}'^* \circ \mathrm{Fr}^* : H^i\Big(\bar{X}, \bar{M}\Big) \to H^i\Big(\bar{X}, \bar{M}\Big)$$

is the identity map for all $i \geq 0$.

Proof. It is easy to see that the corollary holds for H^0. To prove the result for general i, take an exact sequence in $\bar{\mathcal{C}}(\mathfrak{b}) : 0 \to \bar{M} \to \bar{N} \to \bar{Q} \to 0$ such that $H^i(\bar{X}, \bar{N}) = 0$, for all $i \geq 1$. Then, from the surjective map $H^{i-1}(\bar{X}, \bar{Q}) \twoheadrightarrow H^i(\bar{X}, \bar{M})$ and the functoriality of Fr^* and Fr'^*, the corollary for i follows by induction. □

Remark 3.10. (a) As we will see in a subsequent section, Fr'^* is a quantization of the map induced on the cohomology of homogeneous vector bundles from the 'canonical' Frobenius splitting of the flag variety G/B obtained by Mehta-Ramanathan. Thus the key lemma of Mathieu (asserting that a B-canonical splitting of a B-variety Y sends any B-submodule of $H^0(Y, \mathfrak{L}^{\otimes p})$ to a B-submodule of $H^0(Y, \mathfrak{L})$, for any B-equivariant line bundle $\mathfrak{L}$ on Y; cf. [M, Lemma 2.4 and the remark following it]) in this case follows from the fact in Theorem 3.8 that the splitting is a $\bar{U}(\mathfrak{b}^-)$-module (as well as a $\bar{U}(\mathfrak{b})$-module) map.

(b) For $\lambda \notin \ell X$, the map $\mathrm{Fr}'^* : H^0\Big(\mathfrak{X}, \chi_\lambda^\xi\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \to H^0\Big(\bar{X}, (\chi_\lambda^\xi)^{\mathrm{Fr}'}\Big)$ is identically zero. To see this, write $\lambda = \lambda_0 + \ell\lambda_1$ with $0 \leq \lambda_0(h_i) \leq \ell - 1$

for all simple coroots h_i and $0 < \lambda_0(h_{i_o})$ for some h_{i_o}. Now take any $f \in H^0\Big(\mathfrak{X}, \chi^\xi_\lambda\Big)^{1/\ell}$. Then, by Lemma 3.7, $\begin{bmatrix} K_{i_o};0 \\ \lambda_0(h_{i_o}) \end{bmatrix} \cdot f = 0$. This gives that

$$\left(\begin{bmatrix} K_{i_o};0 \\ \lambda_0(h_{i_o}) \end{bmatrix} \cdot f\right)\Big(F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)}\Big) = 0$$

for any nonnegative $m_1, \cdots, m_r$. Hence

$$\begin{aligned} \left(\begin{bmatrix} K_{i_o};0 \\ \lambda_0(h_{i_o}) \end{bmatrix} \cdot f\right)\Big(F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)}\Big) &= f\left(F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)} \begin{bmatrix} K_{i_o};0 \\ \lambda_0(h_{i_o}) \end{bmatrix}\right) \\ &= cf\Big(F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)}\Big) = 0\,, \end{aligned}$$

for some nonzero c. Thus we conclude that $f(F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)}) = 0$, and hence $\mathrm{Fr}'^* f = 0$ (by the definition of Fr'^*).

4. Stronger quantized Frobenius splitting

In this section we abbreviate the homomorphism $\mathrm{Fr}'_{\mathfrak{b}^-}$ *of Theorem* 1.2 *to* Fr'. We also continue to abbreviate $H^i(\mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{b}), -)$ (resp. $H^i(\bar{U}(\mathfrak{g})/\bar{U}(\mathfrak{b}), -)$) to $H^i(\mathfrak{X}, -)$ (resp. $H^i(\bar{X}, -)$). Any reduced decomposition $w_o = s_{i_1} \cdots s_{i_N}$ of the longest element w_o of the Weyl group in terms of the simple reflections gives an indexing of the set Δ_+ of positive roots $\{\beta_1, \cdots, \beta_N\}$, where $\beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ (α_i being the simple root corresponding to the simple reflection s_i).

Definition 4.1. For any $\mathfrak{U}^0$-module $M \in \mathcal{C}^0$, define the map (abbreviating $-2(\ell-1)\rho$ to γ, where ρ is the half sum of positive roots)

$$\psi_\gamma : \mathrm{Hom}_{\mathfrak{U}^0}\Big(\mathfrak{U}(\mathfrak{b}^-), \chi^\xi_\gamma \otimes M\Big)^{\mathrm{Fr}'} \to \mathrm{Hom}_{\bar{U}^0}\Big(\bar{U}(\mathfrak{b}^-), M^{\mathrm{Fr}'}\Big)$$

by

$$(\psi_\gamma f)(a) = f(F_o \mathrm{Fr}'(a)) \otimes v_+ \quad \text{for } a \in \bar{U}(\mathfrak{b}^-)\,,$$

where $F_o := F_{\beta_N}^{(\ell-1)} \cdots F_{\beta_1}^{(\ell-1)}$, F_{β_i} are Lusztig's root vectors [Lu2, §4], v_+ is a $\mathbb{Z}_\xi$-basis vector of the one-dimensional representation $\chi^\xi_{2(\ell-1)\rho}$ and we identify $\chi^\xi_\gamma \otimes M \otimes \chi^\xi_{-\gamma}$ with M (F_o does not depend upon the choice of the reduced decomposition of w_o up to a nonzero scalar multiple, since the corresponding weight space in the quantized restricted enveloping algebra is of rank one). By the following lemma, $\psi_\gamma(f)$ is indeed $\bar{U}^0$-linear.

Lemma 4.2. *With notation and assumptions as above,* $\psi_\gamma(f)$ *is* $\bar{U}^0$*-linear for any* $f \in \mathrm{Hom}_{\mathfrak{U}^0}\Big(\mathfrak{U}(\mathfrak{b}^-), \chi^\xi_\gamma \otimes M\Big)^{\mathrm{Fr}'}$.

Proof. For any $a \in \bar{U}(\mathfrak{b}^-)$ and $h = \binom{h_i}{m} \in \bar{U}^0$,

$$\begin{aligned}
(\psi_\gamma f)(ha) &= f(F_o \,\mathrm{Fr}'(h)\,\mathrm{Fr}'(a)) \otimes v_+ \\
&= f\left(F_o \begin{bmatrix} K_i;0 \\ \ell m \end{bmatrix} \mathrm{Fr}'(a)\right) \otimes v_+ \\
&= f\left(\begin{bmatrix} K_i;2(\ell-1) \\ \ell m \end{bmatrix} F_o\mathrm{Fr}'(a)\right) \otimes v_+, \quad \text{by [Lu2, §6.5]} \\
&= \left(\begin{bmatrix} K_i;2(\ell-1) \\ \ell m \end{bmatrix} \cdot f(F_o\mathrm{Fr}'(a))\right) \otimes v_+ \\
&= \begin{bmatrix} K_i;0 \\ \ell m \end{bmatrix} \cdot \Big(f(F_o\mathrm{Fr}'(a)) \otimes v_+\Big), \text{ since } v_+ \text{ is of weight } 2(\ell-1)\rho \\
&= \begin{pmatrix} h_i \\ m \end{pmatrix} \odot ((\psi_\gamma f)(a)).
\end{aligned}$$

□

LEMMA 4.3. *With notation and assumptions as in §4.1, ψ_γ is $\bar{U}(\mathfrak{b}^-)$-linear.*

Proof. For $a, b \in \bar{U}(\mathfrak{b}^-)$,

$$\begin{aligned}
\Big(\psi_\gamma\Big(b \odot f\Big)\Big)(a) &= \Big(\psi_\gamma\Big(\mathrm{Fr}'(b)\cdot f\Big)\Big)(a) \\
&= \Big((\mathrm{Fr}'(b)\cdot f)(F_o\mathrm{Fr}'(a))\Big) \otimes v_+ \\
&= f(F_o\mathrm{Fr}'(a)\mathrm{Fr}'(b)) \otimes v_+ \\
&= f(F_o\mathrm{Fr}'(ab)) \otimes v_+ \\
&= (\psi_\gamma f)(ab) \\
&= (b \cdot \psi_\gamma f)(a).
\end{aligned}$$

This proves that ψ_γ is $\bar{U}(\mathfrak{b}^-)$-linear. □

We now prove the following crucial proposition.

PROPOSITION 4.4. *For any $\bar{U}(\mathfrak{b})$-module $\bar{M}$ which is a $\bar{U}^0$-weight module, $m \geq 0$ and $f \in \mathrm{Hom}_{\mathfrak{U}(\mathfrak{b})}\Big(\mathfrak{U}(\mathfrak{g}), \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}\Big) \simeq \mathrm{Hom}_{\mathfrak{U}^0}\Big(\mathfrak{U}(\mathfrak{b}^-), \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}\Big)$,*

$$\psi_\gamma\Big(E_i^{(\ell m)} \cdot f\Big) = \bar{E}_i^{(m)} \cdot (\psi_\gamma f), \tag{1}$$

where the action of $\bar{E}_i^{(m)}$ on $\psi_\gamma f$ comes from the similar identification

$$\mathrm{Hom}_{\bar{U}(\mathfrak{b})}\Big(\bar{U}(\mathfrak{g}), \bar{M}\Big) \simeq \mathrm{Hom}_{\bar{U}^0}\Big(\bar{U}(\mathfrak{b}^-), \bar{M}\Big).$$

Proof. Take $a = \bar{F}_{i_r}^{(m_r)} \cdots \bar{F}_{i_1}^{(m_1)}$. Then

$$(2) \qquad \Big(\psi_\gamma(E_i^{(\ell m)} \cdot f)\Big)(a) = (E_i^{(\ell m)} \cdot f)(F_o \mathrm{Fr}'(a)) \otimes v_+ = f\Big(F_o\, F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)}\, E_i^{(\ell m)}\Big) \otimes v_+ .$$

Now, by (1) of Lemma 3.4 and Lemma 3.5, we get

(3)
$$F_o\, F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)} E_i^{(\ell m)} = \sum_t \hat{A}_t\, F_o\, E_i^{(\ell m - \sum_{k=1}^s t_k)} F_{i_r}^{(\ell m_r)} \cdots F_{i_{j_s}}^{(\ell m_{j_s} - t_s)} \cdots F_{i_{j_1}}^{(\ell m_{j_1} - t_1)} \cdots F_{i_1}^{(\ell m_1)},$$

where the summation is over $t = (t_1, \cdots, t_s) \in \mathbb{Z}_+^s$, $1 \le j_1 < \cdots < j_s \le r$ are precisely the indices such that $i_{j_p} = i$ and

$$\hat{A}_t := \begin{bmatrix} K_i^{-1}; -\ell m_{j_1} + \ell m + (\sum_{j > j_1} a_j\, \ell m_j) - 2(\ell - 1) \\ t_1 \end{bmatrix} \times \begin{bmatrix} K_i^{-1}; -\ell m_{j_2} + \ell m + (\sum_{j > j_2} a_j \ell m_j) - t_1 - 2(\ell - 1) \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_i^{-1}; -\ell m_{j_s} + \ell m + (\sum_{j > j_s} a_j \ell m_j) - (t_1 + \cdots + t_{s-1}) - 2(\ell - 1) \\ t_s \end{bmatrix}$$

(where $a_j := -\alpha_{i_j}(h_i)$). Substituting (3) in (2), we get

$$(4) \quad \Big(\psi_\gamma(E_i^{(\ell m)} \cdot f)\Big)(a) = \sum_{t \in \mathbb{Z}_+^s} \bigg(\hat{A}_t \cdot \bigg(f\bigg(F_o E_i^{(\ell m - \sum_{k=1}^s t_k)}\, F_{i_r}^{(\ell m_r)} \cdots F_{i_{j_s}}^{(\ell m_{j_s} - t_s)} \cdots F_{i_{j_1}}^{(\ell m_{j_1} - t_1)} \cdots F_{i_1}^{(\ell m_1)}\bigg)\bigg)\bigg) \otimes v_+ .$$

Since Im $f \subset \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}$, using (3) of Lemma 3.7, the sum in (4) reduces to $(t_1, \cdots, t_s) \in \mathbb{Z}_+^s$ such that each t_k is divisible by ℓ, i.e., denoting $\ell t = (\ell t_1, \cdots, \ell t_s)$, we have

$$(5) \quad \Big(\psi_\gamma(E_i^{(\ell m)} \cdot f)\Big)(a)$$
$$= \sum_{t \in \mathbb{Z}_+^s} \bigg(\hat{A}_{\ell t} \cdot \bigg(f\bigg(F_o E_i^{(\ell m - \sum_{k=1}^s \ell t_k)}\, F_{i_r}^{(\ell m_r)} \cdots F_{i_{j_s}}^{(\ell m_{j_s} - \ell t_s)} \cdots F_{i_{j_1}}^{(\ell m_{j_1} - \ell t_1)} \cdots F_{i_1}^{(\ell m_1)}\bigg)\bigg)\bigg) \otimes v_+$$
$$= \sum_{t \in \mathbb{Z}_+^s} \bigg(\hat{A}_{\ell t} E_i^{(\ell m - \sum_{k=1}^s \ell t_k)} \cdot \bigg(f\bigg(F_o F_{i_r}^{(\ell m_r)} \cdots F_{i_{j_s}}^{(\ell m_{j_s} - \ell t_s)} \cdots F_{i_{j_1}}^{(\ell m_{j_1} - \ell t_1)} \cdots F_{i_1}^{(\ell m_1)}\bigg)\bigg)\bigg) \otimes v_+ , \text{ by the next lemma}$$

$$= \sum_{t\in\mathbb{Z}_+^s} \Big(\tilde{A}_{\ell t} E_i^{(\ell m - \sum_{k=1}^s \ell t_k)}\Big)$$

$$\odot\bigg(f\bigg(F_o F_{i_r}^{(\ell m_r)} \cdots F_{i_{j_s}}^{(\ell m_{j_s} - \ell t_s)} \cdots F_{i_{j_1}}^{(\ell m_{j_1} - \ell t_1)} \cdots F_{i_1}^{(\ell m_1)}\bigg) \otimes v_+\bigg)$$

$$= \sum_{t\in\mathbb{Z}_+^s} \mathrm{Fr}\bigg(\tilde{A}_{\ell t} E_i^{(\ell m - \sum_{k=1}^s \ell t_k)}\bigg)$$

$$\cdot\bigg(f\bigg(F_o F_{i_r}^{(\ell m_r)} \cdots F_{i_{j_s}}^{(\ell m_{j_s} - \ell t_s)} \cdots F_{i_{j_1}}^{(\ell m_{j_1} - \ell t_1)} \cdots F_{i_1}^{(\ell m_1)}\bigg) \otimes v_+\bigg),$$

where

$$\tilde{A}_t \quad := \quad \begin{bmatrix} K_i^{-1}; -\ell m_{j_1} + \ell m + \sum_{j>j_1} a_j \ell m_j \\ t_1 \end{bmatrix}$$

$$\cdots \begin{bmatrix} K_i^{-1}; -\ell m_{j_s} + \ell m + \sum_{j>j_s} a_j \ell m_j - (t_1 + \cdots + t_{s-1}) \\ t_s \end{bmatrix}.$$

We now calculate the right side of (1):

$$(6)\ \Big(\bar{E}_i^{(m)} \cdot (\psi_\gamma f)\Big)(a)$$

$$= (\psi_\gamma f)\Big(\bar{F}_{i_r}^{(m_r)} \cdots \bar{F}_{i_1}^{(m_1)} \bar{E}_i^{(m)}\Big)$$

$$= \sum_{t\in\mathbb{Z}_+^s} (\psi_\gamma f)\bigg(H_t \bar{E}_i^{(m - \sum_{k=1}^s t_k)}\, \bar{F}_{i_r}^{(m_r)} \cdots \bar{F}_{i_{j_s}}^{(m_{j_s} - t_s)} \cdots \bar{F}_{i_{j_1}}^{(m_{j_1} - t_1)} \cdots \bar{F}_{i_1}^{(m_1)}\bigg)$$

$$= \sum_{t\in\mathbb{Z}_+^s} H_t \bar{E}_i^{(m - \sum_{k=1}^s t_k)}$$

$$\cdot\bigg((\psi_\gamma f)\bigg(\bar{F}_{i_r}^{(m_r)} \cdots \bar{F}_{i_{j_s}}^{(m_{j_s} - t_s)} \cdots \bar{F}_{i_{j_1}}^{(m_{j_1} - t_1)} \cdots \bar{F}_{i_1}^{(m_1)}\bigg)\bigg)$$

$$= \sum_{t\in\mathbb{Z}_+^s} H_t \bar{E}_i^{(m - \sum_{k=1}^s t_k)}$$

$$\cdot\bigg(f\bigg(F_o\, F_{i_r}^{(\ell m_r)} \cdots F_{i_{j_s}}^{(\ell m_{j_s} - \ell t_s)} \cdots F_{i_{j_1}}^{(\ell m_{j_1} - \ell t_1)} \cdots F_{i_1}^{(\ell m_1)}\bigg) \otimes v_+\bigg),$$

where $H_t := \mathrm{Fr}(\tilde{A}_{\ell t})$.

Comparing (5) and (6) we get (1). This proves the proposition modulo the next lemma. $\square$

Lemma 4.5. *For any $\bar{U}(\mathfrak{b})$-module $\bar{M}$ such that $\bar{M}$ is a $\bar{U}^0$-weight module, $m \geq 0$ and $f \in \operatorname{Hom}_{\mathfrak{U}(\mathfrak{b})}\Big(\mathfrak{U}(\mathfrak{g}), \chi^{\xi}_{\gamma} \otimes \bar{M}^{\mathrm{Fr}}\Big)$,*

$$f(F_o\, E_i^{(\ell m)}) = f(E_i^{(\ell m)}\, F_o). \tag{1}$$

Thus, replacing f by $x \cdot f$,

$$f(F_o\, E_i^{(\ell m)} x) = f(E_i^{(\ell m)}\, F_o x) \quad \textit{for any } x \in \mathfrak{U}(\mathfrak{g}).$$

Proof. Any such $\bar{M}$ is a quotient of a $\bar{U}(\mathfrak{b})$-module $\bar{Q}$ such that $\bar{Q}$ is a $\bar{U}^0$-weight module and $\bar{Q}$ is $\mathbb{Z}_\xi$-free (since, for any weight vector $v \in \bar{M}$ of weight λ, there exists a $\bar{U}(\mathfrak{b})$-module map $\pi_v : \bar{U}(\mathfrak{b}) \otimes_{\bar{U}^0} \bar{\chi}_\lambda \to \bar{M}$ taking $1 \otimes 1 \mapsto v$). Now the surjective $\bar{U}(\mathfrak{b})$-module map $\theta : \bar{Q} \to \bar{M}$ induces a surjective map

$$\hat{\theta} : \operatorname{Hom}_{\mathfrak{U}(\mathfrak{b})}\Big(\mathfrak{U}(\mathfrak{g}), \chi^{\xi}_{\gamma} \otimes \bar{Q}^{\mathrm{Fr}}\Big) \to \operatorname{Hom}_{\mathfrak{U}(\mathfrak{b})}\Big(\mathfrak{U}(\mathfrak{g}), \chi^{\xi}_{\gamma} \otimes \bar{M}^{\mathrm{Fr}}\Big).$$

Hence, to prove (1), we can (and do) assume that $\bar{M}$ is a $\mathbb{Z}_\xi$-free module.

We first prove (1) for $m = 1$. Since $\bar{M}$ is $\mathbb{Z}_\xi$-free (by assumption), we can replace the ground ring $\mathbb{Z}_\xi$ by $\mathbb{Q}_\xi$. For any $\mathfrak{d} = (p_1, \cdots, p_N) \in \{0, \cdots, \ell - 1\}^N$, $N := |\Delta_+|$, define

$$F^{\mathfrak{d}} = F_{\beta_N}^{(p_N)} \cdots F_{\beta_1}^{(p_1)}.$$

By [Lu2, Lemma 8.5 and Th. 8.3] write

$$F_o\, E_i^{(\ell)} - E_i^{(\ell)} F_o = \sum_{0<m<\ell} E_i^{(m)} x_m + \sum_{\mathfrak{d} \in \{0,\cdots,\ell-1\}^N} c_{\mathfrak{d}}\, F^{\mathfrak{d}}, \tag{2}$$

for some $x_m \in \mathfrak{U}(\mathfrak{b}^-)$ (in fact in the restricted quantized enveloping algebra) and $c_{\mathfrak{d}} \in \mathsf{u}^0_{\mathbb{Q}_\xi}$ (where $\mathsf{u}^0_{\mathbb{Q}_\xi} \subset \mathfrak{U}^0_{\mathbb{Q}_\xi}$ is the $\mathbb{Q}_\xi$-subalgebra generated by $\{K_i^{\pm}; 1 \leq i \leq n\}$).

Applying the anti-automorphism S of $\mathfrak{U}(\mathfrak{g})$ to (2), we get

$$\begin{aligned} (-K_i^{-1}E_i)^{(\ell)} S(F_o) - S(F_o)(-K_i^{-1}E_i)^{(\ell)} \quad &= \quad \sum_{0<m<\ell} S(x_m)(-K_i^{-1}E_i)^{(m)} \\ &\qquad + \sum_{\mathfrak{d}} S(F^{\mathfrak{d}})\, S(c_{\mathfrak{d}}). \end{aligned}$$

Applying the above to a highest weight vector v_+ of $V_\xi(2(\ell-1)\rho)$, we get

$$(-K_i^{-1}E_i)^{(\ell)} S(F_o) v_+ = \sum_{\mathfrak{d}} S(F^{\mathfrak{d}})\, S(c_{\mathfrak{d}})\, v_+. \tag{3}$$

We next show that

$$E_i^{(m)}\, F_o v_+ = 0 \qquad \text{for any } m > 0. \tag{4}$$

Since $F_o v_+$ is a weight vector of weight 0, it suffices to show that

$$F_i^{(m)}\, F_o v_+ = 0 \qquad \text{for any } m > 0: \tag{5}$$

For $0 < m < \ell$, since $F_i^{(m)} F_o = 0$, (5) follows in this case. Further, $F_i^{(m)}$ commutes with F_o (for any $m \geq 0$) as can be seen from [Lu2, 5.8(c), Th. 8.3 and Lemma 8.5] by the weight consideration. Hence $F_i^{(m)} F_o v_+ = F_o F_i^{(m)} v_+$. For any $1 \leq i \leq n$, we can choose a reduced decomposition of w_o starting in s_i (and hence $\beta_1 = \alpha_i$) resulting in the expression $F_o = F_{\beta_N}^{(\ell-1)} \cdots F_{\beta_2}^{(\ell-1)} F_i^{(\ell-1)}$. This gives

$$F_o F_i^{(m)} v_+ = \begin{bmatrix} \ell+m-1 \\ \ell-1 \end{bmatrix}_{d_i}^{\xi} F_{\beta_N}^{(\ell-1)} \cdots F_{\beta_2}^{(\ell-1)} F_i^{(\ell+m-1)} v_+ = 0, \quad \text{for } m \geq \ell,$$

which proves (5) and hence (4). Substituting (4) in (3), we get

$$\text{(6)} \qquad \sum_{\mathfrak{d}} S(F^{\mathfrak{d}})\, S(c_{\mathfrak{d}}) v_+ = 0.$$

Since $\{F^{\mathfrak{d}} v_+\}_{\mathfrak{d} \in \{0,\cdots,\ell-1\}^N}$ are linearly independent, as the same is true already for the Steinberg module $V_\xi((\ell-1)\rho)$ (cf. [Ku, Prop. 4.1]), from (6) we get $\chi_{-\gamma}^{\xi}(S(c_{\mathfrak{d}})) = 0$, for all $\mathfrak{d}$, i.e.,

$$\text{(7)} \qquad \chi_\gamma^\xi(c_{\mathfrak{d}}) = 0, \quad \text{for all } \mathfrak{d}.$$

By (2),

$$\begin{aligned} f(F_o E_i^{(\ell)} - E_i^{(\ell)} F_o) &= \sum_{0<m<\ell} E_i^{(m)} \cdot (f(x_m)) + \sum_{\mathfrak{d} \in \{0,\cdots,\ell-1\}^N} c_{\mathfrak{d}} \cdot (f(F^{\mathfrak{d}})) \\ &= \sum_{\mathfrak{d}} c_{\mathfrak{d}} \cdot (f(F^{\mathfrak{d}}))\,, \text{ since } \ \operatorname{Im} f \subset \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}, \\ &= \sum_{\mathfrak{d}} \chi_\gamma^\xi(c_{\mathfrak{d}})\, f(F^{\mathfrak{d}}), \quad \text{since } c_{\mathfrak{d}} \in \mathsf{u}^0_{\mathbb{Q}_\xi}, \\ &= 0, \qquad \text{by (7)}. \end{aligned}$$

This proves the identity (1) for $m = 1$.

We assume the validity of (1) for m (by induction) and prove it for m replaced by $m+1$: First of all

$$\begin{aligned} \text{(8)} \qquad E_i^{(\ell m)} E_i^{(\ell)} &= \begin{bmatrix} \ell m + \ell \\ \ell \end{bmatrix}_{d_i}^{\xi} E_i^{(\ell m+\ell)}, \qquad \text{by [Lu2, 5.8(c)]} \\ &= (m+1)\, E_i^{(\ell m+\ell)}, \qquad \text{by (3) of Lemma 3.7.} \end{aligned}$$

Thus,

$$\begin{aligned} \text{(9)} \quad (m+1)\, f\Big(F_o E_i^{(\ell m+\ell)}\Big) &= f\Big(F_o E_i^{(\ell m)} E_i^{(\ell)}\Big) \\ &= \Big(E_i^{(\ell)} f\Big)\Big(F_o\, E_i^{(\ell m)}\Big) \end{aligned}$$

$$
\begin{aligned}
&= (E_i^{(\ell)} f)\Big(E_i^{(\ell m)}\, F_o\Big), && \text{by induction,}\\
&= E_i^{(\ell m)} \cdot \Big((E_i^{(\ell)} f)(F_o)\Big)\\
&= E_i^{(\ell m)} \cdot \Big(f(E_i^{(\ell)}\, F_o)\Big), && \text{by the } m=1 \text{ case,}\\
&= f\Big(E_i^{(\ell m)}\, E_i^{(\ell)}\, F_o\Big)\\
&= (m+1)\, f\Big(E_i^{(\ell m+\ell)} F_o\Big), && \text{by (8).}
\end{aligned}
$$

Since $m+1$ is not a zero divisor in $\mathbb{Z}_\xi$ and (by assumption) $\bar{M}$ is $\mathbb{Z}_\xi$-free, we get the validity of (1) for $m+1$ (by virtue of (9)). □

PROPOSITION 4.6. *For any $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{b})$, there exists a functorial $\bar{U}(\mathfrak{b}^-)$-module map*

$$
\mathrm{Fr}_\gamma'^{*} : H^0\Big(\mathfrak{X}, \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} \to H^0\Big(\bar{X}, \bar{M}\Big),
$$

defined by

$$
(\mathrm{Fr}_\gamma'^{*} f)(a) = f(F_o \mathrm{Fr}'(a)) \otimes v_+, \tag{1}
$$

for $a \in \bar{U}(\mathfrak{b}^-)$ and $f \in H^0\Big(\mathfrak{X}, \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'}$.

Moreover, for any $m \geq 0$ and $f \in H^0\Big(\mathfrak{X}, \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}\Big)$,

$$
\bar{E}_i^{(m)} \cdot \Big(\mathrm{Fr}_\gamma'^{*} f\Big) = \mathrm{Fr}_\gamma'^{*}(E_i^{(m\ell)} \cdot f). \tag{2}
$$

Proof. As in the proof of Proposition 3.3, consider the diagram

$$
\begin{array}{ccccc}
H^0\Big(\mathfrak{X}, \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} & \hookrightarrow \mathrm{Hom}_{\mathfrak{U}(\mathfrak{b})}\Big(\mathfrak{U}(\mathfrak{g}), \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} & \simeq & \mathrm{Hom}_{\mathfrak{U}^0}\Big(\mathfrak{U}(\mathfrak{b}^-), \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} \\
 & & & \psi_\gamma \downarrow \\
H^0(\bar{X}, \bar{M}) & \hookrightarrow \mathrm{Hom}_{\bar{U}(\mathfrak{b})}(\bar{U}(\mathfrak{g}), \bar{M}) & \simeq & \mathrm{Hom}_{\bar{U}^0}(\bar{U}(\mathfrak{b}^-), \bar{M}),
\end{array}
$$

where ψ_γ is as defined in §4.1. By combining Lemma 4.3 and Proposition 4.4, we get

$$
\psi_\gamma\Big(H^0\Big(\mathfrak{X}, \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'}\Big) \subset H^0(\bar{X}, \bar{M}).
$$

So define $\mathrm{Fr}_\gamma'^{*}$ as the restriction of ψ_γ to $H^0(\mathfrak{X}, \chi_\gamma^\xi \otimes \bar{M}^{\mathrm{Fr}})^{\mathrm{Fr}'}$. Since ψ_γ is $\bar{U}(\mathfrak{b}^-)$-linear (by Lemma 4.3), so is $\mathrm{Fr}_\gamma'^{*}$ and moreover (2) follows from (1) of Proposition 4.4. □

THEOREM 4.7. *For any* $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{b})$, *there exists a functorial* $\bar{U}(\mathfrak{b}^-)$*-module map* (*for all* $i \geq 0$)

$$\mathrm{Fr}'^*_\gamma : H^i\Big(\mathfrak{X}, \chi^\xi_\gamma \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} \to H^i(\bar{X}, \bar{M});$$

i.e., the following diagram is commutative for any $\bar{U}(\mathfrak{b})$*-module map* $\theta : \bar{M} \to \bar{N}$:

(D)
$$\begin{array}{ccc} H^i\Big(\mathfrak{X}, \chi^\xi_\gamma \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} & \xrightarrow{\mathrm{Fr}'^*_\gamma} & H^i(\bar{X}, \bar{M}) \\ \Big\downarrow & & \Big\downarrow \\ H^i\Big(\mathfrak{X}, \chi^\xi_\gamma \otimes \bar{N}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} & \xrightarrow[\mathrm{Fr}'^*_\gamma]{} & H^i(\bar{X}, \bar{N}), \end{array}$$

where the vertical maps are the canonical maps induced from θ.

Moreover, for any $m \geq 0$ *and* $f \in H^i(\mathfrak{X}, \chi^\xi_\gamma \otimes \bar{M}^{\mathrm{Fr}})$,

(1)
$$\bar{E}_i^{(m)} \cdot (\mathrm{Fr}'^*_\gamma\, f) = \mathrm{Fr}'^*_\gamma(E_i^{(m\ell)} \cdot f).$$

Proof. Consider the standard resolution in category $\bar{\mathcal{C}}(\mathfrak{b})$:

(2)
$$0 \to \bar{M} \to \bar{Q}_0 \xrightarrow{\bar{\varepsilon}_0} \bar{Q}_1 \xrightarrow{\bar{\varepsilon}_1} \cdots.$$

Lifting (2) by Fr and then tensoring with χ^ξ_γ, we get the resolution in $\mathcal{C}(\mathfrak{b})$:

(3)
$$0 \to \chi^\xi_\gamma \otimes \bar{M}^{\mathrm{Fr}} \to \hat{Q}_0 \xrightarrow{\hat{\varepsilon}_0} \hat{Q}_1 \xrightarrow{\hat{\varepsilon}_1} \cdots,$$

where $\hat{Q}_k := \chi^\xi_\gamma \otimes \bar{Q}_k^{\mathrm{Fr}}$ and $\hat{\varepsilon}_k := \mathrm{Id} \otimes \bar{\varepsilon}_k$. By Proposition 4.6, we get the cochain map induced by the $\bar{U}(\mathfrak{b}^-)$-module maps Fr'^*_γ:

(∗)
$$\begin{array}{ccccc} H^0(\mathfrak{X}, \hat{Q}_0)^{\mathrm{Fr}'} & \longrightarrow & H^0(\mathfrak{X}, \hat{Q}_1)^{\mathrm{Fr}'} & \longrightarrow & \cdots \\ \Big\downarrow{\scriptstyle \mathrm{Fr}'^*_\gamma} & & \Big\downarrow{\scriptstyle \mathrm{Fr}'^*_\gamma} & & \\ H^0(\bar{X}, \bar{Q}_0) & \longrightarrow & H^0(\bar{X}, \bar{Q}_1) & \longrightarrow & \cdots. \end{array}$$

By the next lemma, for any $p \geq 0$ and $i > 0$, $H^i(\mathfrak{X}, \hat{Q}_p) = 0$. Hence the i^{th} cohomology of the top cochain complex is equal to $H^i(\mathfrak{X}, \chi^\xi_\gamma \otimes \bar{M}^{\mathrm{Fr}})^{\mathrm{Fr}'}$, whereas the i^{th} cohomology of the bottom cochain complex is $H^i(\bar{X}, \bar{M})$ (cf. [H, Prop. 1.2A, Chap. III]). So, we define the $\bar{U}(\mathfrak{b}^-)$-module map

$$\mathrm{Fr}'^*_\gamma : H^i\Big(\mathfrak{X}, \chi^\xi_\gamma \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} \to H^i(\bar{X}, \bar{M})$$

as the induced map in cohomology from (∗).

Commutativity of diagram (D) follows from the functoriality of all the constructions involved and moreover (1) follows from (2) of Proposition 4.6. So the theorem is proved modulo the following lemma. □

Lemma 4.8. *For any $\bar{M} \in \bar{\mathcal{C}}^0$ and $\lambda \in X$*

$$H^i\Big(\mathfrak{x}, \chi_\lambda^\xi \otimes \Big(H^0(\bar{U}(\mathfrak{b})/\bar{U}^0, \bar{M})\Big)^{\mathrm{Fr}}\Big) = 0,$$

for all $i > 0$.

Proof. The proof is similar to the proof of [APW, Th. 5.4]. By definition of the category $\bar{\mathcal{C}}^0$, $\bar{M} = \bigoplus_{\mu \in X} \bar{M}_\mu$. Since H^* commutes with (possibly infinite) direct sums (cf. [APW, Th. 1.31]), we can assume that $\bar{M} = \bar{M}_\mu$. Since $H^0(\bar{U}(\mathfrak{b})/\bar{U}^0, -)$ is an exact functor (cf. Proposition 1.5) and any $\mathbb{Z}_\xi$-module N admits a free resolution for some $d \geq 0$ (since $\mathbb{Z}_\xi$ has finite global homological dimension):

$$0 \to F_d \to \cdots \to F_1 \to F_0 \to N,$$

we can assume that $\bar{M}_\mu$ is $\mathbb{Z}_\xi$-free of rank 1, i.e., $\bar{M}_\mu = \bar{\chi}_\mu$. By a result of Cline-Parshall-Scott (cf. [CPS, Proposition 5.5], [APW, Lemma 5.3]), there is a $\bar{U}(\mathfrak{b})$-module isomorphism:

$$H^0(\bar{U}(\mathfrak{b})/\bar{U}^0, \bar{\chi}_\mu) \simeq \varinjlim_{m \geq 0} H^0(\bar{X}, \bar{\chi}_{-m\rho}) \otimes \bar{\chi}_{-m\rho+\mu},$$

where the right side is a directed union. Since the cohomology commutes with directed unions (cf. [APW, Proof of Th. 5.4]), to prove the lemma, it suffices to show that

$$H^i\Big(\mathfrak{x}, H^0\Big(\bar{X}, \bar{\chi}_{-m\rho}\Big)^{\mathrm{Fr}} \otimes \chi^\xi_{-\ell m\rho + \ell\mu + \lambda}\Big) = 0,\ \forall i > 0,\ m \gg 0.$$

(We have used here the fact that Fr commutes with the coproduct.)

Now, by [APW, Prop. 2.19 (ii)] (since $H^0(\bar{X}, \bar{\chi}_{-m\rho})$ is $\mathbb{Z}_\xi$-free; cf. [APW, Corollary 3.3(i)] for the corresponding result in the quantum case),

$$\begin{aligned} H^i\Big(\mathfrak{x}, H^0(\bar{X}, \bar{\chi}_{-m\rho})^{\mathrm{Fr}} \otimes \chi^\xi_{-\ell m\rho+\ell\mu+\lambda}\Big) & \\ \simeq\ & H^0(\bar{X}, \bar{\chi}_{-m\rho})^{\mathrm{Fr}} \otimes H^i(\mathfrak{x}, \chi^\xi_{-\ell m\rho+\ell\mu+\lambda}) \\ =\ & 0,\ \text{for}\, m \gg 0\,, \end{aligned}$$

by the following quantized analogue of the Serre vanishing theorem.

For any $\lambda \in X^+, \mu \in X$ and $i > 0$,

$$(1)\qquad H^i(\mathfrak{x}, \chi^\xi_{-m(\lambda+\rho)+\mu}) = 0\,,\ \textit{for all}\ m \gg 0.$$

The ring $\mathbb{Z}_\xi$ has projective dimension one (cf. [Mi, Lemma 1.5]). Hence (1) follows by the base change [APW, (8) of §3.6] and [AW, Th. 2.6].

This completes the proof of the lemma, thereby completing the proof of Theorem 4.7. □

LEMMA 4.9. *For any $M \in \mathcal{C}(\mathfrak{b})$ such that M is $\mathbb{Z}_\xi$-flat and $\sigma \in H^0(\mathfrak{X}, M)$, there exists a functorial map, for any $N \in \mathcal{C}(\mathfrak{b})$,*

$$m_\sigma : H^0(\mathfrak{X}, N) \to H^0(\mathfrak{X}, M \otimes N)$$

given by

$$(1) \qquad (m_\sigma f)\, a = \sum_i \sigma(a_i') \otimes f(a_i''), \quad \text{for } a \in \mathfrak{U}(\mathfrak{g}),$$

where $\Delta a = \sum_i a_i' \otimes a_i''$.

Moreover, m_σ gives rise to a functorial map (again denoted by)

$$(2) \qquad m_\sigma : H^i(\mathfrak{X}, N) \to H^i(\mathfrak{X}, M \otimes N).$$

Proof. It is easy to see that (1) defines a map

$$m_\sigma : \mathrm{Hom}_{\mathfrak{U}(\mathfrak{b})}(\mathfrak{U}(\mathfrak{g}), N) \to \mathrm{Hom}_{\mathfrak{U}(\mathfrak{b})}(\mathfrak{U}(\mathfrak{g}), M \otimes N).$$

Moreover, $m_\sigma(f) \in H^0(\mathfrak{X}, M \otimes N)$ for any $f \in H^0(\mathfrak{X}, N)$ (as is easy to see). The existence of m_σ at the higher cohomology follows from the consideration of the standard resolution of N in $\mathcal{C}(\mathfrak{b})$: $0 \to N \to Q_0 \to Q_1 \to \cdots$. (Observe that, since M is $\mathbb{Z}_\xi$-flat, by [APW, Prop. 2.16(i)], $0 \to M \otimes N \to M \otimes Q_0 \to M \otimes Q_1 \to \cdots$ is the standard resolution of $M \otimes N$.) □

LEMMA 4.10. *Let $\mathbb{Z}_\xi v_+$ be the highest weight subspace of $V_\xi(2(\ell-1)\rho)$. Then $S(F_o)\,\mathbb{Z}_\xi v_+$ is a $\mathbb{Z}_\xi$-module direct summand of $V_\xi(2(\ell-1)\rho)$.*

Proof. It suffices to construct $f \in \mathrm{Hom}_{\mathbb{Z}_\xi}(V_\xi(2(\ell-1)\rho), \mathbb{Z}_\xi)$ such that $f(S(F_o)v_+)$ is an invertible element of $\mathbb{Z}_\xi$. Consider the $\mathfrak{U}(\mathfrak{g})$-module homomorphism

$$\delta : V_\xi(2(\ell-1)\rho) \to V_\xi((\ell-1)\rho) \otimes V_\xi((\ell-1)\rho),$$

taking $v_+ \mapsto w_+ \otimes w_+$, where w_+ is a primitive highest weight vector of $V_\xi((\ell-1)\rho)$. Since S is an (anti)automorphism which keeps the restricted enveloping algebra stable, by the weight consideration, $S(F_o) = F_o a$, for an invertible element $a \in u^0$, where u^0 is the $\mathbb{Z}_\xi$-subalgebra of $\mathfrak{U}^0$ generated by $\{k_i; 1 \le i \le n\}$. Thus

$$S(F_o)v_+ = F_o a v_+ = x F_o v_+\,,$$

for an invertible element $x \in \mathbb{Z}_\xi$. Write

$$(1) \qquad \delta(S(F_o)v_+) = x w_+ \otimes F_o w_+ + v\,,$$

for $v \in V_\xi((\ell-1)\rho)^+ \otimes V_\xi((\ell-1)\rho)$, where $V_\xi((\ell-1)\rho)^+$ (resp. $V_\xi((\ell-1)\rho)^-$) is the sum of all the weight spaces of $V_\xi((\ell-1)\rho)$ except the highest (resp.

lowest) weight space. Now, by [Ku, Prop. 4.1] and [Lu2, Th. 8.3], $F_o w_+$ is a primitive vector in $V_\xi((\ell-1)\rho)$. The decomposition

$$\begin{aligned} &V_\xi((\ell-1)\rho)\otimes V_\xi((\ell-1)\rho)\\ &\quad=\Big(V_\xi((\ell-1)\rho)^+\otimes V_\xi((\ell-1)\rho)+V_\xi((\ell-1)\rho)\otimes V_\xi((\ell-1)\rho)^-\Big)\\ &\qquad\oplus \mathbb{Z}_\xi(w_+\otimes F_o w_+)\end{aligned}$$

gives rise to the map $\tilde{f}: V_\xi((\ell-1)\rho)\otimes V_\xi((\ell-1)\rho)\to\mathbb{Z}_\xi$ by projecting on the last factor. Finally, let f be the linear form $\tilde{f}\circ\delta: V_\xi(2(\ell-1)\rho)\to\mathbb{Z}_\xi$. Then, by (1), $f(S(F_o)v_+)=x$. Hence $\mathbb{Z}_\xi S(F_o)v_+$ is a $\mathbb{Z}_\xi$-module direct summand in $V_\xi(2(\ell-1)\rho)$. □

Decompose

$$V_\xi(2(\ell-1)\rho)=S(F_o)\,\mathbb{Z}_\xi v_+\oplus M\,,$$

where M is a weight subspace. Define $\hat{\sigma}_o\in V_\xi(2(\ell-1)\rho)^*$ by $\hat{\sigma}_o(S(F_o)v_+)=1$ and $\hat{\sigma}_{o_{|M}}\equiv 0$.

Proposition 4.11. *For any $\bar{M}\in\bar{\mathcal{C}}(\mathfrak{b})$, the composite*

$$(1)\qquad H^i\Big(\bar{X},\bar{M}\Big)\xrightarrow{\mathrm{Fr}^*}H^i\Big(\mathfrak{X},\bar{M}^{\mathrm{Fr}}\Big)\xrightarrow{m_{\sigma_o}}H^i\Big(\mathfrak{X},\chi^\xi_\gamma\otimes\bar{M}^{\mathrm{Fr}}\Big)\xrightarrow{\mathrm{Fr}'^*_\gamma}H^i\Big(\bar{X},\bar{M}\Big)$$

is the identity map, where $\sigma_o\in H^0\Big(\mathfrak{X},\chi^\xi_\gamma\Big)$ is given by $\sigma_o:=\beta(\hat{\sigma}_o)$, and β is as defined in Section 1.

Proof. From the functoriality of all the maps involved, it suffices to prove the lemma for H^0 (cf. the argument in the proof of Corollary 3.9). Take $f\in H^0(\bar{X},\bar{M}),\bar{y}\in\bar{U}(\mathfrak{n}^-)$ and write $\Delta(\mathrm{Fr}'\bar{y})=\sum_i y_i'\otimes y_i''$. Also write $\Delta(F_o)=\sum_j F_j'\otimes F_j''$. Then

$$\begin{aligned}&(\mathrm{Fr}'^*_\gamma\, m_{\sigma_o}\mathrm{Fr}^* f)\,\bar{y}\\ &\quad=(m_{\sigma_o}\mathrm{Fr}^* f)(F_o\mathrm{Fr}'(\bar{y}))\otimes v_+\\ &\quad=\sum_{i,j}\sigma_o(F_j'y_i')\otimes f(\mathrm{Fr}(F_j'')\,\mathrm{Fr}(y_i''))\otimes v_+\\ &\quad=\sum_i\sigma_o(F_o y_i')\otimes f(\mathrm{Fr}(y_i''))\otimes v_+,\ \text{since } \mathrm{Fr}(F_j'')=0 \text{ unless } F_j''\in\mathfrak{U}^0\\ &\quad=\sigma_o(F_o)\otimes f(\mathrm{Fr}\,\mathrm{Fr}'\bar{y})\otimes v_+,\ \text{since } \sigma_o(F_o y_i')=0 \text{ unless } y_i'\in\mathfrak{U}^0\\ &\quad=v_+^*\otimes f(\bar{y})\otimes v_+\\ &\quad=f(\bar{y}).\end{aligned}$$

This proves the proposition. □

Remark 4.12. (a) By the same argument as that in the proof of Proposition 4.6, we obtain the fact that for any $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{b})$, there exists a functorial $\bar{U}(\mathfrak{b}^-)$-module map

$$\Theta : H^0\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} \to H^0\Big(\bar{X}, \bar{M}\Big),$$

defined by

$$(\Theta f)(a) = f(\mathrm{Fr}'(a)), \text{ for } a \in \bar{U}(\mathfrak{b}^-).$$

From the following lemma, we see that the map

$$\mathrm{Fr}'^{*} : H^0\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'} \to H^0\Big(\bar{X}, \bar{M}\Big)$$

defined in Proposition 3.3 coincides with Θ. (Observe however that in Proposition 3.3, the map Fr'^{*} was defined for an arbitrary $M \in \mathcal{C}(\mathfrak{b})$.)

(b) It is not clear if the composite $\mathrm{Fr}'^{*}_{\gamma} \circ m_{\sigma_o}$ of the last two maps in the above proposition is the map Fr'^{*} of Proposition 3.3.

LEMMA 4.13. *For any $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{b})$, the map*

$$\mathrm{Fr}'^{*} : H^0\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}}\Big) \to H^0(\bar{X}, \bar{M})$$

satisfies $\mathrm{Fr}'^{*} f = 0$ *if f is a weight vector of weight $\lambda \notin \ell X$.*

Proof. Choose an i such that $\lambda(h_i) = \lambda_0 + \ell\lambda_1$, $0 < \lambda_0 < \ell$. For $a = \begin{bmatrix} K_i; -\ell\lambda_1 \\ \lambda_0 \end{bmatrix}$ and any $\bar{y} = \bar{F}_{i_r}^{(m_r)} \cdots \bar{F}_{i_1}^{(m_1)}$,

$$(1) \qquad \begin{aligned} f\Big((\mathrm{Fr}'\bar{y})a\Big) &= f\left(F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)} \begin{bmatrix} K_i; -\ell\lambda_1 \\ \lambda_0 \end{bmatrix}\right) \\ &= f\Big(F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)}\Big). \end{aligned}$$

Also

$$F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)} \begin{bmatrix} K_i; -\ell\lambda_1 \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} K_i; -\ell\lambda_1 + \sum_k \ell m_k\, a_{i_k} \\ \lambda_0 \end{bmatrix} F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)},$$

by [Lu2, §6.5], where $a_{i_k} := \alpha_{i_k}(h_i)$. So

$$(2) \qquad f\Big((\mathrm{Fr}'\bar{y})a\Big) = 0, \qquad \text{since} \quad \mathrm{Image}\, f \subset \bar{M}^{\mathrm{Fr}}.$$

Comparing (1) and (2), we get

$$(\mathrm{Fr}'^{*} f)(\bar{y}) = f\Big(F_{i_r}^{(\ell m_r)} \cdots F_{i_1}^{(\ell m_1)}\Big) = 0\,.$$

This proves the lemma. $\square$

Remark 4.14. (a) Observe that to prove Theorem 4.7, we needed the quantized version of Serre vanishing (cf. (1) of Lemma 4.8), whereas the corresponding results for Fr^* and Fr'^* (as in Theorems 2.3 and 3.8 respectively) did not require this. This is due to the fact that Proposition 4.6 (and hence Theorem 4.7) is available only for $\bar{M}^{\mathrm{Fr}}$, where $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{b})$. In fact, the analogue of Proposition 4.6 for an arbitrary $M \in \mathcal{C}(\mathfrak{b})$ (as in Proposition 3.3) is *false*, as can be seen already in the case of $\mathfrak{g} = sl(2)$.

(b) Fr'^*_γ is a quantization of the stronger *Frobenius* $\mathcal{L}(\gamma)$*-splitting* of the flag variety G/B (proved by Ramanan-Ramanathan) for the homogeneous line bundle $\mathcal{L}(\gamma)$ on G/B associated to the character γ of B (see §6 for further details).

5. The Kempf vanishing theorem

In this section we assume that $\ell = p$ is an odd prime. We further assume that $p \neq 3$ if G_2 is a factor of $\mathfrak{g}$. Let $\mathbb{F}_p$ be the prime field with p elements and let k be any field containing $\mathbb{F}_p$. Let G be the connected simply-connected semisimple algebraic group defined and split over k corresponding to $\mathfrak{g}$ and let B be its Borel subgroup defined over k (corresponding to $\mathfrak{b}$). Consider the base change $c : \mathbb{Z}_\xi \to \mathbb{F}_p \subset k$ which takes $\xi \mapsto 1$.

For $\lambda \in X$, we denote by $\mathcal{L}(\lambda)$ the line bundle on the flag variety G/B corresponding to the character e^λ of B. More generally, for any $M \in \mathcal{C}(\mathfrak{b})$, we denote by $\mathcal{L}(M)$ the homogeneous vector bundle on G/B associated to the B-module induced by the $\mathfrak{U}_k(\mathfrak{b})$-module $M_k := M \otimes_{\mathbb{Z}_\xi} k$, where $\mathfrak{U}_k(\mathfrak{b}) := \mathfrak{U}(\mathfrak{b}) \otimes_{\mathbb{Z}_\xi} k$ (cf. [Lu2, §8.15] and [CPS, Th. 9.4]).

We recall the following result due to [APW, Prop. 3.7].

PROPOSITION 5.1. *For any* $M \in \mathcal{C}(\mathfrak{b})$, *there exists a canonical isomorphism*:

$$H^i(\mathfrak{U}_k(\mathfrak{g})/\mathfrak{U}_k(\mathfrak{b}), M_k) \simeq H^i(G/B, \mathcal{L}(M)) \qquad \textit{for all } i \geq 0. \tag{1}$$

Similarly, for $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{b})$, *there exists a canonical isomorphism*:

$$H^i(\bar{U}_k(\mathfrak{g})/\bar{U}_k(\mathfrak{b}), \bar{M}_k) \simeq H^i(G/B, \mathcal{L}(\bar{M})) \qquad \textit{for all } i \geq 0. \tag{2}$$

As a corollary of the strong Frobenius splitting as in Section 4 (cf. Theorem 4.7 and Proposition 4.11), we obtain the following Kempf vanishing theorem [K] (if we use the usual Serre vanishing theorem).

THEOREM 5.2. *For any* $\lambda \in X$ *such that* $\lambda + \rho \in X^+$,

$$H^i(G/B, \mathcal{L}(-\lambda)) = 0 \qquad \textit{for all } i > 0.$$

Proof. The constructions and results of Section 4 are compatible under base change. Hence, by Proposition 4.11, the map

$$m_{\sigma_o}\mathrm{Fr}^* : H^i\Big(\bar{U}_k(\mathfrak{g})/\bar{U}_k(\mathfrak{b}), (\bar{\chi}_{-\lambda})_k\Big) \to H^i\Big(\mathfrak{U}_k(\mathfrak{g})/\mathfrak{U}_k(\mathfrak{b}), (\chi^{\xi}_{-2(p-1)\rho-p\lambda})_k\Big)$$

is injective.

Applying Proposition 5.1, we get an injective map

$$H^i(G/B, \mathcal{L}(-\lambda)) \hookrightarrow H^i(G/B, \mathcal{L}(-2(p-1)\rho - p\lambda)).$$

Iterating m-times, we get an injection

$$H^i(G/B, \mathcal{L}(-\lambda)) \hookrightarrow H^i(G/B, \mathcal{L}(-p^m(\lambda + 2\rho) + 2\rho)).$$

Now, using the Serre vanishing theorem for the cohomology of ample line bundles on G/B [H, Chap. III, Prop. 5.3], we obtain that $H^i(G/B, \mathcal{L}(-\lambda)) = 0$, for all $i > 0$. ☐

6. Sheafification: Frobenius splitting of G/B and Schubert varieties

We follow the same notation and conventions as in Section 5. In particular, $\ell = p$ is an odd prime, and $p > 3$ if $\mathfrak{g}$ has a simple component of type $\mathtt{G}_2$. Let k be an algebraically closed field of characteristic p.

Definition 6.1. The *absolute Frobenius morphism* of a scheme X over k is the identity map on the underlying point space and arises to the p^{th} power locally on the functions. Observe that the absolute Frobenius morphism is *not* a morphism of k-schemes. To remedy this, let X' be the scheme with the same underlying topological space as that of X and the same structure sheaf $\mathcal{O}_X$ of rings, only the scalar multiplication of k on $\mathcal{O}_{X'}$ is twisted as:

$$z \odot f = z^p f\,, \text{ for } z \in k \text{ and } f \in \mathcal{O}_{X'}\,.$$

Thus we get a morphism of k-schemes $F : X' \to X$, which at the point set level is the identity map and at the sheaf level corresponds to the morphism $\mathcal{O}_X \to \mathcal{O}_{X'}, f \mapsto f^p$.

Following Mehta-Ramanathan [MR1], a scheme X is called *Frobenius split* if the homomorphism $\mathcal{O}_X \to F_*(\mathcal{O}_{X'})$ of $\mathcal{O}_X$-modules is split. Then clearly an $\mathcal{O}_X$-module homomorphism $\sigma : F_*(\mathcal{O}_{X'}) \to \mathcal{O}_X$ is a splitting of $\mathcal{O}_X \to F_*(\mathcal{O}_{X'})$ if and only if $\sigma(1) = 1$. A splitting σ is said to *split a closed subscheme* $Y \subset X$ *compatibly* if $\sigma(F_*(\mathcal{I}_{Y'})) \subset \mathcal{I}_Y$, where $\mathcal{I}_Y$ is the ideal sheaf of Y.

Let $(\mathcal{D}, \phi)$ be a line bundle on X with a section ϕ. Then, pulling back via F, we get the line bundle (denoted) $\mathcal{D}'$, and the section ϕ clearly gives rise to an $\mathcal{O}_X$-linear morphism $\bar{\phi} : F_*(\mathcal{O}_{X'}) \to F_*(\mathcal{D}')$. Following Ramanan-Ramanathan [RR, §2], a scheme X is called *Frobenius* $(\mathcal{D}, \phi)$*-split* (or less precisely *Frobenius* $\mathcal{D}$*-split*) if there exists an $\mathcal{O}_X$-linear morphism

$$\sigma^{\mathcal{D}} : F_*(\mathcal{D}') \to \mathcal{O}_X\,,$$

such that $\sigma^{\mathcal{D}} \circ \bar{\phi}$ is a splitting. A closed subscheme $Y \subset X$ is called *compatibly* $(\mathcal{D}, \phi)$*-split* (or *compatibly* $\mathcal{D}$*-split*) if $\sigma^{\mathcal{D}} \circ \bar{\phi}$ compatibly splits Y and moreover on no irreducible component of Y, ϕ is identically zero.

Now, we come to the corresponding 'local version'. Let K be a k-algebra. Recall (cf. [M], [Ka, §4.3]) that a *Frobenius-linear endomorphism* of K is a map $\sigma : K \to K$ such that for all $f, g \in K$:

a) additivity: $\sigma(f + g) = \sigma(f) + \sigma(g)$, and

b) $\sigma(f^p g) = f\sigma(g)$.

Observe that by (b), σ cannot be k-linear. The k-space of all the Frobenius-linear endomorphisms of K is denoted by $\mathrm{End}_F(K)$.

A Frobenius-linear endomorphism σ is called a *splitting* if $\sigma(f^p) = f$, for all $f \in K$. Hence a Frobenius-linear endomorphism σ is a splitting if and only if $\sigma(1) = 1$. Let I be an ideal of K. Then a splitting σ is said to *compatibly split* I if and only if $\sigma I \subset I$.

For $K = \oplus_{\lambda \in \mathbb{Z}_+^n} K_\lambda$ a graded k-algebra, a Frobenius-linear endomorphism σ is called *graded* if for all $\lambda \in \mathbb{Z}_+^n$, $\sigma(K_{p\lambda}) \subset K_\lambda$ and $\sigma(K_\lambda) = 0$, if p does not divide λ.

For a smooth scheme X over k with an affine open cover $\{X_j\}$, giving a Frobenius splitting of X is equivalent to giving splittings σ_j of the affine coordinate rings $k[X_j]$ and $\sigma_{\{j,j'\}}$ of $k[X_j \cap X_{j'}]$ (for unordered pairs $\{j, j'\}$) such that the following compatibility is satisfied:

$$\sigma_j(f)_{|X_j \cap X_{j'}} = \sigma_{\{j,j'\}}(f_{|X_j \cap X_{j'}}) \,, \text{ for all } f \in k[X_j]. \tag{1}$$

Moreover, under this correspondence, a closed subscheme $Y \subset X$ is compatibly split if and only if

$$\sigma_j(\mathcal{I}_{X_j \cap Y}) \subset \mathcal{I}_{X_j \cap Y} \,, \text{ for all } j \,, \tag{2}$$

where $\mathcal{I}_{X_j \cap Y} \subset k[X_j]$ is the ideal of $X_j \cap Y$.

For a smooth scheme X with an affine open cover $\{X_j\}$ and a line bundle together with a section $(\mathcal{D}, \phi)$ on X as above, giving a Frobenius $\mathcal{D}$-splitting of X is equivalent to giving additive maps $\sigma_j^{\mathcal{D}} : \Gamma(X_j, \mathcal{D}) \to k[X_j]$ and

$$\sigma_{\{j,j'\}}^{\mathcal{D}} : \Gamma(X_j \cap X_{j'}, \mathcal{D}) \to k[X_j \cap X_{j'}]$$

satisfying the following three properties (3)–(5):

$$\sigma_j^{\mathcal{D}}(f^p s) = f\sigma_j^{\mathcal{D}}(s) \,, \text{ for } f \in k[X_j] \text{ and } s \in \Gamma(X_j, \mathcal{D}) \tag{3}$$

and similarly for $\sigma_{\{j,j'\}}^{\mathcal{D}}$,

$$\sigma_j^{\mathcal{D}}(\phi_{|X_j}) = 1 \,, \tag{4}$$

and for any pair $\{j, j'\}$,

$$\sigma_j^{\mathcal{D}}(s)_{|X_j \cap X_{j'}} = \sigma_{\{j,j'\}}^{\mathcal{D}}(s_{|X_j \cap X_{j'}}) \,, \text{ for } \quad \text{all } s \in \Gamma(X_j, \mathcal{D}), \tag{5}$$

where $\Gamma(X_j, \mathcal{D})$ denotes the $k[X_j]$-module of all the regular sections of $\mathcal{D}$ on X_j.

Let H be an algebraic group over k and let H act on a k-algebra K algebraically via k-algebra automorphisms. Then we define an H-action on $\mathrm{End}_F(K)$ by

$$(h * \sigma)(a) = h \cdot \sigma(h^{-1} \cdot a), \text{ for } h \in H, \sigma \in \mathrm{End}_F(K), \text{ and } a \in K .$$

Let B be a Borel subgroup of a connected, simply-connected, semisimple algebraic group G over k and let $T \subset B$ be a maximal torus. Let B act on K algebraically via k-algebra automorphisms. Then a splitting σ of K is called *B-canonical* if

c) $t * \sigma = \sigma$, for all $t \in T$, and

d) for each simple root α_i, there exist Frobenius-linear endomorphisms $\{\sigma_{m,\alpha_i}\}_{0 \le m < p}$ of K such that

$$(x_{\alpha_i}(z) * \sigma)(a) = \sum_{m=0}^{p-1} \sigma_{m,\alpha_i}(z^m a), \text{ for all } z \in k \text{ and } a \in K,$$

where $x_{\alpha_i}(z)$ is the one-parameter subgroup of B corresponding to the simple root α_i.

We can easily sheafify and extend the notion of B-canonical splitting for a variety X with an action of B (cf. [Ka, p. 42], [M]).

Mehta-Ramanathan [MR1] have shown that the flag variety G/P (for any parabolic subgroup $P \subset G$) is Frobenius split, and the Schubert subvarieties $X(w)_P \subset G/P$ are compatibly split. Moreover, this splitting is B-canonical. The aim of this section is to show that the map Fr'^* constructed in Section 3 induces such a splitting in the case $P = B$. The general case will be handled in Section 9.

Recall from Lemma 2.2 (b) and Proposition 3.3 that, for any $\lambda \in X$, there are maps

$$\mathrm{Fr}^* : H^0\Big(\bar{X}, \bar{\chi}_\lambda\Big)^{\mathrm{Fr}} \to H^0\Big(\mathfrak{X}, \chi^\xi_{p\lambda}\Big), \quad \mathrm{Fr}'^* : H^0\Big(\mathfrak{X}, \chi^\xi_{p\lambda}\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \to H^0\Big(\bar{X}, \bar{\chi}_\lambda\Big).$$

(In fact Fr'^* is defined for any $\mu \in X$, not only for $p\lambda$, but unless μ is divisible by p it is identically zero by Remark 3.10 (b).)

For a vector space V over k, by $V^{[1]}$ we mean the same additive group, but the scalar multiplication is twisted as $z \odot v := z^p v$. Let $G(\mathbb{F}_p)$ be the connected, simply-connected, semisimple algebraic group defined and split over $\mathbb{F}_p$ corresponding to $\mathfrak{g}$ and let $B(\mathbb{F}_p)$ be its Borel subgroup (corresponding to $\mathfrak{b}$) defined over $\mathbb{F}_p$ (cf. §1 for the notation $\mathfrak{g}$ and $\mathfrak{b}$). We denote by G and B the corresponding k-rational points. Consider the base change $c : \mathbb{Z}_\xi \to \mathbb{F}_p$ which takes $\xi \mapsto 1$. Then, as in Proposition 5.1, we have canonical k-linear isomorphisms for any $\lambda \in X$,

$$\begin{aligned} \theta : H^0(\mathfrak{U}_k(\mathfrak{g})/\mathfrak{U}_k(\mathfrak{b}), (\chi^\xi_\lambda)_k) &\simeq H^0(G/B, \mathcal{L}(\lambda))^{[1]} , \text{ and} \\ \bar{\theta} : H^0(\bar{U}_k(\mathfrak{g})/\bar{U}_k(\mathfrak{b}), (\bar{\chi}_\lambda)_k) &\simeq H^0(G/B, \mathcal{L}(\lambda)) , \end{aligned}$$

obtained by first taking the base field $\mathbb{F}_p$ in Proposition 5.1 and then extending k-linearly (with respect to the twisted k-linear structure on $H^0(G/B,\mathcal{L}(\lambda))^{[1]}$ in the first case). Thus, under the above identifications, the maps Fr^* and Fr'^* (after the base change) become k-linear maps (for any $\lambda \in X^+$):

$$\mathcal{F}_\lambda : H^0(G/B,\mathcal{L}(-\lambda))^{\mathrm{Fr}} \to H^0(G/B,\mathcal{L}(-p\lambda))^{[1]}\,, \text{ and}$$

$$\mathcal{F}'_\lambda : \Big(H^0(G/B,\mathcal{L}(-p\lambda))^{[1]}\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \to H^0(G/B,\mathcal{L}(-\lambda))\,.$$

We have shown in [KL, Th. 1] that $\mathcal{F}_\lambda$ is the map $s \mapsto s^p$ sending a section to its p^{th} power, and $\mathcal{F}'_\lambda$ provides a splitting of this map by Corollary 3.9.

We consider the X^+–graded algebra $K := \bigoplus_{\mu\in X^+} H^0\Big(G/B,\mathcal{L}(-\mu)\Big)$ (under the multiplication of sections). Then G acts algebraically on K via k-algebra automorphisms; in particular, so does B.

Let $\mathcal{F}' : K \to K$ be the map defined by $\mathcal{F}'(f) = 0$ for $f \in H^0\Big(G/B,\mathcal{L}(-\mu)\Big)$ if $\mu \notin pX^+$, and $\mathcal{F}' : H^0\Big(G/B,\mathcal{L}(-p\mu)\Big) \to H^0\Big(G/B,\mathcal{L}(-\mu)\Big)$ is the splitting map $\mathcal{F}'_\mu$ defined above (as maps of abelian groups; without regard to the k-linear or $\bar{U}(\mathfrak{b}^-)$-module structures).

As a first step we show:

Proposition 6.2. *$\mathcal{F}'$ is a Frobenius-linear graded endomorphism of K. Moreover, it is a splitting. Further, for any $q \in \mathbb{Z}_+$, $\lambda \in X^+$ and $f \in H^0\Big(G/B,\mathcal{L}(-\lambda)\Big)$:*

$$\mathcal{F}'(\bar{E}_i^{(pq)} f) = \bar{E}_i^{(q)}\mathcal{F}'(f)\,, \quad \textit{and} \tag{1}$$

$$\mathcal{F}'(\bar{F}_i^{(pq)} f) = \bar{F}_i^{(q)}\mathcal{F}'(f), \tag{2}$$

where the action of $\bar{E}_i^{(m)}$ and $\bar{F}_i^{(m)}$ comes from the canonical action of G on $H^0\Big(G/B,\mathcal{L}(-\lambda)\Big)$.

In particular, the splitting $\mathcal{F}'$ is B-canonical.

Proof. The map $\mathcal{F}'$ is clearly additive by definition. Further, by Proposition 3.3 and [Lu2, §8.15], $\mathcal{F}'$ satisfies (1) and (2). Next we prove the condition (b) of §6.1: Take $f \in H^0(G/B,\mathcal{L}(-\lambda))$ and $g \in H^0(G/B,\mathcal{L}(-\mu))$. Thus $f^p g \in H^0(G/B,\mathcal{L}(-(p\lambda+\mu)))$, so if μ is not divisible by p, then we have $\mathcal{F}'(f^p g) = 0 = f\mathcal{F}'(g)$, which proves the claim in this case. Now, we consider the case when μ is divisible by p: It is easy to show that the following diagram is commutative:

$$\begin{tikzcd}
& H^0(\mathfrak{X}, \chi^{\xi}_{-(p\lambda+p\mu)}) \arrow[dl, "\mathrm{Fr}'^*"] & H^0(\mathfrak{X}, \chi^{\xi}_{-p\lambda}) \otimes H^0(\mathfrak{X}, \chi^{\xi}_{-p\mu}) \arrow[l, "m"'] \\
H^0(\bar{X}, \bar{\chi}_{-(\lambda+\mu)}) & & \\
& H^0(\bar{X}, \bar{\chi}_{-\lambda}) \otimes H^0(\bar{X}, \bar{\chi}_{-\mu}) \arrow[ul, "\bar{m}"] & H^0(\bar{X}, \bar{\chi}_{-\lambda}) \otimes H^0(\mathfrak{X}, \chi^{\xi}_{-p\mu}) \arrow[l, "\mathrm{Id}\otimes\mathrm{Fr}'^*"'] \arrow[uu, "\mathrm{Fr}^*\otimes\mathrm{Id}"']
\end{tikzcd}$$

where $m(f \otimes g) := m_f(g)$ (cf. Lemma 4.9) and $\bar{m}$ is defined similarly and Id is the identity map. The commutativity of the diagram after base change implies (b) of §6.1 since $\mathcal{F}_\lambda f = f^p$. Next $\mathcal{F}'(1) = 1$ (as is easy to see), showing that $\mathcal{F}'$ is a splitting.

Finally we prove that $\mathcal{F}'$ is B-canonical. For any simple root α_i, the corresponding one-parameter subgroup $x_{\alpha_i}(z)$ in B can be written as $x_{\alpha_i}(z) = \sum_{m\geq 0} z^m \bar{E}_i^{(m)}$. Then, for any $a \in K$,

$$\begin{aligned}
(3)\quad (x_{\alpha_i}(z) * \mathcal{F}')(a) &\\
&= x_{\alpha_i}(z) \cdot \mathcal{F}'(x_{\alpha_i}(-z) \cdot a)\\
&= x_{\alpha_i}(z) \cdot \mathcal{F}'\Big(\sum_{m\geq 0} (-z)^m \bar{E}_i^{(m)} \cdot a\Big)\\
&= x_{\alpha_i}(z) \cdot \mathcal{F}' \left(\left(\left(\sum_{n\geq 0} (-z)^{np} \bar{E}_i^{(np)}\right) \left(\sum_{m=0}^{p-1} (-z)^m \bar{E}_i^{(m)}\right)\right) \cdot a\right),\\
&\qquad \text{since } \bar{E}_i^{(np)} \bar{E}_i^{(m)} = \bar{E}_i^{(np+m)} \text{ over } k, \text{ for } 0 \leq m < p\\
&= \left(x_{\alpha_i}(z)\Big(\sum_{n\geq 0} (-z)^n\, \bar{E}_i^{(n)}\Big)\right) \cdot \mathcal{F}'\left(\sum_{m=0}^{p-1} (-z)^m\, \bar{E}_i^{(m)} \cdot a\right), \quad \text{by (1)}\\
&= \sum_{m=0}^{p-1} \mathcal{F}'((-z)^m\, \bar{E}_i^{(m)} \cdot a).
\end{aligned}$$

Define $\sigma_m(a) = \mathcal{F}'((-1)^m \bar{E}_i^{(m)} \cdot a)$. Then

$$\sigma_m(z^p a) = z\sigma_m(a). \tag{4}$$

Now since k is an infinite field and $x_{\alpha_i}(z) * \mathcal{F}' \in \mathrm{End}_F(K)$, it is easy to see from (3)–(4) that $\sigma_m \in \mathrm{End}_F(K)$. This proves the defining property (d) of canonical splitting as in §6.1. The proof that, for any $t \in T$, $t * \sigma = \sigma$ is easy from the fact that $\mathrm{Fr}'^*\Big(H^0(\mathfrak{X}, \chi^{\xi}_{-p\mu})_{p\lambda}\Big) \subset H^0(\bar{X}, \bar{\chi}_{-\mu})_\lambda$. □

Fix $\lambda \in X^{++}$ (where X^{++} is the set of dominant regular weights). Replacing λ by $d\lambda$ for $d \in \mathbb{Z}_+$ big enough, we may assume that the embedding of $X = G/B \hookrightarrow \mathbb{P}(\bar{V}_k(\lambda))$, taking gB to the line $[g\bar{v}_+]$, is projectively normal (cf. [H, Chap. II, Ex. 5.14]), where $\bar{v}_+$ is a highest weight vector of $\bar{V}_k(\lambda)$,

$\mathbb{P}(\bar{V}_k(\lambda))$ denotes the space of lines in $\bar{V}_k(\lambda)$ and $\bar{V}_k(\lambda) := H^0(G/B, \mathcal{L}(-\lambda))^*$ is the Weyl module. (Actually, the embedding is always projectively normal. This has been shown in [RR] as a consequence of the Frobenius splitting of X, or in [Li] as a consequence of standard monomial theory; but we do not need this result, in fact we will derive this.) The homogeneous coordinate ring of $X \subset \mathbb{P}(\bar{V}_k(\lambda))$ is hence $k[X] = \bigoplus_{n\geq 0} H^0(X, \mathcal{L}(-n\lambda))$.

For $\tau \in W$, let $p_\tau \in H^0(G/B, \mathcal{L}(-\lambda))$ be a nonzero section of weight $-\tau(\lambda)$ (which is unique up to scalar multiples). Let X_τ be the affine open subset defined by $X_\tau := \{x \in X \mid p_\tau(x) \neq 0\}$. Note that X_τ depends only on τ and not on the choice of λ. (In fact, $X_\tau = \tau U^- B/B \subset G/B$, where U^- is the unipotent radical of the opposite Borel subgroup.)

The affine coordinate ring $k[X_c]$ of the affine open subset X_τ is the degree 0 part of the localization $k[X]_{(p_\tau)}$ of $k[X]$ at p_τ:

$$k[X_\tau] = \bigcup_{m\in\mathbb{Z}_+} \{\frac{f}{p_\tau^m} \mid f \in H^0(X, \mathcal{L}(-m\lambda))\}.$$

Here f/p_τ^m is equivalent to $f'/p_\tau^{m'}$ if there exists a q and nonzero $s \in H^0(X, \mathcal{L}(-q\lambda))$ such that $s(fp_\tau^{m'} - f'p_\tau^m) = 0$. Since X is irreducible, this is equivalent to $f'p_\tau^{m-m'} = f$, if $m \geq m'$. We define now a splitting $\mathcal{F}'_\tau : k[X_\tau] \to k[X_\tau]$ as follows:

$$\mathcal{F}'_\tau(\frac{f}{p_\tau^m}) := \frac{\mathcal{F}'(p_\tau^r f)}{p_\tau^{(r+m)/p}}, \quad \text{where } r \in \mathbb{Z}_+ \text{ is such that } p|(r+m).$$

To see that the map $\mathcal{F}'_\tau$ is well-defined, note first that if we fix an $r \in \mathbb{Z}_+$ such that $p|(r+m)$, then

$$\begin{aligned} p_\tau^r f \in H^0\Big(X, \mathcal{L}(-(r+m)\lambda)\Big) \quad &\Rightarrow \quad \mathcal{F}'(p_\tau^r f) \in H^0\Big(X, \mathcal{L}(-((r+m)/p)\lambda)\Big) \\ &\Rightarrow \quad \frac{\mathcal{F}'(p_\tau^r f)}{p_\tau^{(r+m)/p}} \in k[X_\tau]. \end{aligned}$$

Next, since $\mathcal{F}'$ is a Frobenius-linear endomorphism of K, it is easy to see that the definition is independent of the choice of r and of the chosen representative $\frac{f}{p_\tau^m}$, and moreover $\mathcal{F}'_\tau$ is a Frobenius-linear endomorphism. Further, observe that $\mathcal{F}'_\tau$ is indeed a splitting:

$$\mathcal{F}'_\tau((\frac{f}{p_\tau^m})^p) = \frac{\mathcal{F}'(f^p)}{p_\tau^m} = \frac{f}{p_\tau^m}.$$

It is easy to see that $\mathcal{F}'_\tau$ does not depend upon the choice of p_τ. However, a priori, the definition of the map $\mathcal{F}'_\tau$ seems to depend on the choice of λ, but:

LEMMA 6.3. *The definition of the splitting $\mathcal{F}'_\tau : k[X_\tau] \to k[X_\tau]$ is independent of the choice of $\lambda \in X^{++}$ (such that $X \subset \mathbb{P}(\bar{V}_k(\lambda))$ is projectively normal).*

Proof. Let $\mu \in X^{++}$ be such that the embedding $X \subset \mathbb{P}(\bar{V}_k(\mu))$ is projectively normal. Let $q_\tau \in H^0(X, \mathcal{L}(-\mu))$ be a nonzero weight vector of weight $-\tau(\mu)$. Then, as mentioned above, X_τ coincides with $\{x \in X \mid q_\tau(x) \neq 0\}$, and its affine coordinate ring can be identified as the degree 0 part of the localization of $\bigoplus_{n\geq 0} H^0(X, \mathcal{L}(-n\mu))$ at q_τ.

Take a function $h \in k[X_\tau]$ and write $h = \frac{f}{p_\tau^n} = g/q_\tau^m$ with $f \in H^0(X, \mathcal{L}-(n\lambda))$ and $g \in H^0(X, \mathcal{L}(-m\mu))$. By replacing f/p_τ^n with fp_τ^r/p_τ^{n+r} and g/q_τ^m with gq_τ^t/q_τ^{m+t} if necessary, we may assume that $n = m$ and n is divisible by p. Hence

$$\begin{aligned} \frac{f}{p_\tau^n} &= \frac{g}{q_\tau^n} \Leftrightarrow fq_\tau^n = gp_\tau^n \quad \text{ on the whole of } X \\ \Rightarrow \mathcal{F}'(fq_\tau^n) &= \mathcal{F}'(gp_\tau^n) \Rightarrow q_\tau^{n/p}\mathcal{F}'(f) = p_\tau^{n/p}\mathcal{F}'(g). \end{aligned}$$

The last equality finishes the proof of the lemma. □

To glue $\{\mathcal{F}'_\tau\}_{\tau\in W}$ together to a global splitting, it remains to show that the definitions are compatible on the intersections; i.e., we need to check condition (1) of Definition 6.1:

Consider the affine open set $X_{\{\tau,\kappa\}} := X_\tau \cap X_\kappa = \{x \in X \mid (p_\tau p_\kappa)(x) \neq 0\}$. We define, as above, the map $\mathcal{F}'_{\{\tau,\kappa\}} : k[X_{\{\tau,\kappa\}}] \to k[X_{\{\tau,\kappa\}}]$; i.e., expressing

$$k[X_{\{\tau,\kappa\}}] = \bigcup_{m\in\mathbb{Z}_+} \{\frac{f}{(p_\tau p_\kappa)^m} \mid f \in H^0(X, \mathcal{L}(-2m\lambda))\},$$

we define

$$\mathcal{F}'_{\{\tau,\kappa\}}(\frac{f}{(p_\tau p_\kappa)^m}) := \frac{\mathcal{F}'((p_\tau p_\kappa)^r f)}{(p_\tau p_\kappa)^{(r+m)/p}}, \quad \text{where } r \in \mathbb{Z}_+ \text{ is such that } p|(r+m).$$

As above, one sees that $\mathcal{F}'_{\{\tau,\kappa\}}$ is a well defined Frobenius-linear endomorphism of $k[X_{\{\tau,\kappa\}}]$ and is a splitting.

Take a regular function h on $X_{\{\tau,\kappa\}}$ which is the restriction of a regular function on X_τ. So we can represent h as f/p_τ^n as well as $g/(p_\tau p_\kappa)^m$ (for $f \in H^0(X, \mathcal{L}(-n\lambda))$ and $g \in H^0(X, \mathcal{L}(-2m\lambda))$), and these functions coincide on $X_{\{\tau,\kappa\}}$. Since X is irreducible, this implies $f(p_\tau p_\kappa)^m = gp_\tau^n$ on the whole of X. By multiplying f/p_τ^n with p_τ^a/p_τ^a and $g/(p_\tau p_\kappa)^m$ with $(p_\tau p_\kappa)^b/(p_\tau p_\kappa)^b$ if necessary, we may assume that n, m are divisible by p. Now,

$$(p_\tau p_\kappa)^{m/p}\mathcal{F}'(f) = \mathcal{F}'((p_\tau p_\kappa)^m f) = \mathcal{F}'(p_\tau^n g) = p_\tau^{n/p}\mathcal{F}'(g),$$

and hence, as functions on $X_{\{\tau,\kappa\}}$,

$$\mathcal{F}'_\tau(\frac{f}{p_\tau^n}) = \frac{\mathcal{F}'(f)}{p_\tau^{n/p}} = \frac{\mathcal{F}'(g)}{(p_\tau p_\kappa)^{m/p}} = \mathcal{F}'_{\{\tau,\kappa\}}(\frac{g}{(p_\tau p_\kappa)^m}).$$

This proves the compatibility condition (1) of Definition 6.1.

Since the subsets X_τ, $\tau \in W$, form an affine open cover of X, the above compatibility implies that one can glue the splittings $\mathcal{F}'_\tau$ together to get a Frobenius splitting Θ on the whole of X. Moreover, since $\mathcal{F}'$ is B-canonical, so is Θ.

Summarizing, we have:

THEOREM 6.4. *The Frobenius-linear graded endomorphism $\mathcal{F}'$ of K* (*cf. Proposition* 6.2) *induces a Frobenius splitting Θ of the flag variety G/B by the method described above. Moreover*, *this splitting is B-canonical.* $\square$

By an argument similar to the proof of the above theorem, using Proposition 4.6 instead of Proposition 3.3, we obtain the following stronger result originally due to Ramanan-Ramanathan [RR]. (The commutativity of the diagram, analogous to that in the proof of Proposition 6.2, follows from ideas similar to the proof of Proposition 4.11.)

THEOREM 6.5. *The flag variety G/B is Frobenius $\mathcal{D}$-split*, *where $\mathcal{D}$ is the line bundle $\mathcal{L}(-2(p-1)\rho)$ together with the section $\phi := \theta(\sigma_o)$*, *where σ_o is as defined in* §4.11, *and θ is the isomorphism as in* §6.1. $\square$

For an element $w \in W$ denote by $e_w := w.\mathrm{id} \in G/B$ the corresponding T-fixed point. The closure $\overline{B.e_w}$ of the B-orbit $B.e_w$ is called a *Schubert variety* in G/B and denoted $X(w)$. The closure $\overline{B^-.e_w}$ of the orbit with respect to the opposite Borel subgroup B^- is called an *opposite Schubert variety* in G/B and denoted $X(w)^-$. Using representation theoretic arguments, we show below that the above splitting Θ is compatible with all the Schubert varieties and opposite Schubert varieties.

As in Section 1, for $\lambda \in X^+$, let $V_\xi(\lambda)$ be the Weyl module of highest weight λ for $\mathfrak{U}(\mathfrak{g})$ (over $\mathbb{Z}_\xi$). Similarly, let $\bar{V}(\lambda)$ be the Weyl module for $\bar{U}(\mathfrak{g})$ (again over $\mathbb{Z}_\xi$). For a base change $\mathbb{Z}_\xi \to \mathcal{B}$, we denote $\mathcal{B} \otimes V_\xi(\lambda)$ by $V_\mathcal{B}(\lambda)$ and similarly $\bar{V}_\mathcal{B}(\lambda)$. In particular, we have $V_k(\lambda)$ and $\bar{V}_k(\lambda)$ for the base change $\mathbb{Z}_\xi \to k$ ($\xi \mapsto 1$). Then $V_k(\lambda)$ and $\bar{V}_k(\lambda)$ are canonically isomorphic modules for $\bar{U}_k(\mathfrak{g})$ via [Lu2, §8.15].

For any $w \in W$, the *Demazure module* $V_\xi(\lambda)_w \subset V_\xi(\lambda)$ is, by definition, the $\mathfrak{U}(\mathfrak{b})$–submodule $\mathfrak{U}(\mathfrak{b})v_{w\lambda}$ generated by a primitive extremal weight vector $v_{w\lambda} \in V_\xi(\lambda)$ of weight $w(\lambda)$, and the *opposite Demazure module* is the $\mathfrak{U}(\mathfrak{b}^-)$-submodule $V_\xi(\lambda)^-_w := \mathfrak{U}(\mathfrak{b}^-)v_{w\lambda} \subset V_\xi(\lambda)$. Similarly, define the Demazure module $\bar{V}(\lambda)_w$ and $\bar{V}(\lambda)^-_w \subset \bar{V}(\lambda)$.

As in Section 3, for a $\mathfrak{U}(\mathfrak{g})$-module which is a $\mathfrak{U}^0$-weight module $V = \bigoplus_{\mu \in X} V_\mu$, denote by $V^{\frac{1}{p}}$ the direct sum of weight spaces $\bigoplus_{\mu \in pX} V_\mu$. It is easy to see that $V^{\frac{1}{p}}$ is equipped with a $\bar{U}(\mathfrak{b})$-module structure via $\mathrm{Fr}'_\mathfrak{b}$ (as well as a $\bar{U}(\mathfrak{b}^-)$-module structure via $\mathrm{Fr}'_{\mathfrak{b}^-}$).

By Proposition 3.3, the map $\mathrm{Fr}'^* : H^0(\mathfrak{X}, \chi^{\xi}_{-p\lambda})^{\mathrm{Fr}'_{\mathfrak{b}}} \to H^0(\bar{X}, \bar{\chi}_{-\lambda})$, under the identification β ((1) of §1.4) and a similar identification $\bar{\beta}$, decomposes as the composite of the restriction $V_\xi(p\lambda)^* \to (V_\xi(p\lambda)^{\frac{1}{p}})^*$ followed by a map $(V_\xi(p\lambda)^{\frac{1}{p}})^* \to \bar{V}(\lambda)^*$. We get the dual maps $\bar{V}(\lambda) \hookrightarrow V_\xi(p\lambda)^{\frac{1}{p}} \hookrightarrow V_\xi(p\lambda)$. (The injectivity of the first map follows, e.g., from Corollary 3.9.) These maps are $\bar{U}(\mathfrak{b})$ as well as $\bar{U}(\mathfrak{b}^-)$-module maps (where the latter two modules are equipped with $\bar{U}(\mathfrak{b}), \bar{U}(\mathfrak{b}^-)$-module structures via $\mathrm{Fr}'_{\mathfrak{b}}$ and $\mathrm{Fr}'_{\mathfrak{b}^-}$ respectively); in particular, the μ-weight space (for any $\mu \in X$) $\bar{V}(\lambda)_\mu$ is mapped to $V_\xi(p\lambda)_{p\mu}$. For any $w \in W$, since the extremal weight space $V_\xi(p\lambda)_{pw\lambda}$ is of rank 1, we have the inclusions of Demazure modules (obtained by the restrictions of the above inclusions):

$$(1)\quad \bar{V}(\lambda)_w \hookrightarrow (V_\xi(p\lambda)_w)^{\frac{1}{p}} \hookrightarrow V_\xi(p\lambda)_w, \quad \bar{V}(\lambda)^-_w \hookrightarrow (V_\xi(p\lambda)^-_w)^{\frac{1}{p}} \hookrightarrow V_\xi(p\lambda)^-_w\,,$$

and the associated dual maps

$$(2)\qquad \begin{aligned} &\Big(V_\xi(p\lambda)/V_\xi(p\lambda)_w\Big)^* \to \Big(\bar{V}(\lambda)/\bar{V}(\lambda)_w\Big)^*, \text{ and} \\ &\Big(V_\xi(p\lambda)/V_\xi(p\lambda)^-_w\Big)^* \to \Big(\bar{V}(\lambda)/\bar{V}(\lambda)^-_w\Big)^*, \end{aligned}$$

which are restrictions of Fr'^* under the identifications β and $\bar{\beta}$.

LEMMA 6.6. *Let $\lambda \in X^+$. Then the composite map $r_w \circ \bar{\theta} \circ \bar{\beta}$:*

$$\bar{V}_k(\lambda)^* \overset{\bar{\beta}}{\underset{\sim}{\to}} H^0\Big(\bar{U}_k(\mathfrak{g})/\bar{U}_k(\mathfrak{b}), (\bar{\chi}_{-\lambda})_k\Big) \overset{\bar{\theta}}{\underset{\sim}{\to}} H^0(G/B, \mathcal{L}(-\lambda)) \xrightarrow{r_w} H^0(X_w, \mathcal{L}(-\lambda))$$

has kernel precisely equal to $(\bar{V}_k(\lambda)/\bar{V}_k(\lambda)_w)^*$, where r_w is the restriction map and $\bar{\theta}$ is the isomorphism of §6.1.

Similarly, the composite map $r_w \circ \theta \circ \beta$:

$$V_k(\lambda)^* \overset{\beta}{\underset{\sim}{\to}} H^0\Big(\mathfrak{U}_k(\mathfrak{g})/\mathfrak{U}_k(\mathfrak{b}), (\chi^{\xi}_{-\lambda})_k\Big) \overset{\theta}{\underset{\sim}{\to}} H^0(G/B, \mathcal{L}(-\lambda))^{[1]} \xrightarrow{r_w} H^0(X_w, \mathcal{L}(-\lambda))^{[1]}$$

has kernel precisely equal to $(V_k(\lambda)/V_k(\lambda)_w)^*$.

Similar statements are true with X_w replaced by X^-_w and $\bar{V}_k(\lambda)_w$, $V_k(\lambda)_w$ replaced by $\bar{V}_k(\lambda)^-_w$, $V_k(\lambda)^-_w$ respectively.

Proof. We prove the first assertion. The remaining ones are proved similarly. It is easy to see that the composite $\bar{\gamma} = \bar{\theta} \circ \bar{\beta}$ is given by $\bar{\gamma}(f)(gB) = (g, f(g\bar{v}_\lambda)\bar{v}^*_\lambda)$, for $f \in \bar{V}_k(\lambda)^*$, where $\bar{v}_\lambda$ is a nonzero vector of $\bar{V}_k(\lambda)_\lambda$ and $\bar{v}^*_\lambda \in (\bar{\chi}_{-\lambda})_k = (\bar{V}_k(\lambda)_\lambda)^*$ is given by $\bar{v}^*_\lambda(\bar{v}_\lambda) = 1$. So $r_w \circ \bar{\gamma}(f) = 0 \Leftrightarrow f(bw\bar{v}_\lambda) = 0$ for all $b \in B \Leftrightarrow f(\bar{V}_k(\lambda)_w) \equiv 0$, since the B-module span and $\bar{U}_k(\mathfrak{b})$-module span of $w\bar{v}_\lambda$ are the same. This proves the lemma. □

The following result is originally due to Mehta-Ramanathan [MR1].

THEOREM 6.7. *Let $Z \subset G/B$ be a subscheme obtained from $\{X(w), X(w)^-\}_{w\in W}$ by repeatedly taking scheme theoretic unions, intersections and irreducible components. Then $Z \subset G/B$ is compatibly split under the splitting Θ of G/B given in Theorem* 6.4. *In particular, Z is a reduced scheme.*

Proof. We first show that, for any $w \in W$, $X(w) \subset G/B$ is compatibly split: Fix $\tau \in W$ such that $X_\tau \cap X(w) \neq \phi$ and let $I(w)_\tau \subset k[X_\tau]$ be the ideal of $X_\tau \cap X(w)$ in $k[X_\tau]$. Choose $\lambda \in X^{++}$ such that the embedding $X \hookrightarrow \mathbb{P}(\bar{V}_k(\lambda))$ is projectively normal and let $p_\tau \in H^0(X, \mathcal{L}(-\lambda))$ be a nonzero section of weight $-\tau(\lambda)$. Then, by Lemma 6.6, for $f \in H^0(X, \mathcal{L}(-m\lambda))$, $\frac{f}{p_\tau^m} \in I(w)_\tau \Leftrightarrow f \in \gamma\Big((V_k(m\lambda)/V_k(m\lambda)_w)^*\Big)$, where $\gamma := \theta \circ \beta$ is as in Lemma 6.6. Hence, if $p|m$, for $\frac{f}{p_\tau^m} \in I(w)_\tau$,

$$\mathcal{F}'_\tau(\frac{f}{p_\tau^m}) = \frac{\mathcal{F}'(f)}{p_\tau^{m/p}} \in I(w)_\tau\,,$$

by (2) of §6.5 (with the base change $\mathbb{Z}_\xi \to k$). This proves that $I(w)_\tau$ is stable under $\mathcal{F}'_\tau$. The same argument shows that the ideal $I(w)_\tau^-$ of $X(w)_\tau^-$ in $k[X_\tau]$ is again stable under $\mathcal{F}'_\tau$. Hence, any repeated sum and intersection of these ideals is stable under $\mathcal{F}'_\tau$. This shows that the splitting Θ of G/B compatibly splits any unions and intersections of $X(w)$ and $X(w)^-$. Since any irreducible component of a compatibly split subscheme is again compatibly split (cf. [R2, Prop. 1.9]), the theorem follows. $\square$

Remark 6.8. The map Fr'^* (resp. Fr'^*_γ) can hence be viewed as a characteristic zero lift of the Frobenius splitting (resp. Frobenius $\mathcal{D}$-splitting, for the line bundle $\mathcal{D} := \mathcal{L}(-2(p-1)\rho)$) of G/B on the level of quantum groups. In fact, on the level of quantum groups, as we saw in Sections 3 and 4, the maps Fr'^* and Fr'^*_γ are defined for any odd integer $\ell > 1$ (not necessarily prime numbers). Also see [Li], where a 'standard monomial' basis of $H^0(X(w), \mathcal{L}(\lambda))$ has been constructed. There the maps $\mathcal{F}'_\tau$ are used to define the "ℓ^{th} root" of a product of extremal weight vectors in $V_\xi(\lambda)^*_w$.

7. Splitting of the diagonal in $G/B \times G/B$

As in Sections 1 to 4, we continue to assume that $\ell > 1$ is an odd integer, which, in addition, is coprime to 3 if G_2 is a component of $\mathfrak{g}$.

Definition 7.1. Let $\bar{M}, \bar{N} \in \bar{\mathcal{C}}(\mathfrak{b})$ be two $\bar{U}(\mathfrak{b})$-modules. By Theorem 2.3, we have a $\mathfrak{U}(\mathfrak{b})$-module map (for any $j \geq 0$):

$$\pi_1 : [\bar{M} \otimes H^j(\bar{X}, \bar{N})]^{\mathrm{Fr}} \longrightarrow \bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}, \bar{N}^{\mathrm{Fr}}),\ a \otimes f \mapsto a \otimes \mathrm{Fr}^*(f).$$

Similarly, by Theorem 3.8, we have a $\bar{U}(\mathfrak{b})$-module map:

$$\pi_2 : [\bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})]^{\mathrm{Fr}'_{\mathfrak{b}}} \longrightarrow \bar{M} \otimes H^j(\bar{X}, \bar{N}),\ a \otimes g \mapsto a \otimes \mathrm{Fr}'^{*}(g).$$

The composition of the above two maps is the identity map by Corollary 3.9. By Theorem 2.3 and inducing π_1 we get, for all $i \in \mathbb{Z}_+$, $\mathfrak{U}(\mathfrak{g})$-module maps:

$$\begin{tikzcd}
H^i\Big(\bar{X}, \bar{M} \otimes H^j(\bar{X}, \bar{N})\Big)^{\mathrm{Fr}} \arrow[dr, "\mathrm{Fr}^*"] \arrow[dd, "\mathrm{Fr}^*_{\Delta}"'] & \\
& H^i\Big(\mathfrak{X}, [\bar{M} \otimes H^j(\bar{X}, \bar{N})]^{\mathrm{Fr}}\Big) \arrow[dl, "\hat{\pi}_1"] \\
H^i\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})\Big), &
\end{tikzcd}$$

where $\hat{\pi}_1$ is induced from π_1 and Fr^*_{Δ} is by definition the composite map $\hat{\pi}_1 \circ \mathrm{Fr}^*$.

Similarly, by inducing the map π_2 and the splitting given by Theorem 3.8, we get $\bar{U}(\mathfrak{b})$-module maps:

$$\begin{tikzcd}
H^i\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})\Big)^{\mathrm{Fr}'_{\mathfrak{b}}} \arrow[dr, "\mathrm{Fr}'^{*}"] \arrow[dd, "\mathrm{Fr}'^{*}_{\Delta}"'] & \\
& H^i\Big(\bar{X}, [\bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})]^{\mathrm{Fr}'_{\mathfrak{b}}}\Big) \arrow[dl, "\hat{\pi}_2"] \\
H^i\Big(\bar{X}, \bar{M} \otimes H^j(\bar{X}, \bar{N})\Big), &
\end{tikzcd}$$

where we denote by $\mathrm{Fr}'^{*}_{\Delta}$ the composition of the two maps Fr'^{*} and $\hat{\pi}_2$.

From the functoriality of Fr'^{*}, we get $\mathrm{Fr}'^{*} \circ \hat{\pi}_1 = \hat{\bar{\pi}}_1 \circ \mathrm{Fr}'^{*}$, where the last Fr'^{*} is the map $H^i\Big(\mathfrak{X}, [\bar{M} \otimes H^j(\bar{X}, \bar{N})]^{\mathrm{Fr}}\Big) \to H^i\Big(\bar{X}, \bar{M} \otimes H^j(\bar{X}, \bar{N})\Big)$ and $\bar{\pi}_1$ is the $\bar{U}(\mathfrak{b})$-module map $\bar{M} \otimes H^j(\bar{X}, \bar{N}) \to [\bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})]^{\mathrm{Fr}'_{\mathfrak{b}}}$ obtained by applying $[\ ]^{\mathrm{Fr}'_{\mathfrak{b}}}$ to π_1. Hence

$$\mathrm{Fr}'^{*}_{\Delta} \circ \mathrm{Fr}^*_{\Delta} = \hat{\pi}_2 \circ \hat{\bar{\pi}}_1 \circ \mathrm{Fr}'^{*} \circ \mathrm{Fr}^* = \mathrm{Id}\,,$$

by application of Corollary 3.9 twice. Summarizing, we have:

Proposition 7.2. *For any $\bar{M}, \bar{N} \in \bar{\mathcal{C}}(\mathfrak{b})$ and $i, j \in \mathbb{Z}_+$, there exists a functorial $\bar{U}(\mathfrak{b})$-module map*

$$\mathrm{Fr}'^{*}_{\Delta} : H^i\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})\Big)^{\mathrm{Fr}'_{\mathfrak{b}}} \to H^i\Big(\bar{X}, \bar{M} \otimes H^j(\bar{X}, \bar{N})\Big)$$

which is a splitting of the functorial $\mathfrak{U}(\mathfrak{g})$-module map

$$\mathrm{Fr}^*_{\Delta} : H^i\Big(\bar{X}, \bar{M} \otimes H^j(\bar{X}, \bar{N})\Big)^{\mathrm{Fr}} \to H^i\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})\Big).$$

Moreover, for any $m \geq 0$, $1 \leq t \leq n$ and $f \in H^i\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})\Big)$,

$$\bar{F}_t^{(m)} \cdot (\mathrm{Fr}'^{*}_{\Delta} f) = \mathrm{Fr}'^{*}_{\Delta}(F_t^{(m\ell)} \cdot f), \tag{1}$$

where the action of $F_t^{(m)}$ *on* $H^i\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})\Big)$ *comes from its action on the cohomology* $H^i\Big(\mathfrak{X}, M\Big)$ *for any* $M \in \mathcal{C}(\mathfrak{b})$ *and similarly for the action of* $\bar{F}_i^{(m)}$. □

From now on, to the end of this section, use the same notation and assumptions as in Sections 5 and 6. In particular, assume that $\ell = p$ is an odd prime and $p > 3$ if $\mathfrak{g}$ has a component of type $\mathtt{G}_2$, and let k be an algebraically closed field of characteristic p. Also let G and B be as in Section 6.

We view $Y := G/B \times G/B$ as a G-variety via the diagonal action. Note that the canonical map $G \times_B G/B \to Y$, defined by $(g, g'B) \mapsto (gB, gg'B)$, is a G-equivariant isomorphism. It follows immediately that the G-orbits in Y are precisely the images of $G \times_B C(w)$, where $C(w) := Be_w$ is the B-orbit of the T-fixed point $e_w \in G/B$ for $w \in W$, and the closure $Y(w)$ of the image of $G \times_B C(w)$ in Y is the image of $G \times_B X(w)$, where $X(w) \subset G/B$ is the Schubert variety $C(w)$. The diagonal $G/B \subset G/B \times G/B$ corresponds here to $G \times_B X(\mathrm{id})$, where $X(\mathrm{id}) = e_{\mathrm{id}}$ is the one point Schubert variety. The varieties $Y(w)$ are called the *G-Schubert varieties* in Y.

Of course, $G \times G$ satisfies the conditions of the preceding section, so the flag variety $(G \times G)/(B \times B) \simeq G/B \times G/B$ is Frobenius split. However, in general, the G-Schubert varieties $Y(w)$ are not compatibly split with respect to this splitting. So we need to consider a different splitting.

Recall [APW, Prop. 2.16] that for $\mathfrak{U}(\mathfrak{g})$-module $M \in \mathcal{C}(\mathfrak{g})$ which is $\mathbb{Z}_\xi$-flat and $N \in \mathcal{C}(\mathfrak{b})$, there is a $\mathfrak{U}(\mathfrak{g})$-module isomorphism

$$\delta : H^0(\mathfrak{X}, N) \otimes M \xrightarrow{\sim} H^0(\mathfrak{X}, N \otimes M)$$

given by

$$\delta(f \otimes m)(x) = \sum_i f(x_i') \otimes (x_i'' m),$$

for $f \in H^0(\mathfrak{X}, N)$, $m \in M$ and $x \in \mathfrak{U}(\mathfrak{g})$, where $\Delta x = \sum_i x_i' \otimes x_i''$.

There is a similar $\bar{U}(\mathfrak{g})$-module isomorphism for $\bar{N} \in \bar{\mathcal{C}}(\mathfrak{b})$ and $\mathbb{Z}_\xi$-flat $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{g})$:

$$\bar{\delta} : H^0(\bar{X}, \bar{N}) \otimes \bar{M} \xrightarrow{\sim} H^0(\bar{X}, \bar{N} \otimes \bar{M}).$$

In particular, for $\bar{M}, \bar{N} \in \bar{\mathcal{C}}(\mathfrak{b})$ such that $H^0(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})$ and $H^0(\bar{X}, \bar{N})$ are both $\mathbb{Z}_\xi$-flat, the $\bar{U}(\mathfrak{b})$-module map Fr'^*_Δ of §7.1 (for $i = j = 0$) under the identifications δ and $\bar{\delta}$ can be rewritten as

$$\mathrm{Fr}'^*_\Delta : [H^0(\mathfrak{X}, \bar{M}^{\mathrm{Fr}}) \otimes H^0(\mathfrak{X}, \bar{N}^{\mathrm{Fr}})]^{\mathrm{Fr}'_{\mathfrak{b}}} \to H^0(\bar{X}, \bar{M}) \otimes H^0(\bar{X}, \bar{N}).$$

Similarly, we can rewrite the $\mathfrak{U}(\mathfrak{g})$-module map Fr^*_Δ as

$$\mathrm{Fr}^*_\Delta : H^0(\bar{X}, \bar{M})^{\mathrm{Fr}} \otimes H^0(\bar{X}, \bar{N})^{\mathrm{Fr}} \to H^0(\mathfrak{X}, \bar{M}^{\mathrm{Fr}}) \otimes H^0(\mathfrak{X}, \bar{N}^{\mathrm{Fr}}).$$

In particular, taking $\bar{M} = \bar{\chi}_{-\lambda}, \bar{N} = \bar{\chi}_{-\mu}$ (for $\lambda, \mu \in X^+$) and considering the base change $\mathbb{Z}_\xi \to k\,(\xi \mapsto 1)$, the maps Fr'^*_Δ and Fr^*_Δ under the identifications θ and $\bar{\theta}$ of §6.1, correspond respectively to the maps

$$\begin{aligned}\mathcal{F}'_{(\lambda,\mu)} : &[H^0(G/B, \mathcal{L}(-p\lambda))^{[1]} \otimes H^0(G/B, \mathcal{L}(-p\mu))^{[1]}]^{\mathrm{Fr}'_{\mathfrak{b}}} \\ &\longrightarrow H^0(G/B, \mathcal{L}(-\lambda)) \otimes H^0(G/B, \mathcal{L}(-\mu))\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_{(\lambda,\mu)} : &H^0(G/B, \mathcal{L}(-\lambda))^{\mathrm{Fr}} \otimes H^0(G/B, \mathcal{L}(-\mu))^{\mathrm{Fr}} \\ &\longrightarrow H^0(G/B, \mathcal{L}(-p\lambda))^{[1]} \otimes H^0(G/B, \mathcal{L}(-p\mu))^{[1]}\,.\end{aligned}$$

Moreover, $\mathcal{F}'_{(\lambda,\mu)} \circ \mathcal{F}_{(\lambda,\mu)} = \mathrm{Id}$. Observe that the map $\mathcal{F}_{(\lambda,\mu)}$ is a $\mathfrak{U}(\mathfrak{g})$-module map and $\mathcal{F}'_{(\lambda,\mu)}$ is a $\bar{U}(\mathfrak{b})$-module map (under the diagonal actions).

Consider the $X^+ \times X^+$-graded algebra

$$K_\Delta := \bigoplus_{\lambda,\mu \in X^+} H^0(G/B, \mathcal{L}(-\lambda)) \otimes H^0(G/B, \mathcal{L}(-\mu)),$$

under the multiplication $(s \otimes t) \cdot (f \otimes g) := (sf) \otimes tg$. We abbreviate

$$H^0(G/B, \mathcal{L}(-\lambda)) \otimes H^0(G/B, \mathcal{L}(-\mu))$$

to $K_{\lambda,\mu}$. Denote by $\mathcal{F}'_\Delta : K_\Delta \to K_\Delta$ the graded map defined by $\mathcal{F}'_\Delta(f \otimes g) = 0$ for $f \otimes g \in K_{\lambda,\mu}$ if $(\lambda, \mu) \notin pX^+ \times pX^+$, and let $\mathcal{F}'_{\Delta|K_{p\lambda,p\mu}}$ be the splitting map $\mathcal{F}'_{(\lambda,\mu)}$ (as maps of abelian groups; without regard to the k-linear or $\bar{U}(\mathfrak{b})$-module structures).

Similar to Proposition 6.2, we have the following:

PROPOSITION 7.3. *$\mathcal{F}'_\Delta$ is a Frobenius-linear graded endomorphism of K_Δ. Moreover, it is a splitting. Further, for any $q \in \mathbb{Z}_+$ and $f \otimes g \in K_\Delta$,*

$$\mathcal{F}'_\Delta(\bar{E}_i^{(pq)} \cdot (f \otimes g)) = \bar{E}_i^{(q)} \cdot \mathcal{F}'_\Delta(f \otimes g)\,, \text{ and} \tag{1}$$

$$\mathcal{F}'_\Delta(\bar{F}_i^{(pq)} \cdot (f \otimes g)) = \bar{F}_i^{(q)} \cdot \mathcal{F}'_\Delta(f \otimes g)\,, \tag{2}$$

where $\bar{E}_i^{(q)}$ and $\bar{F}_i^{(q)}$ act diagonally.

In particular, $\mathcal{F}'_\Delta$ is B-canonical for the diagonal action of B on K_Δ.

Proof. The map $\mathcal{F}'_\Delta$ is clearly additive and the properties (1) and (2) follow from Proposition 7.2. We proceed now to prove property (b) of Definition 6.1 (following the proof of Proposition 6.2):

Let $s \otimes t \in K_{\lambda,\mu}$ and $f \otimes g \in K_{\eta,\nu}$. Since $s^p f \otimes t^p g \in K_{p\lambda+\eta, p\mu+\nu}$,

$$\mathcal{F}'_\Delta(s^p f \otimes t^p g) = 0 = (s \otimes t) \cdot \mathcal{F}'_\Delta(f \otimes g) \quad \text{if} \quad (\eta, \nu) \notin pX^+ \times pX^+.$$

Assume now that $f \otimes g \in K_{p\eta,p\nu}$. Consider the commutative diagram:

$$\begin{array}{ccc}
H(\bar{X}, \bar{\chi}_{-\lambda} \otimes H(\bar{\chi}_{-\mu})) \otimes H(\mathfrak{X}, \chi^{\xi}_{-p\eta} \otimes H(\chi^{\xi}_{-p\nu})) & \xrightarrow{\mathrm{Fr}^{*}_{\Delta} \otimes \mathrm{Id}} & H(\mathfrak{X}, \chi^{\xi}_{-p\lambda} \otimes H(\chi^{\xi}_{-p\mu})) \otimes H(\mathfrak{X}, \chi^{\xi}_{-p\eta} \otimes H(\chi^{\xi}_{-p\nu})) \\
\Big\downarrow {\scriptstyle \mathrm{Id} \otimes \mathrm{Fr}'^{*}_{\Delta}} & & \Big\downarrow {\scriptstyle m} \\
H(\bar{X}, \bar{\chi}_{-\lambda} \otimes H(\bar{\chi}_{-\mu})) \otimes H(\bar{X}, \bar{\chi}_{-\eta} \otimes H(\bar{\chi}_{-\nu})) & & H(\mathfrak{X}, \chi^{\xi}_{-p\lambda} \otimes H(\chi^{\xi}_{-p\mu}) \otimes \chi^{\xi}_{-p\eta} \otimes H(\chi^{\xi}_{-p\nu})) \\
\Big\downarrow {\scriptstyle \bar{\zeta}} & & \Big\downarrow {\scriptstyle \hat{m}} \\
H(\bar{X}, \bar{\chi}_{-\lambda-\eta} \otimes H(\bar{\chi}_{-\mu-\nu})) & \xleftarrow{\mathrm{Fr}'^{*}_{\Delta}} & H(\mathfrak{X}, \chi^{\xi}_{-p\lambda-p\eta} \otimes H(\chi^{\xi}_{-p\mu-p\nu})),
\end{array}$$

where H denotes H^0, $H(\chi^{\xi}_{\lambda})$ denotes $H^0(\mathfrak{X}, \chi^{\xi}_{\lambda})$ (similarly $H(\bar{\chi}_{\lambda})$), $\hat{m}$ is the map induced from the $\mathfrak{U}(\mathfrak{b})$-module map

$$\chi^{\xi}_{-p\lambda} \otimes H^0(\mathfrak{X}, \chi^{\xi}_{-p\mu}) \otimes \chi^{\xi}_{-p\eta} \otimes H^0(\mathfrak{X}, \chi^{\xi}_{-p\nu}) \to \chi^{\xi}_{-p\lambda-p\eta} \otimes H^0(\mathfrak{X}, \chi^{\xi}_{-p\mu-p\nu})$$

taking $(a \otimes b) \otimes (c \otimes d) \mapsto (a \otimes c) \otimes m_b(d)$ (where m_b is the map defined in Lemma 4.9) and $m(\sigma \otimes \sigma') = m_\sigma(\sigma')$. Let ζ be the composite map $\hat{m} \circ m$; $\bar{\zeta}$ is analogously defined as $\hat{\bar{m}} \circ \bar{m}$. (Checking the commutativity of the above diagram is routine if we keep track of the definitions of various maps involved.)

The commutativity of the above diagram after base change implies property (b) of Definition 6.1 since

$$\mathcal{F}_{(\lambda,\mu)}(s \otimes t) = s^p \otimes t^p, \tag{3}$$

for $s \in H^0(G/B, \mathcal{L}(-\lambda))$ and $t \in H^0(G/B, \mathcal{L}(-\mu))$. Observe that (3) follows from the corresponding property: $\mathcal{F}_\lambda(s) = s^p$ (cf. §6.1) together with the identity:

$$\mathcal{F}_{(\lambda,\mu)}(s \otimes t) = \mathcal{F}_\lambda(s) \otimes \mathcal{F}_\mu(t). \tag{4}$$

Next, it is easy to see that $\mathcal{F}'_\Delta(1) = 1$ and hence $\mathcal{F}'_\Delta$ is a splitting. The assertion that $\mathcal{F}'_\Delta$ is B-canonical follows from (1) by the same argument as that used in the proof of Proposition 6.2. □

Analogous to Lemma 6.6, we get the following:

LEMMA 7.4. *Let $\lambda, \mu \in X^+$. Then the composite map*:

$$\begin{aligned}
[V_k(\mu) \otimes V_k(\lambda)]^* \overset{\nu}{\xrightarrow{\sim}} V_k(\lambda)^* \otimes V_k(\mu)^* &\overset{\tilde{\gamma}}{\xrightarrow{\sim}} H^0(G/B \times G/B, \mathcal{L}(-\lambda) \boxtimes \mathcal{L}(-\mu))^{[1]} \\
&\to H^0(Y(w), \mathcal{L}(-\lambda) \boxtimes \mathcal{L}(-\mu))^{[1]}
\end{aligned}$$

has kernel precisely equal to $[(V_k(\mu) \otimes V_k(\lambda))/(\mathfrak{U}_k(\mathfrak{g}).(v_{w\mu} \otimes v_\lambda))]^*$, *where* $\mathfrak{U}_k(\mathfrak{g})$ *acts diagonally*, $v_{w\mu}$ *is a nonzero vector of weight* $w\mu$, *the inverse of the first isomorphism* ν *is given by* $\nu^{-1}(f \otimes g)(y \otimes x) = f(x)g(y)$ *for* $f \in V_k(\lambda)^*, g \in V_k(\mu)^*, x \in V_k(\lambda)$, $y \in V_k(\mu)$, *and* $\tilde{\gamma}$ *is induced by the isomorphism* $\gamma = \theta \circ \beta$ *as in Lemma* 6.6.

A similar statement is true with V_k replaced by $\bar{V}_k$. □

Dualize the map

$$\mathrm{Fr}'^{*}_{\Delta} : H^0(\mathfrak{X}, \chi^{\xi}_{-p\lambda}) \otimes H^0(\mathfrak{X}, \chi^{\xi}_{-p\mu}) \to H^0(\bar{X}, \bar{\chi}_{-\lambda}) \otimes H^0(\bar{X}, \bar{\chi}_{-\mu})$$

to get the map

$$\kappa : \bar{V}(\mu) \otimes \bar{V}(\lambda) \to V_\xi(p\mu) \otimes V_\xi(p\lambda).$$

The map κ commutes with $\bar{U}(\mathfrak{b})$ and $\bar{U}(\mathfrak{b}^-)$-actions, where $V_\xi(p\mu) \otimes V_\xi(p\lambda)$ is equipped with the diagonal $\bar{U}(\mathfrak{b})$ and $\bar{U}(\mathfrak{b}^-)$-actions via Fr'. It can be seen that

$$\mathrm{Fr}'^{*}_{\Delta}(\beta(v^*_{p\lambda}) \otimes g) = \bar{\beta}(\bar{v}^*_\lambda) \otimes \mathrm{Fr}'^{*} g\,, \text{ for any } g \in H^0(\mathfrak{X}, \chi^{\xi}_{-p\mu}),$$

where $v^*_{p\lambda} \in V_\xi(p\lambda)^*$ is defined by $v^*_{p\lambda}(v_{p\lambda}) = 1$ and $v^*_{p\lambda}(v) = 0$ for any weight vector of weight $\neq p\lambda$ ($\bar{v}^*_\lambda$ is defined similarly). Dualizing, we obtain:

$$\kappa(\bar{v}_{w\mu} \otimes \bar{v}_\lambda) \in \mathbb{Z}_\xi(v_{pw\mu} \otimes v_{p\lambda})\,.$$

Hence

$$\kappa(\bar{U}(\mathfrak{g}) \cdot (\bar{v}_{w\mu} \otimes \bar{v}_\lambda)) \subset \mathfrak{U}(\mathfrak{g}) \cdot (v_{pw\mu} \otimes v_{p\lambda}).$$

By the same proof as that of Theorems 6.4 and 6.7, we obtain the following from Proposition 7.3 and Lemma 7.4. It was first proved by Mehta-Ramanathan [MR2] that $G/B \times G/B$ (more generally $G/P \times G/P'$) admits a Frobenius splitting which compatibly splits all the G-Schubert subvarieties.

THEOREM 7.5. *Let Y be the G-variety $G/B \times G/B$ (under the diagonal action of G). Then the Frobenius-linear graded endomorphism $\mathcal{F}'_\Delta$ of K_Δ (cf. Prop.* 7.3*) induces a Frobenius splitting Θ_Y of Y by a similar method given in Section* 6*. Moreover, this splitting is B-canonical.*

Further, any subscheme $Z \subset Y$ obtained from the G-Schubert varieties $\{Y(w)\}_{w\in W}$ by repeatedly taking scheme theoretic unions, intersections and irreducible components is compatibly split. In particular, Z is a reduced scheme.

□

8. Frobenius splitting of quantized Bott-Samelson desingularization

In this section, as in Sections 1 to 4, $\ell > 1$ is an odd integer which is assumed to be coprime to 3 if G_2 is a component of $\mathfrak{g} = \mathfrak{g}(A)$. For any $1 \le i \le n$, let $\mathfrak{b} \subset \mathfrak{p}_i$ be the minimal parabolic subalgebra of $\mathfrak{g}$.

Definition 8.1 [APW, §5.1]. For any sequence of simple reflections $\mathfrak{w} = (s_{i_1}, \cdots, s_{i_m})$, define the functor $D_{\mathfrak{w}} : \mathcal{C}(\mathfrak{b}) \to \mathcal{C}(\mathfrak{b})$ inductively by

$$D_{\mathfrak{w}}(M) = H^0(\mathfrak{U}(\mathfrak{p}_{i_1})/\mathfrak{U}(\mathfrak{b}), D_{\mathfrak{w}'}(M)),$$

where $\mathfrak{w}'$ is the subsequence $(s_{i_2}, \cdots, s_{i_m})$.

Similarly, define $\bar{D}_{\mathfrak{w}} : \bar{\mathcal{C}}(\mathfrak{b}) \to \bar{\mathcal{C}}(\mathfrak{b})$ by

$$\bar{D}_{\mathfrak{w}}(M) = H^0(\bar{U}(\mathfrak{p}_{i_1})/\bar{U}(\mathfrak{b}), \bar{D}_{\mathfrak{w}'}(M)).$$

Both the functors $D_{\mathfrak{w}}$ and $\bar{D}_{\mathfrak{w}}$ are left exact. Denote their right derived functors respectively by $H^i(\mathcal{Z}_{\mathfrak{w}},-)$ and $H^i(\bar{Z}_{\mathfrak{w}},-)$. Let k be a field which is a $\mathbb{Z}_\xi$-algebra $\mathbb{Z}_\xi \to k$. Let $\mathcal{C}_k(\mathfrak{b})$ (resp. $\bar{\mathcal{C}}_k(\mathfrak{b})$) be the analogue of the category $\mathcal{C}(\mathfrak{b})$ (resp. $\bar{\mathcal{C}}(\mathfrak{b})$), where the base ring $\mathbb{Z}_\xi$ is replaced by k. For any $M \in \mathcal{C}_k(\mathfrak{b})$ we can similarly define $H^*(\mathcal{Z}^k_{\mathfrak{w}}, M)$ as the derived functors of $D^k_{\mathfrak{w}}(M) := H^0(\mathfrak{U}_k(\mathfrak{p}_{i_1})/\mathfrak{U}_k(\mathfrak{b}), D^k_{\mathfrak{w}'}(M))$. (Also, $H^*(\bar{Z}^k_{\mathfrak{w}}, \bar{M})$, for $\bar{M} \in \bar{\mathcal{C}}_k(\mathfrak{b})$, is defined analogously.)

By [APW, Cor. 2.13] (actually their setting differs a bit, but the same proof works), for any $M \in \mathcal{C}^0_k$, the $\mathfrak{U}_k(\mathfrak{b})$-module $N := H^0(\mathfrak{U}_k(\mathfrak{b})/\mathfrak{U}^0_k, M)$ is an injective object of $\mathcal{C}_k(\mathfrak{b})$. In particular, it is acyclic for the functor $D^k_{\mathfrak{w}}$, i.e.,

$$H^i(\mathcal{Z}^k_{\mathfrak{w}}, N) = 0, \text{ for all } i > 0. \tag{1}$$

Similarly, for $\bar{M} \in \bar{\mathcal{C}}^0_k$,

$$H^i(\bar{Z}^k_{\mathfrak{w}}, \bar{N}) = 0, \text{ for all } i > 0, \tag{2}$$

where

$$\bar{N} := H^0(\bar{U}_k(\mathfrak{b})/\bar{U}^0_k, \bar{M}).$$

Remark 8.2. It is very likely that (1) and (2) above remain true for any $\mathbb{Z}_\xi$-algebra k (not only when k is a field).

THEOREM 8.3. *For any sequence $\mathfrak{w} = (s_{i_1}, \cdots, s_{i_m})$ and any $\bar{M} \in \bar{\mathcal{C}}_k(\mathfrak{b})$, there exists a functorial $\mathfrak{U}_k(\mathfrak{b})$-module map*

$$\mathrm{Fr}^*_{\mathfrak{w}} : H^i(\bar{Z}^k_{\mathfrak{w}}, \bar{M})^{\mathrm{Fr}} \to H^i(\mathcal{Z}^k_{\mathfrak{w}}, \bar{M}^{\mathrm{Fr}}).$$

Proof. Consider the standard resolution of $\bar{M}$ in the category $\bar{\mathcal{C}}_k(\mathfrak{b})$ (cf. $(*)$ of (1.5)):

$$0 \to \bar{M} \to \bar{Q}_0 \to \bar{Q}_1 \to \cdots, \tag{1}$$

and also the standard resolution of $\bar{M}^{\mathrm{Fr}}$ in the category $\mathcal{C}_k(\mathfrak{b})$:

$$0 \to \bar{M}^{\mathrm{Fr}} \to Q_0 \to Q_1 \to \cdots. \tag{2}$$

As in the proof of Theorem 2.3, there are $\mathfrak{U}_k(\mathfrak{b})$-module homomorphisms $\theta_i : \bar{Q}^{\mathrm{Fr}}_i \to Q_i$, for all $i \geq 0$, making the diagram (D′) of the proof of Theorem 2.3 commutative.

For any $\bar{Q} \in \bar{\mathcal{C}}_k(\mathfrak{b})$, we first construct a functorial $\mathfrak{U}_k(\mathfrak{b})$-module map

$$\mathrm{Fr}^*_{\mathfrak{w}} : H^0(\bar{Z}^k_{\mathfrak{w}}, \bar{Q})^{\mathrm{Fr}} \to H^0(\mathcal{Z}^k_{\mathfrak{w}}, \bar{Q}^{\mathrm{Fr}}),$$

by induction on $\ell(\mathfrak{w}) = m$.

By definition,

$$H^0(\bar{Z}^k_{\mathfrak{w}}, \bar{Q}) = H^0\Big(\bar{U}_k(\mathfrak{p}_{i_1})/\bar{U}_k(\mathfrak{b}), H^0(\bar{Z}^k_{\mathfrak{w}'}, \bar{Q})\Big),$$

where $\mathfrak{w}' := (s_{i_2}, \cdots, s_{i_m})$. By Lemma 2.2(b) (with $\mathfrak{g}$ replaced by $\mathfrak{p}_{i_1}$), we have a $\mathfrak{U}_k(\mathfrak{b})$ (in fact a $\mathfrak{U}_k(\mathfrak{p}_{i_1})$)-module homomorphism

$$\begin{aligned}\varphi' : H^0\Big(\bar{U}_k(\mathfrak{p}_{i_1})/\bar{U}_k(\mathfrak{b}), H^0(\bar{Z}^k_{\mathfrak{w}'}, \bar{Q})\Big)^{\mathrm{Fr}} \\ \to H^0\Big(\mathfrak{U}_k(\mathfrak{p}_{i_1})/\mathfrak{U}_k(\mathfrak{b}), H^0(\bar{Z}^k_{\mathfrak{w}'}, \bar{Q})^{\mathrm{Fr}}\Big).\end{aligned}$$

Also, by induction, we have a $\mathfrak{U}_k(\mathfrak{b})$-module homomorphism

$$\mathrm{Fr}^*_{\mathfrak{w}'} : H^0(\bar{Z}^k_{\mathfrak{w}'}, \bar{Q})^{\mathrm{Fr}} \to H^0(\mathcal{Z}^k_{\mathfrak{w}'}, \bar{Q}^{\mathrm{Fr}}),$$

which induces a $\mathfrak{U}_k(\mathfrak{p}_{i_1})$ (in particular $\mathfrak{U}_k(\mathfrak{b})$)-module homomorphism

$$\varphi'' : H^0\Big(\mathfrak{U}_k(\mathfrak{p}_{i_1})/\mathfrak{U}_k(\mathfrak{b}), H^0(\bar{Z}^k_{\mathfrak{w}'}, \bar{Q})^{\mathrm{Fr}}\Big) \to H^0(\mathcal{Z}^k_{\mathfrak{w}}, \bar{Q}^{\mathrm{Fr}}).$$

Now we set $\mathrm{Fr}^*_{\mathfrak{w}}$ as the composition $\varphi := \varphi'' \circ \varphi'$ (which is a $\mathfrak{U}_k(\mathfrak{p}_{i_1})$-module homomorphism)

$$\varphi : H^0(\bar{Z}^k_{\mathfrak{w}}, \bar{Q})^{\mathrm{Fr}} \to H^0(\mathcal{Z}^k_{\mathfrak{w}}, \bar{Q}^{\mathrm{Fr}}).$$

This completes the induction.

Replacing $\bar{Q}$ by $\bar{Q}_i$, we get $\mathfrak{U}_k(\mathfrak{p}_{i_1})$-module homomorphisms

$$H^0(\bar{Z}^k_{\mathfrak{w}}, \bar{Q}_i)^{\mathrm{Fr}} \xrightarrow{\varphi_i} H^0(\mathcal{Z}^k_{\mathfrak{w}}, \bar{Q}^{\mathrm{Fr}}_i) \xrightarrow{\theta^*_i} H^0(\mathcal{Z}^k_{\mathfrak{w}}, Q_i),$$

where θ^*_i is induced from θ_i. The resolutions (1) and (2) give rise to the cochain complexes

$$H^0(\bar{Z}^k_{\mathfrak{w}}, \bar{Q}_0)^{\mathrm{Fr}} \to H^0(\bar{Z}^k_{\mathfrak{w}}, \bar{Q}_1)^{\mathrm{Fr}} \to \cdots, \quad \text{and} \tag{3}$$

$$H^0(\mathcal{Z}^k_{\mathfrak{w}}, Q_0) \to H^0(\mathcal{Z}^k_{\mathfrak{w}}, Q_1) \to \cdots. \tag{4}$$

The maps $\theta^*_i \circ \varphi_i$ give a cochain map from the cochain complex (3) to the cochain complex (4). Taking cohomology, we get the desired map

$$\mathrm{Fr}^*_{\mathfrak{w}} : H^i(\bar{Z}^k_{\mathfrak{w}}, \bar{M})^{\mathrm{Fr}} \to H^i(\mathcal{Z}^k_{\mathfrak{w}}, \bar{M}^{\mathrm{Fr}}). \qquad \square$$

THEOREM 8.4. *For any sequence $\mathfrak{w}$ and any $M \in \mathcal{C}_k(\mathfrak{b})$, there exists a functorial $\bar{U}_k(\mathfrak{b})$-module map*

$$\mathrm{Fr}'^*_{\mathfrak{w}} : H^i(\mathcal{Z}^k_{\mathfrak{w}}, M)^{\mathrm{Fr}'_{\mathfrak{b}}} \to H^i\Big(\bar{Z}^k_{\mathfrak{w}}, M^{\mathrm{Fr}'_{\mathfrak{b}}}\Big).$$

Proof. We first define the map $\mathrm{Fr}'^*_{\mathfrak{w}}$ at the H^0-level. Applying Proposition 3.3 for $\mathfrak{g}$ replaced by $\mathfrak{p}_{i_1}$, we get the $\bar{U}_k(\mathfrak{b})$-module map

$$\begin{aligned}\beta_1 : H^0(\mathcal{Z}^k_{\mathfrak{w}}, M)^{\mathrm{Fr}'_{\mathfrak{b}}} &= H^0\Big(\mathfrak{U}_k(\mathfrak{p}_{i_1})/\mathfrak{U}_k(\mathfrak{b}), H^0(\mathcal{Z}^k_{\mathfrak{w}'}, M)\Big)^{\mathrm{Fr}'_{\mathfrak{b}}} \\ &\to H^0\Big(\bar{U}_k(\mathfrak{p}_{i_1})/\bar{U}_k(\mathfrak{b}), H^0(\mathcal{Z}^k_{\mathfrak{w}'}, M)^{\mathrm{Fr}'_{\mathfrak{b}}}\Big).\end{aligned}$$

By induction on $\ell(\mathfrak{w})$, we have the $\bar{U}_k(\mathfrak{b})$-module map

$$\mathrm{Fr}'^*_{\mathfrak{w}'} : H^0(\mathcal{Z}^k_{\mathfrak{w}'}, M)^{\mathrm{Fr}'_\mathfrak{b}} \to H^0(\bar{Z}^k_{\mathfrak{w}'}, M^{\mathrm{Fr}'_\mathfrak{b}}).$$

Inducing $\mathrm{Fr}'^*_{\mathfrak{w}'}$, we get the $\bar{U}_k(\mathfrak{p}_{i_1})$-module map

$$\beta_2 : H^0\Big(\bar{U}_k(\mathfrak{p}_{i_1})/\bar{U}_k(\mathfrak{b}), H^0(\mathcal{Z}^k_{\mathfrak{w}'}, M)^{\mathrm{Fr}'_\mathfrak{b}}\Big) \to H^0\Big(\bar{U}_k(\mathfrak{p}_{i_1})/\bar{U}_k(\mathfrak{b}), H^0(\bar{Z}^k_{\mathfrak{w}'}, M^{\mathrm{Fr}'_\mathfrak{b}})\Big).$$

Composing $\beta_2 \circ \beta_1$, we get the desired map $\mathrm{Fr}'^*_{\mathfrak{w}}$ at the H^0-level. Now, by a proof parallel to that of the proof of Theorem 3.8, we define $\mathrm{Fr}'^*_{\mathfrak{w}}$ for an arbitrary H^i. □

Corollary 8.5. *For any $\bar{M} \in \bar{\mathcal{C}}_k(\mathfrak{b})$ and sequence $\mathfrak{w}$,*

$$\mathrm{Fr}'^*_{\mathfrak{w}} \circ \mathrm{Fr}^*_{\mathfrak{w}} : H^i(\bar{Z}^k_{\mathfrak{w}}, \bar{M}) \to H^i(\bar{Z}^k_{\mathfrak{w}}, \bar{M})$$

is the identity map.

Proof. From the definition of the maps involved, it is easy to see by induction on $\ell(\mathfrak{w})$ that the corresponding property is true at the H^0-level. Now the validity of the corollary for general i follows by the same argument as that of the proof of Corollary 3.9. □

Definition 8.6. Let $\mathfrak{w} = (s_{i_1}, \cdots, s_{i_m})$ be any sequence of simple reflections. We need a certain generalization of the functor $D_\mathfrak{w}$, still denoted by $D_\mathfrak{w} : \mathcal{C}_k(\mathfrak{b})^{\times m} \to \mathcal{C}_k(\mathfrak{b})$, defined as follows:

$$D_\mathfrak{w}(M_1, \cdots, M_m) = H^0\Big(\mathfrak{U}_k(\mathfrak{p}_{i_1})/\mathfrak{U}_k(\mathfrak{b}), M_1 \otimes D_{\mathfrak{w}'}(M_2, \cdots, M_m)\Big),$$

where $\mathfrak{w}' := (s_{i_2}, \cdots, s_{i_m})$. We similarly define $\bar{D}_\mathfrak{w} : \bar{\mathcal{C}}_k(\mathfrak{b})^{\times m} \to \bar{\mathcal{C}}_k(\mathfrak{b})$.

These functors are again left exact. Denote their right derived functors respectively by $H^i\Big(\mathcal{Z}^k_\mathfrak{w}, M_1 \boxtimes \cdots \boxtimes M_m\Big)$ and $H^i\Big(\bar{Z}^k_\mathfrak{w}, \bar{M}_1 \boxtimes \cdots \boxtimes \bar{M}_m\Big)$, for $M_i \in \mathcal{C}_k(\mathfrak{b})$ and $\bar{M}_i \in \bar{\mathcal{C}}_k(\mathfrak{b})$. These are respectively $\mathfrak{U}_k(\mathfrak{b})$ and $\bar{U}_k(\mathfrak{b})$-modules. If $M_1 = \cdots = M_{m-1} = k$ is the trivial representation, then

$$(1) \qquad \begin{aligned} D_\mathfrak{w}(k, \cdots, k, M_m) &\cong D_\mathfrak{w}(M_m), \qquad \text{and} \\ H^i\Big(\mathcal{Z}^k_\mathfrak{w}, k \boxtimes \cdots \boxtimes k \boxtimes M_m\Big) &\cong H^i(\mathcal{Z}^k_\mathfrak{w}, M_m). \end{aligned}$$

Analogous to the definition of $\mathrm{Fr}^*_\mathfrak{w}$ (cf. Theorem 8.3), we can define the $\mathfrak{U}(\mathfrak{b})$-module map for any $\bar{M}_i \in \bar{\mathcal{C}}_k(\mathfrak{b})$,

$$\mathrm{Fr}^*_\mathfrak{w} : H^i\Big(\bar{Z}^k_\mathfrak{w}, \bar{M}_1 \boxtimes \cdots \boxtimes \bar{M}_m\Big)^{\mathrm{Fr}} \to H^i\Big(\mathcal{Z}^k_\mathfrak{w}, \bar{M}_1^{\mathrm{Fr}} \boxtimes \cdots \boxtimes \bar{M}_m^{\mathrm{Fr}}\Big).$$

Similarly, we can define the $\bar{U}_k(\mathfrak{b})$-module map (cf. Theorem 8.4)

$$\mathrm{Fr}'^*_\mathfrak{w} : H^i\Big(\mathcal{Z}^k_\mathfrak{w}, \bar{M}_1^{\mathrm{Fr}} \boxtimes \cdots \boxtimes \bar{M}_m^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'_\mathfrak{b}} \to H^i\Big(\bar{Z}^k_\mathfrak{w}, \bar{M}_1 \boxtimes \cdots \boxtimes \bar{M}_m\Big).$$

Then

$$\mathrm{Fr}'^*_\mathfrak{w} \circ \mathrm{Fr}^*_\mathfrak{w} = \mathrm{Id} \text{ on } H^i\Big(\bar{Z}^k_\mathfrak{w}, \bar{M}_1 \boxtimes \cdots \boxtimes \bar{M}_m\Big).$$

From now on, to the end of this section, assume that $\ell = p$ is a prime and k is an algebraically closed field of characteristic p which is a $\mathbb{Z}_\xi$ algebra under $\xi \mapsto 1$.

Let G, B be as in Section 6 and, for any $1 \le i \le n$, let $B \subset P_i$ be the minimal parabolic subgroup containing the simple reflection s_i. Recall that the Bott-Samelson-Demazure-Hansen variety $Z_\mathfrak{w}$ is defined as $P_{i_1} \times \cdots \times P_{i_m}/B^{\times m}$ where $B^{\times m}$ acts on $P_{i_1} \times \cdots \times P_{i_m}$ from the right under

$$(p_1, \cdots, p_m) \cdot (b_1, \cdots, b_m) := (p_1 b_1, b_1^{-1} p_2 b_2, \cdots, b_{m-1}^{-1} p_m b_m),$$

for $p_j \in P_{i_j}$ and $b_j \in B$. Then $Z_\mathfrak{w}$ is a smooth projective variety over k. For any $\lambda_1, \cdots, \lambda_m \in X$, the character $e^{\lambda_1} \boxtimes \cdots \boxtimes e^{\lambda_m}$ of $B^{\times m}$ gives rise to the line bundle $\mathcal{L}_\mathfrak{w}(\lambda_1 \boxtimes \cdots \boxtimes \lambda_m)$ on $Z_\mathfrak{w}$.

Consider the embedding $Z_\mathfrak{w} \hookrightarrow (G/B)^{\times m}$ defined by

$$(p_1, \cdots, p_m) \bmod B^{\times m} \mapsto (p_1 B, p_1 p_2 B, \cdots, p_1 \cdots p_m B).$$

Then the line bundle $\mathcal{L}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda_m)$ on $G/B^{\times m}$ restricts to the line bundle $\mathcal{L}_\mathfrak{w}(\lambda_1 \boxtimes \cdots \boxtimes \lambda_m)$ on $Z_\mathfrak{w}$. In particular, if each of $\lambda_1, \cdots, \lambda_m$ is dominant and regular, then $\mathcal{L}_\mathfrak{w}(-\lambda_1 \boxtimes \cdots \boxtimes -\lambda_m)$ is ample on $Z_\mathfrak{w}$.

Define the k-algebra (under the multiplication of sections)

$$K_\mathfrak{w} := \bigoplus_{(\lambda_1, \cdots, \lambda_m) \in (X^+)^{\times m}} H^0\Big(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(-\lambda_1 \boxtimes \cdots \boxtimes -\lambda_m)\Big).$$

Analogous to the map θ of §6.1, by induction on $\ell(\mathfrak{w})$, using the Leray spectral sequence for the fibration $Z_\mathfrak{w} \to P_{i_1}/B$, we get

$$\theta_\mathfrak{w} : H^0\Big(\mathcal{Z}^k_\mathfrak{w}, (\chi^\xi_{-\lambda_1})_k \boxtimes \cdots \boxtimes (\chi^\xi_{-\lambda_m})_k\Big) \simeq H^0\Big(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(-\lambda_1 \boxtimes \cdots \boxtimes -\lambda_m)\Big)^{[1]}.$$

Similarly,

$$\bar\theta_\mathfrak{w} : H^0\Big(\bar Z^k_\mathfrak{w}, (\bar\chi_{-\lambda_1})_k \boxtimes \cdots \boxtimes (\bar\chi_{-\lambda_m})_k\Big) \simeq H^0\Big(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(-\lambda_1 \boxtimes \cdots \boxtimes -\lambda_m)\Big).$$

Under the above identifications, the map $\mathrm{Fr}'^*_\mathfrak{w}$ gives rise to the k-linear map

$$\mathcal{F}'_\mathfrak{w}(\lambda_1, \cdots, \lambda_m) : \Big(\quad H^0\Big(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(-p\lambda_1 \boxtimes \cdots \boxtimes -p\lambda_m)\Big)^{[1]}\Big)^{\mathrm{Fr}'_\mathfrak{b}} \\ \to H^0\Big(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(-\lambda_1 \boxtimes \cdots \boxtimes -\lambda_m)\Big).$$

Combining these, we get the map (as maps of abelian groups; without regard to the $\bar U(\mathfrak{b})$ or k-linear structures) $\mathcal{F}'_\mathfrak{w} : K_\mathfrak{w} \to K_\mathfrak{w}$, where we take $\mathcal{F}'_\mathfrak{w}|_{H^0(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(-\lambda_1 \boxtimes \cdots \boxtimes -\lambda_m))} \equiv 0$ unless p divides each of $\lambda_1, \cdots, \lambda_m$.

By an argument similar to the proofs of Proposition 6.2, Theorem 6.4 and Proposition 7.3, we get the following. It was first proved by Mehta-Ramanathan [MR1] that $Z_\mathfrak{w}$ is Frobenius split.

THEOREM 8.7. *The map $\mathcal{F}'_{\mathfrak{w}} : K_{\mathfrak{w}} \to K_{\mathfrak{w}}$ is a Frobenius-linear graded endomorphism. Moreover, it is a splitting. Further, for any $q \in \mathbb{Z}_+$ and $f \in K_{\mathfrak{w}}$,*

$$\mathcal{F}'_{\mathfrak{w}}(\bar{E}_i^{(pq)} \cdot f) = \bar{E}_i^{(q)} \cdot \mathcal{F}'_{\mathfrak{w}}(f),$$

where the action of $\bar{E}_i^{(m)}$ comes from the canonical action of B on

$$H^0\Big(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(-\lambda_1 \boxtimes \cdots \boxtimes -\lambda_m)\Big).$$

In particular, the splitting $\mathcal{F}'_{\mathfrak{w}}$ is B-canonical.

The splitting $\mathcal{F}'_{\mathfrak{w}}$ induces a B-canonical Frobenius splitting of the variety $Z_{\mathfrak{w}}$ by a method similar to that in Section 6.

Remark 8.8. 1) For any reduced decomposition of the longest element of the Weyl group $w_o = s_{i_1} \cdots s_{i_N}$, consider the sequence $\mathfrak{w}_o = (s_{i_1}, s_{i_2}, \cdots, s_{i_N})$. Then the above splitting of $Z_{\mathfrak{w}_o}$ 'descends' via the map $Z_{\mathfrak{w}_o} \to G/B$, $(p_1, \dots, p_N)$ mod $B^{*N} \mapsto p_1 \cdots p_N$ mod B, to give the splitting of G/B given in Theorem 6.4.

2) For any subsequence $\mathfrak{v}$ of $\mathfrak{w}$, the subvariety $Z_{\mathfrak{v}} \subset Z_{\mathfrak{w}}$ is compatibly split under the splitting of $Z_{\mathfrak{w}}$ given in the above theorem.

9. Extension of results to the parabolic case

The aim of this section is to extend various results obtained in the earlier sections for the Borel case to an arbitrary parabolic case. We formulate the extensions but omit the proofs as they are similar to the proofs given earlier (of the corresponding results in the Borel case).

Let ℓ be as in Sections 1–4 (i.e., it is an odd integer > 1 assumed to be coprime to 3 if G_2 is a factor of $\mathfrak{g}$).

For a subset $I \subset \{1, \cdots, n\}$, let $\bar{U}(\mathfrak{p}_I)$ be the parabolic subalgebra of $\bar{U}(\mathfrak{g})$ generated by $\bar{U}(\mathfrak{b})$ and $\{\bar{F}_i^{(m)};\ i \in I \text{ and } m \geq 0\}$. Similarly, let $\mathfrak{U}(\mathfrak{p}_I)$ be the parabolic subalgebra of $\mathfrak{U}(\mathfrak{g})$ generated by $\mathfrak{U}(\mathfrak{b})$ and $\{F_i^{(m)};\ i \in I \text{ and } m \geq 0\}$. Then Δ and S keep $\mathfrak{U}(\mathfrak{p}_I)$ stable.

For any subset I as above and $\mathfrak{U}(\mathfrak{p}_I)$-module M, we can analogously define (cf. Definition 1.3)

$$F_{\mathfrak{p}_I}(M) := \{v \in F_{\mathfrak{b}}(M) : F_i^{(m)} v = 0, \text{ for all } m \geq m(v) \text{ and } i \in I\}$$

and thus the category $\mathcal{C}(\mathfrak{p}_I)$. Similarly, we can define the category $\bar{\mathcal{C}}(\mathfrak{p}_I)$. Then Proposition 1.5 is true with $\mathcal{C}(\mathfrak{b})$ (resp. $\mathfrak{U}(\mathfrak{b})$) replaced by $\mathcal{C}(\mathfrak{p}_I)$ (resp. $\mathfrak{U}(\mathfrak{p}_I)$) and $\bar{\mathcal{C}}(\mathfrak{b})$ (resp. $\bar{U}(\mathfrak{b})$) replaced by $\bar{\mathcal{C}}(\mathfrak{p}_I)$ (resp. $\bar{U}(\mathfrak{p}_I)$). Hence, we can define the cohomology $H^i(\mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{p}_I), M)$, for $M \in \mathcal{C}(\mathfrak{p}_I)$. Similarly, we can define $H^i(\bar{U}(\mathfrak{g})/\bar{U}(\mathfrak{p}_I), \bar{M})$, for $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{p}_I)$. We abbreviate $H^i(\mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{p}_I), M)$ to $H^i(\mathfrak{X}_I, M)$ and similarly $H^i(\bar{U}(\mathfrak{g})/\bar{U}(\mathfrak{p}_I), \bar{M})$ to $H^i(\bar{X}_I, \bar{M})$.

For any $M \in \mathcal{C}(\mathfrak{p}_I)$, there is the canonical map $\pi_I : H^i(\mathfrak{X}_I, M) \to H^i(\mathfrak{X}, M)$ and similarly for $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{p}_I)$ the map $\bar{\pi}_I : H^i(\bar{X}_I, \bar{M}) \to H^i(\bar{X}, \bar{M})$.

Analogous to Theorems (2.3) and (3.8), we have the following:

Theorem 9.1. *For any $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{p}_I)$, there exists a functorial $\mathfrak{U}(\mathfrak{g})$-module map*

$$\mathrm{Fr}_I^* : H^i(\bar{X}_I, \bar{M})^{\mathrm{Fr}} \longrightarrow H^i\Big(\mathfrak{X}_I, \bar{M}^{\mathrm{Fr}}\Big)$$

compatible with π_I in the sense that the following diagram is commutative:

$$\begin{array}{ccc} H^i\Big(\bar{X}_I, \bar{M}\Big)^{\mathrm{Fr}} & \xrightarrow{\mathrm{Fr}_I^*} & H^i\Big(\mathfrak{X}_I, \bar{M}^{\mathrm{Fr}}\Big) \\ \Big\downarrow{\scriptstyle \bar{\pi}_I} & & \Big\downarrow{\scriptstyle \pi_I} \\ H^i\Big(\bar{X}, \bar{M}\Big)^{\mathrm{Fr}} & \xrightarrow[\mathrm{Fr}^*]{} & H^i\Big(\mathfrak{X}, \bar{M}^{\mathrm{Fr}}\Big). \end{array}$$

Theorem 9.2. *For any $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{p}_I)$, there exists a functorial $\bar{U}(\mathfrak{b}^-)$-module map*

$$\mathrm{Fr}_I'^* : H^i\Big(\mathfrak{X}_I, \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \to H^i\Big(\bar{X}_I, \bar{M}\Big)$$

such that $\mathrm{Fr}_I'^* \circ \mathrm{Fr}_I^* = \mathrm{Id}$.

Further,

$$\bar{E}_i^{(m)} \cdot (\mathrm{Fr}_I'^* f) = \mathrm{Fr}_I'^*(E_i^{(m\ell)} \cdot f),$$

for any $f \in H^i\Big(\mathfrak{X}_I, \bar{M}^{\mathrm{Fr}}\Big)$.

Moreover, $\mathrm{Fr}_I'^*$ *is compatible with* π_I.

Let $\gamma_I := -2(\ell-1)\rho_I$, where $\rho_I := \sum_{i \notin I} \omega_i$ and ω_i is the i^{th} fundamental weight defined by $\omega_i(h_j) = \delta_{i,j}$. Observe that the $\mathfrak{U}(\mathfrak{b})$-module $\chi^{\xi}_{\gamma_I}$ is, in fact, a module for $\mathfrak{U}(\mathfrak{p}_I)$. Let $F_o^I := F_{\beta_{i_m}}^{(\ell-1)} \cdots F_{\beta_{i_1}}^{(\ell-1)}$, where $i_1 < \cdots < i_m$, $\{\beta_{i_1}, \cdots, \beta_{i_m}\} = \Delta_+ \backslash \Delta_+(I)$, and $\Delta_+(I) := \Delta_+ \cap \sum_{i \in I} \mathbb{Z}_+ \alpha_i$ (cf. §4.1). Then observe that F_o^I is of weight γ_I. Decompose $V_\xi(-\gamma_I) = S(F_o^I)\,\mathbb{Z}_\xi v_+ \oplus M$, where $\mathbb{Z}_\xi v_+$ is the highest weight space and M is a weight subspace of $V_\xi(-\gamma_I)$. Let $\widehat{\sigma_o^I} \in V_\xi(-\gamma_I)^*$ be defined by $\widehat{\sigma_o^I}(S(F_o^I)v_+) = 1$ and $\widehat{\sigma^I_{o}}_{|M} \equiv 0$.

Now, replacing F_o by F_o^I in Section 4, we get the following parabolic analogue of Theorem 4.7 and Proposition 4.1.1.

Theorem 9.3. *For any $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{p}_I)$, there exists a functorial $\bar{U}(\mathfrak{b}^-)$-module map*

$$\mathrm{Fr}_{\gamma_I}'^* : H^i\Big(\mathfrak{X}_I, \chi^{\xi}_{\gamma_I} \otimes \bar{M}^{\mathrm{Fr}}\Big)^{\mathrm{Fr}'_{\mathfrak{b}^-}} \to H^i(\bar{X}_I, \bar{M}).$$

Further,

$$\bar{E}_i^{(m)} \cdot (\mathrm{Fr}_{\gamma_I}'^* f) = \mathrm{Fr}_{\gamma_I}'^*(E_i^{(m\ell)} \cdot f),$$

for any $f \in H^i\Big(\mathfrak{X}_I, \chi^{\xi}_{\gamma_I} \otimes \bar{M}^{\mathrm{Fr}}\Big)$.

Moreover, the composite $\mathrm{Fr}'^*_{\gamma_I} \circ m_{\sigma_o^I} \circ \mathrm{Fr}^*_I$ *is the identity map, where* $\sigma_o^I \in H^0\Big(\mathfrak{X}_I, \chi^{\xi}_{\gamma_I}\Big)$ *is given as* $\beta(\widehat{\sigma_o^I})$ *(cf. Prop.* 4.11), *and* $m_{\sigma_o^I} : H^i\Big(\mathfrak{X}_I, \bar{M}^{\mathrm{Fr}}\Big) \to H^i\Big(\mathfrak{X}_I, \chi^{\xi}_{\gamma_I} \otimes \bar{M}^{\mathrm{Fr}}\Big)$ *is defined similarly to Lemma* 4.9.

Similar to Proposition 7.2, we have the following:

PROPOSITION 9.4. *For any subsets* $I, I' \subset \{1, \cdots, n\}$, $\bar{M} \in \bar{\mathcal{C}}(\mathfrak{p}_I)$, $\bar{N} \in \bar{\mathcal{C}}(\mathfrak{p}_{I'})$ *and* $i, j \in \mathbb{Z}_+$, *there exists a functorial* $\bar{U}(\mathfrak{b})$*-module map*

$$\mathrm{Fr}'^*_{\Delta}(I, I') : H^i\Big(\mathfrak{X}_I, \bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}_{I'}, \bar{N}^{\mathrm{Fr}})\Big)^{\mathrm{Fr}'_{\mathfrak{b}}} \to H^i\Big(\bar{X}_I, \bar{M} \otimes H^j(\bar{X}_{I'}, \bar{N})\Big)$$

which is a splitting of the functorial $\mathfrak{U}(\mathfrak{g})$*-module map*

$$\mathrm{Fr}^*_{\Delta}(I, I') : H^i\Big(\bar{X}_I, \bar{M} \otimes H^j(\bar{X}_{I'}, \bar{N})\Big)^{\mathrm{Fr}} \to H^i\Big(\mathfrak{X}_I, \bar{M}^{\mathrm{Fr}} \otimes H^j(\mathfrak{X}_{I'}, \bar{N}^{\mathrm{Fr}})\Big).$$

Moreover, the analogue of (1) *of Proposition* 7.2 *holds.*

From now on, until the end of this section, we take $\ell = p$ to be a prime and k an algebraically closed field of characteristic p, which is a $\mathbb{Z}_\xi$-algebra under $\mathbb{Z}_\xi \to k$, $\xi \mapsto 1$. Let G, B and the Schubert varieties $X(w) \subset G/B$ be as in Section 6. For any subset $I \subset \{1, \cdots, n\}$, let $B \subset P = P_I$ be the parabolic subgroup containing the simple reflections $\{s_i\}_{i \in I}$. (In particular, when I is the singleton $\{i\}$, P_I is the minimal parabolic subgroup P_i.) Let X_P be the character group of P. Then $X_P \cong \{\lambda \in X : \lambda(h_i) = 0, \text{ for all } i \in I\}$. We set $X_P^+ = X_P \cap X^+$. For any $w \in W$, let

$$X(w)_P := \overline{BwP/P} \subset G/P$$

be the Schubert subvariety. Also define the opposite Schubert variety

$$X(w)^-_P := \overline{B^- wP/P} \subset G/P.$$

For any $\lambda \in X_P$, the associated homogeneous line bundle on G/P is denoted by $\mathcal{L}_P(\lambda)$. When there is no cause for confusion, we denote its restriction to $X(w)_P$ again by $\mathcal{L}_P(\lambda)$.

Definition 9.5. Analogous to the isomorphisms θ and $\bar{\theta}$ of §6.1, we have the isomorphisms (for any $\lambda \in X_P$)

$$\begin{aligned} \theta_I : H^0(\mathfrak{U}_k(\mathfrak{g})/\mathfrak{U}_k(\mathfrak{p}_I), (\chi^{\xi}_{\lambda})_k) &\simeq H^0(G/P, \mathcal{L}_P(\lambda))^{[1]} \text{ and} \\ \bar{\theta}_I : H^0(\bar{U}_k(\mathfrak{g})/\bar{U}_k(\mathfrak{p}_I), (\bar{\chi}_{\lambda})_k) &\simeq H^0(G/P, \mathcal{L}_P(\lambda)). \end{aligned}$$

Define the X_P^+-graded algebra

$$K_I := \bigoplus_{\mu \in X_P^+} H^0(G/P, \mathcal{L}_P(-\mu)),$$

and let $\mathcal{F}'_I : K_I \to K_I$ be the map defined by $\mathcal{F}'_{I|H^0(G/P,\mathcal{L}_P(-\mu))} \equiv 0$ if $\mu \notin pX_P^+$ and $\mathcal{F}'_I : H^0(G/P, \mathcal{L}_P(-p\mu)) \to H^0(G/P, \mathcal{L}_P(-\mu))$ is the splitting map $\mathrm{Fr}_I'^*$ (as maps of abelian groups) under the identifications θ_I and $\bar{\theta}_I$.

Then analogous to Proposition 6.2, Theorems 6.4, 6.5 and 6.7, we obtain the following.

THEOREM 9.6. *The map $\mathcal{F}'_I$ is a Frobenius-linear graded endomorphism of K_I. Moreover, it is a B-canonical splitting.*

This induces a B-canonical Frobenius splitting of the flag variety G/P which compatibly splits any subscheme $Z_P \subset G/P$ obtained from $\{X(w)_P, X(w)_P^-\}_{w\in W}$ by repeatedly taking scheme theoretic unions, intersections and irreducible components.

In fact, G/P is Frobenius $\mathcal{D}_I$-split, where $\mathcal{D}_I$ is the line bundle $\mathcal{L}_P(\gamma_I)$ together with the section $\phi_I := \theta_I(\sigma_0^I)$.

Remark 9.7. Since G/P is Frobenius $\mathcal{L}_P(\gamma_I)$-split, in particular, it is Frobenius $\mathcal{L}_P(-(p-1)\rho_I)$-split with respect to an appropriate section of $H^0\Big(G/P, \mathcal{L}_P(-(p-1)\rho_I)\Big)$. Further, choosing the section appropriately, we can easily show that each Schubert variety $X(w)_P$ is compatibly $\mathcal{L}_P(-(p-1)\rho_I)$-split (under the splitting of G/P obtained above). This was originally proved by Ramanathan (cf. [R2, Th. 3.5]).

For any $I, I' \subset \{1, \cdots, n\}$, define the k-algebra

$$K_\Delta(I, I') := \bigoplus_{\lambda\in X_P,\, \mu\in X_{P'}} H^0(G/P, \mathcal{L}_P(-\lambda)) \otimes H^0(G/P', \mathcal{L}_{P'}(-\mu)),$$

where $P := P_I$ and $P' := P_{I'}$.

Analogous to Proposition 7.3, using the maps $\mathrm{Fr}_\Delta'^*(I, I')$ of §9.4, we obtain the Frobenius-linear graded endomorphism

$$\mathcal{F}'_\Delta(I, I') : K_\Delta(I, I') \to K_\Delta(I, I').$$

Moreover, it is a B-canonical splitting under the diagonal action of B on $K_\Delta(I, I')$.

Define the G-Schubert variety $Y(w)_{P,P'}$ as the image of $Y(w)$ under the canonical projection map $G/B \times G/B \to G/P \times G/P'$. Analogous to Theorem 7.5, we obtain the following:

THEOREM 9.8. *The endomorphism $\mathcal{F}'_\Delta(I, I')$ induces a B-canonical Frobenius splitting of $G/P \times G/P'$.*

Further, any subscheme $Z \subset G/P \times G/P'$ obtained from the G-Schubert varieties $\{Y(w)_{P,P'}\}_{w\in W}$ by repeatedly taking unions, intersections and irreducible components is compatibly split.

Appendix: Applications

We follow the notation and assumptions of Section 9 (just above Definition 9.5). In particular, $\ell = p$ is a prime and k is an algebraically closed field of characteristic p.

For completeness, we collect some important (and standard) consequences of Frobenius splitting of the flag varieties and their Schubert subvarieties (cf. [MR1], [RR], [R1], [R2]; and also [A], [Jo], [S]).

THEOREM A.1. *For any $w \in W$ and $\lambda \in X_P^+$,*

(a) $H^i(X(w)_P, \mathcal{L}_P(-\lambda)) = 0$, *for all $i > 0$.*

(b) *The restriction map $H^0(G/P, \mathcal{L}_P(-\lambda)) \to H^0(X(w)_P, \mathcal{L}_P(-\lambda))$ is surjective.*

Proof. Since G/P is Frobenius $\mathcal{L}_P(-(p-1)\rho_I)$-split and $X(w)_P$ is compatibly $\mathcal{L}_P(-(p-1)\rho_I)$-split by Remark 9.7, the theorem follows immediately from the standard properties of Frobenius splitting (cf. [R2, Prop. 1.13(ii)]). □

See [RR, Th. 3] for the following result.

THEOREM A.2. *Any Schubert variety $X(w)_P \subset G/P$ is normal.*

Moreover, for any homogeneous ample line bundle $\mathcal{L} = \mathcal{L}_P(-\lambda)$ on G/P, $X(w)_P$ is projectively normal in the projective embedding given by $\mathcal{L}$.

Proof. To prove the normality, we can of course assume that $P = B$. We prove the normality of $X(w)$ by induction on the length $\ell(w)$. If $\ell(w) = 0$, there is nothing to prove. So take $\ell(w) > 0$ and write $w = w's_i$ for a simple reflection s_i such that $w' < w$. Under the canonical map $\pi : G/B \to G/P_i$, $X(w)$ and $X(w')$ have the same image $X(w)_{P_i}$. Moreover, $\pi_{|_{X(w)}} : X(w) \to X(w)_{P_i}$ is a $\mathbb{P}^1$-fibration and $\pi_{|_{X(w')}} : X(w') \to X(w)_{P_i}$ is a birational map. By induction, $X(w')$ is normal. For any $\lambda \in X_{P_i}^+$, we have the commutative diagram:

$$\begin{array}{ccc} H^0\Big(G/P_i, \mathcal{L}_{P_i}(-\lambda)\Big) & \twoheadrightarrow & H^0\Big(X(w)_{P_i}, \mathcal{L}_{P_i}(-\lambda)\Big) \\ \downarrow\wr & & \downarrow\pi^* \\ H^0\Big(G/B, \mathcal{L}(-\lambda)\Big) & \twoheadrightarrow & H^0\Big(X(w'), \mathcal{L}(-\lambda)\Big), \end{array}$$

where the vertical maps are induced by π and the horizontal maps are induced by inclusions. The horizontal maps are surjective by Theorem A.1 and clearly the left vertical map is an isomorphism, hence the right vertical map π^* is surjective. Since π^* is surjective for all $\lambda \in X_{P_i}^+$ (in particular, for all large enough positive powers of an ample line bundle $\mathcal{L}$ on $X(w)_{P_i}$), and $H^1(X(w)_{P_i}, \mathcal{L}_{P_i}(-\lambda)) = 0$ (by Theorem A.1), we get $(\pi_{|_{X(w')}})_*\mathcal{O}_{X(w')} = \mathcal{O}_{X(w)_{P_i}}$. But since $X(w')$ is normal, so is $X(w)_{P_i}$ and hence $X(w)$ is normal.

Now we come to the projective normality: It suffices to show that the multiplication map

$$(1) \qquad H^0\Big(X(w)_P, \mathcal{L}^{\otimes m}\Big) \otimes H^0\Big(X(w)_P, \mathcal{L}^{\otimes n}\Big) \longrightarrow H^0\Big(X(w)_P, \mathcal{L}^{\otimes(n+m)}\Big)$$

is surjective for all $m, n \geq 1$ (cf. [H, Chap. II, Ex. 5.14(d)]).

By the compatible Frobenius splitting of the diagonal $G/P \hookrightarrow G/P \times G/P$ (cf. Th. 9.8), we get that

$$(2) \qquad H^0\Big(G/P, \mathcal{L}^{\otimes m}\Big) \otimes H^0\Big(G/P, \mathcal{L}^{\otimes n}\Big) \twoheadrightarrow H^0\Big(G/P, \mathcal{L}^{\otimes(n+m)}\Big)$$

is surjective. Now (1) follows from (2) by Theorem A.1(b). $\square$

For any sequence of simple reflections $\mathfrak{w} = (s_{i_1}, \cdots, s_{i_m})$, consider the B-equivariant morphism $\psi_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$ given by $\psi_{\mathfrak{w}}\Big((p_1, \cdots, p_m) \bmod B^{\times m}\Big) = p_1 \cdots p_m B$. For any $\lambda \in X$, let $\mathcal{L}_{\mathfrak{w}}(\lambda)$ be the pull-back line bundle $\psi^*_{\mathfrak{w}}(\mathcal{L}(\lambda))$ on $Z_{\mathfrak{w}}$. For $\mathfrak{w}$ as above, let $\theta(\mathfrak{w}) := s_{i_1} \cdots s_{i_m} \in W$. A sequence $\mathfrak{w}$ is called *reduced* if the above decomposition of $\theta(\mathfrak{w})$ is reduced. If $\mathfrak{w}$ is reduced, Image $\psi_{\mathfrak{w}} = X(\theta(\mathfrak{w}))$ and $\psi_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X(\theta(\mathfrak{w}))$ is birational. In the notation of §8.6, $\mathcal{L}_{\mathfrak{w}}(\lambda) \simeq \mathcal{L}_{\mathfrak{w}}(0 \boxtimes \cdots \boxtimes 0 \boxtimes \lambda)$.

LEMMA A.3. *For any $\lambda \in X$ and any reduced $\mathfrak{w}$,*

$$(1) \qquad H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(-\lambda)) \simeq H^0(X(\theta(\mathfrak{w})), \mathcal{L}(-\lambda)).$$

Moreover, for any $\lambda \in X^+$,

$$(2) \qquad H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(-\lambda)) = 0, \qquad \text{for all } i > 0$$

In particular, by Kempf's lemma (cf., e.g., [D, §5, *Prop.* 2]), $R^i\psi_{\mathfrak{w}*}(\mathcal{O}_{Z_{\mathfrak{w}}}) = 0$ *for all $i > 0$.*

Proof. Since $X(\theta(\mathfrak{w}))$ is normal and $\psi_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X(\theta(\mathfrak{w}))$ is birational, (1) follows.

Consider the fibration $\eta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to P_{i_1}/B$ with fibre $Z_{\mathfrak{w}'}$, where $\mathfrak{w}' := (s_{i_2}, \cdots, s_{i_m})$. Assume, by induction on m, that $H^i(Z_{\mathfrak{w}'}, \mathcal{L}_{\mathfrak{w}'}(-\lambda)) = 0$ for all $i > 0$. Hence, by the degenerate Leray spectral sequence, we get:

$$(3) \qquad H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(-\lambda)) \simeq H^i\Big(P_{i_1}/B, \mathcal{L}(H^0(Z_{\mathfrak{w}'}, \mathcal{L}_{\mathfrak{w}'}(-\lambda)))\Big),$$

where, for a B-module M, $\mathcal{L}(M)$ denotes the associated homogeneous vector bundle.

Now, by Theorem A.1(b), we have the surjective map

$$(4) \qquad H^0(G/B, \mathcal{L}(-\lambda)) \twoheadrightarrow H^0(X(w'), \mathcal{L}(-\lambda)) \simeq H^0(Z_{\mathfrak{w}'}, \mathcal{L}_{\mathfrak{w}'}(-\lambda))$$

with kernel, say K, where $w' := \theta(\mathfrak{w}')$. From the long exact cohomology sequence associated to (4), we get

(5) $$H^1\Big(P_{i_1}/B, \mathcal{L}(H^0(G/B, \mathcal{L}(-\lambda)))\Big) \twoheadrightarrow H^1\Big(P_{i_1}/B, \mathcal{L}(H^0(Z_{\mathfrak{w}'}, \mathcal{L}_{\mathfrak{w}'}(-\lambda)))\Big),$$

since $H^2(P_{i_1}/B, \mathcal{L}(K)) = 0$ from the dimensional consideration. But since $H^0(G/B, \mathcal{L}(-\lambda))$ is a G-module, $H^1\Big(P_{i_1}/B,\ \mathcal{L}(H^0(G/B,\ \mathcal{L}(-\lambda)))\Big) = 0$. So, by (3) and (5), we get

$$H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(-\lambda)) = 0$$

for all $i > 0$. This proves (2). □

THEOREM A.4. *Any Schubert variety $X(w)_P \subset G/P$ is Cohen-Macaulay. Moreover, for any homogeneous ample line bundle $\mathcal{L} = \mathcal{L}_P(-\lambda)$ on G/P, $X(w)_P$ is projectively Cohen-Macaulay in the projective embedding given by $\mathcal{L}$.*

Proof. We first prove that $X(w)_P$ is Cohen-Macaulay. We can clearly assume that $P = B$. By the standard characterization of Cohen-Macaulay schemes (cf. [H, Chap. III, Th. 7.6 and its proof]), it suffices to show that

(1) $$H^i(X(w), \mathcal{L}(\lambda)) = 0,$$

for all $i < \ell(w)$ and all dominant regular λ.

Take a reduced sequence $\mathfrak{w}$ with $\theta(\mathfrak{w}) = w$. Then, by Lemma A.3,

(2) $$H^i(X(w), \mathcal{L}(\lambda)) \simeq H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)).$$

Assume, by induction, that $H^i(Z_{\mathfrak{w}'}, \mathcal{L}_{\mathfrak{w}'}(\lambda)) = 0$ for all $i < m-1$, where $m := \ell(w)$. Then, by the Leray spectral sequence for the fibration $Z_{\mathfrak{w}} \to P_{i_1}/B$,

(3) $$H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = 0 \ \text{ unless } i = m-1 \ \text{ or } m.$$

Now, by Serre duality,

(4) $$H^{m-1}(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \simeq H^1(Z_{\mathfrak{w}}, K_{Z_{\mathfrak{w}}} \otimes \mathcal{L}_{\mathfrak{w}}(-\lambda))^*,$$

where $K_{Z_{\mathfrak{w}}}$ is the canonical bundle of $Z_{\mathfrak{w}}$. By [R1, Prop. 2],

$$K_{Z_{\mathfrak{w}}} \simeq \mathcal{O}_{Z_{\mathfrak{w}}}[-\partial Z_{\mathfrak{w}}] \otimes \mathcal{L}_{\mathfrak{w}}(\rho),$$

where ρ is the half sum of positive roots, $\partial Z_{\mathfrak{w}} := \psi_{\mathfrak{w}}^{-1}(\partial X(w))$, and $\partial X(w) := X(w)\backslash(BwB/B)$.

From the sheaf exact sequence:

$$0 \to \mathcal{O}_{Z_{\mathfrak{w}}}[-\partial Z_{\mathfrak{w}}] \to \mathcal{O}_{Z_{\mathfrak{w}}} \to \mathcal{O}_{\partial Z_{\mathfrak{w}}} \to 0$$

tensored with $\mathcal{L}_{\mathfrak{w}}(\rho - \lambda)$, we get the exact sequence:

(5) $$H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\rho-\lambda)) \xrightarrow{r} H^0\Big(\partial Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\rho-\lambda)_{|\partial Z_{\mathfrak{w}}}\Big) \to H^1\Big(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}[-\partial Z_{\mathfrak{w}}] \otimes \mathcal{L}_{\mathfrak{w}}(\rho-\lambda)\Big) \to H^1(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\rho-\lambda)) = 0.$$

(By Lemma A.3, the last term is 0.) We now prove that the restriction map r is surjective:

By the following lemma (and Theorem 8.7, Remark 8.8(2) and Theorem 6.7), the map $\psi'_{\mathfrak{w}} : \partial Z_{\mathfrak{w}} \to \partial X(w)$, gotten by restricting $\psi_{\mathfrak{w}}$, satisfies

(6) $$(\psi'_{\mathfrak{w}})_* \mathcal{O}_{\partial Z_{\mathfrak{w}}} = \mathcal{O}_{\partial X(w)}.$$

(Observe that $\psi'_{\mathfrak{w}}$ has connected fibres since $\psi_{\mathfrak{w}}$ has connected fibres by Zariski's main theorem, as $X(w)$ is normal by Theorem A.2.)

From (6) we get (for any $\mu \in X$)

$$H^0\Big(\partial Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\mu)_{|\partial Z_{\mathfrak{w}}}\Big) \simeq H^0\Big(\partial X(w), \mathcal{L}(\mu)_{|\partial X(w)}\Big).$$

Now the surjectivity of r follows from (1) of Lemma A.3, since G/B is Frobenius $\mathcal{L}(-(p-1)\rho)$-split and $\partial X(w)$ is compatibly $\mathcal{L}(-(p-1)\rho)$-split (cf. Remark 9.7). Thus, from the exact sequence (5), we get

$$H^1\Big(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}[-\partial Z_{\mathfrak{w}}] \otimes \mathcal{L}_{\mathfrak{w}}(\rho - \lambda)\Big) = 0.$$

So, from (3) and (4), we get

$$H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = 0 \ \text{ unless } i = m.$$

By (2), this proves (1) and hence $X(w)$ is Cohen-Macaulay.

To prove that $X(w)_P$ is projectively Cohen-Macaulay, it suffices to show (in view of Theorem A.2) that

(7) $$H^i(X(w)_P, \mathcal{L}^n) = 0 \ \text{ for all } 0 < i < \dim X(w)_P \text{ and all } n \in \mathbb{Z}.$$

Since $X(w)_P$ is Cohen-Macaulay, we get (7) for all $0 \le i < \dim X(w)_P$ and all $n < 0$ (by [H, Chap. III, Th. 7.6(b)]). The vanishing (7) for $n \ge 0$ follows from Theorem A.1. (a) This proves the theorem. $\square$

We recall the following simple lemma due to Mehta-Srinivas [MS, Lemma 2].

Lemma A.5. *Let $\pi : X \to Y$ be a proper and surjective morphism between Frobenius split schemes such that all its fibres are connected. Assume further that for each irreducible component Y' of Y there exists a component X' of X such that $\pi_{|X'} : X' \to Y'$ is birational. Then $\pi_* \mathcal{O}_X = \mathcal{O}_Y$.*

Remark A.6. The above proof (of Theorem A.4) is a minor simplification of the proof given in [MS].

Corollary A.7 (of Theorem A.4). *For any reduced $\mathfrak{w}$, the resolution $\psi_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X(w)$ is rational, where $w := \theta(\mathfrak{w})$.*

Proof. In view of Lemma A.3, it suffices to show that

(1) $$R^q \psi_{\mathfrak{w}*}(K_{Z_{\mathfrak{w}}}) = 0 \ \text{ for all } 0 < q.$$

Fix an ample line bundle $\mathcal{L}(-\lambda)$ on G/B. To prove (1), it suffices to show that

$$H^0\Big(X(w), R^q\psi_{\mathfrak{w}*}(K_{Z_{\mathfrak{w}}} \otimes \mathcal{L}_{\mathfrak{w}}(-n\lambda))\Big) = 0 \text{ for all } n \gg 0. \tag{2}$$

We choose n_o large enough so that for all $n \geq n_o$ and $q \geq 0$

$$H^p\Big(X(w), R^q\psi_{\mathfrak{w}*}(K_{Z_{\mathfrak{w}}} \otimes \mathcal{L}_{\mathfrak{w}}(-n\lambda))\Big) = 0 \text{ for all } p > 0,$$

(cf. [H, Chap. III, Prop. 5.3]). Then, by the degenerate Leray spectral sequence for $\psi_{\mathfrak{w}}$, we get (for all $n \geq n_o$)

$$H^q\Big(Z_{\mathfrak{w}}, K_{Z_{\mathfrak{w}}} \otimes \mathcal{L}_{\mathfrak{w}}(-n\lambda)\Big) \simeq H^0\Big(X(w), R^q\psi_{\mathfrak{w}*}(K_{Z_{\mathfrak{w}}} \otimes \mathcal{L}_{\mathfrak{w}}(-n\lambda))\Big). \tag{3}$$

By the Serre duality,

$$\begin{aligned} H^q\Big(Z_{\mathfrak{w}}, K_{Z_{\mathfrak{w}}} \otimes \mathcal{L}_{\mathfrak{w}}(-n\lambda)\Big) &\simeq H^{\ell(w)-q}\Big(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(n\lambda)\Big)^* \\ &\simeq H^{\ell(w)-q}\Big(X(w), \mathcal{L}(n\lambda)\Big)^*, \text{ by Lemma A.3} \\ &= 0, \text{ by (1) of Theorem A.4.} \end{aligned}$$

Combining this with (3), we get (2), proving the corollary. □

Remark A.8(a). It is possible that the restriction on ℓ in this paper (i.e. $\ell > 1$ is an odd integer and coprime to 3 if G_2 is a factor of $\mathfrak{g}$) can be removed by using the results of Kaneda [Kan] and Andersen-Paradowski [AP] (also see [Li] and [KL, Remark 2]).

(b) It is natural to ask if the results of this paper can be extended to the symmetrizable Kac-Moody algebras.

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