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Frobenius splitting in characteristic zero and the quantum Frobenius map

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Abstract

The aim of the article is to show that Lusztig's Frobenius map (for quantum groups at roots of unity) can be, after dualizing, viewed as a characteristic zero lift of the geometric Frobenius splitting of G/B (in char $p > 0$) introduced by Mehta and Ramanathan. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

In the representation theory of a semisimple algebraic group G, the Schubert varieties $X(w)$, w an element of the Weyl group W of G, play a prominent rôle. An important breakthrough in the analysis of the geometry of these subvarieties of the flag variety G/B was the introduction by Mehta–Ramanathan of the notion of a Frobenius split variety and compatibly split subvarieties (for varieties defined over a field of char. $p > 0$). They proved [13] that G/B (more generally any G/P for a parabolic subgroup P) is a Frobenius split variety such that all of the Schubert subvarieties are compatibly split, in particular, one immediately obtains the Kodaira–Kempf

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vanishing theorem. Using this tool, it was shown for example that Schubert varieties are normal, Cohen–Macaulay and have rational singularities. Moreover, they are projectively normal, projectively Cohen–Macaulay, and are defined by quadratic relations in any embedding given by an ample homogeneous line bundle on G/B (cf. [14–16]). Of course, as is well known, the normality of Schubert varieties is equivalent to the validity of the Demazure's character formula. Further, the Frobenius splitting was used by Mathieu to give a uniform proof that the category of finite-dimensional G-representations (over char. $p>0$) admitting a good filtration is stable under tensor product and more generally under the restriction to the semisimple part of a Levi subgroup (cf. [2,11]).

Earlier, a different way to analyze the geometry of Schubert varieties was suggested by Seshadri and his school. They proposed to construct a standard monomial theory for the homogeneous coordinate ring of an embedding $G/P \hookrightarrow \mathbb{P}(V(\omega))$ given by the orbit of a highest weight vector in a fundamental representation $V(\omega)$ (see for example [4,5] for comments on the development). In this approach, the extremal weight vectors play a prominent role. Using the quantum Frobenius map defined by Lusztig, the second author defined the 'ℓth root' of a product of extremal weight vectors in the quantum Demazure module $(V_v(\lambda)_w)^*$ at an ℓth root of unity v. It turned out that this method presented the perfect tool to develop a standard monomial theory for arbitrary embeddings $X(w) \hookrightarrow \mathbb{P}(V(\lambda))$ of Schubert varieties, avoiding all case by case considerations. Many of the results proved by Frobenius splitting methods follow then also naturally by standard monomial theory, see [4,6,7].

The aim of this article is to systematically begin unifying these two approaches. For the Borel subgroup B of G, let $\mathfrak{b}$ be its Lie algebra over the complex numbers $\mathbb{C}$ and $\mathfrak{b}^-$ be the opposite Borel subalgebra. We first establish a duality between the algebra $U_{\mathbb{Z}}(\mathfrak{b}^-)$ (resp. its quantum analog $U_v(\mathfrak{b}^-)$), and the direct sum of the dual modules of all Weyl modules $\bigoplus_{\lambda\in P^+} V_{\mathbb{Z}}(\lambda)^*$ (resp. its quantum analog $\bigoplus_{\lambda\in P^+} V_v(\lambda)^*$) (cf. Propositions 1 and 2), where v as earlier is an ℓth root of unity, $U_{\mathbb{Z}}(\mathfrak{b}^-)$ is the Kostant's $\mathbb{Z}$-lattice of the enveloping algebra $U(\mathfrak{b}^-)$, $U_v(\mathfrak{b}^-)$ is the Lusztig's $\tilde{\mathbb{Z}}$-lattice of the quantized algebra $U_q(\mathfrak{b}^-)$ and $\tilde{\mathbb{Z}}$ is the ring obtained from $\mathbb{Z}$ by adjoining all the roots of unities. Now Lusztig defined for ℓ an odd integer (ℓ coprime to 3 if G_2 is a factor of G) a certain Frobenius homomorphism $\mathrm{Fr}: U_v(\mathfrak{b}^{-1}) \to U_{\tilde{\mathbb{Z}}}(\mathfrak{b})$ and also a certain splitting of it on the '$\mathfrak{n}^-$-part' (which we shall refer to as *Frobenius splitting* homomorphism) $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}(\mathfrak{n}^-) \to U_v(\mathfrak{n}^-)$, where $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-) := U_{\mathbb{Z}}(\mathfrak{b}^-) \bigotimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$, $\mathfrak{n}^-$ is the nilradical of $\mathfrak{b}^-$, and $U_{\tilde{\mathbb{Z}}}(\mathfrak{n}^-), U_v(\mathfrak{n}^-)$ have meaning similar to that of the corresponding $\mathfrak{b}^-$. We extend the definition of Fr' to $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$ (cf. Lemma 3). By using the duality mentioned above, we get maps $\mathrm{Fr}^* : \bigoplus_{\lambda\in P^+} V_{\tilde{\mathbb{Z}}}(\lambda)^* \to \bigoplus_{\lambda\in P^+} V_v(\lambda)^*$ respectively $\mathrm{Fr}' : \bigoplus_{\lambda\in P^+} V_v(\lambda)^* \to \bigoplus_{\lambda\in P^+} V_{\tilde{\mathbb{Z}}}(\lambda)^*$ (cf. Propositions 4 and 5).

Next we assume that ℓ is a prime, and show that, after a base change with an algebraically closed field of char. $p=\ell$, the map Fr^* has a natural interpretation as the pth power map on the space of sections $H^0(G/B, \mathscr{L}_\lambda)$, and Fr'^* can naturally be interpreted as a splitting of this map. This is our principal result of the paper (cf. Theorem 1). So on the level of quantum groups, the map Fr'^* can be considered as a char. zero lift of the Frobenius splitting of G/B in char. p, and it is exactly this map which has been used in [7] to define the 'ℓth root' of certain sections.

In a subsequent paper we will use Lusztig's Fr (resp. Fr'^*) to define the Frobenius map (resp. Frobenius splitting map) at 'higher cohomology' level as well and show that Fr'^* after base change provides a canonical (in the sense of Mathieu) splitting of G/B compatibly splitting all of the Schubert subvarieties. In particular, this will provide a purely algebraic proof (via the quantum groups at roots of unity) of results of Mehta–Ramanathan mentioned above and also various results on the geometry of Schubert varieties mentioned in the first paragraph.

1. A pairing and its quantum analogue

For a complex semisimple Lie algebra $\mathfrak{g}$, associated to a Cartan matrix $C = (c_{i,j})_{1\le i,j\le n}$, fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{b}$. Let $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ be the associated triangular decomposition and set $\mathfrak{b}^- = \mathfrak{n}^-\oplus\mathfrak{h}$. We write U, U^+, U^- for the enveloping algebras $U(\mathfrak{g}), U(\mathfrak{n}^+), U(\mathfrak{n}^-)$, respectively.

Let $\omega_1,\dots,\omega_n$ be the set of fundamental weights in the weight lattice P of $\mathfrak{g}$. Denote by P^+ the set of non-negative linear combinations $\mathbb{N}\omega_1+\cdots+\mathbb{N}\omega_n$, $\mathbb{N}$ being the set of non-negative integers. We write $V(\lambda)$ for the complex irreducible representation of highest weight $\lambda\in P^+$. As an $\mathfrak{h}$-module, $V(\lambda)$ decomposes into the direct sum $\bigoplus_{\mu\in P} V(\lambda)_\mu$ of weight spaces. Here $V(\lambda)_\mu = \{v\in V(\lambda)\mid hv=\mu(h)v\ \forall h\in\mathfrak{h}\}$.

Let $\alpha_1,\dots,\alpha_n$ be the set of simple roots (corresponding to the Borel subalgebra $\mathfrak{b}$). For a simple root α_i choose $X_i\in\mathfrak{g}_{\alpha_i}$, $Y_i\in\mathfrak{g}_{-\alpha_i}$ and $H_i\in\mathfrak{h}$ such that $[H_i,X_j]=a_{i,j}X_j$, $[H_i,Y_j]=-a_{i,j}Y_j$, $[X_i,Y_j]=\delta_{i,j}H_i$. We denote by $U_\mathbb{Z}$ the Kostant form of U over the ring of integers $\mathbb{Z}$, generated by the divided powers $X_i^{(k)} := X_i^k/k!$ and $Y_i^{(k)} := Y_i^k/k!$, $k\ge 0$. Let $U_\mathbb{Z}^+, U_\mathbb{Z}^-$ be the corresponding Kostant forms of U^+ and U^-, generated respectively by $\{X_i^{(k)}\}$ and $\{Y_i^{(k)}\}$, and let $U_\mathbb{Z}(\mathfrak{b}^-)$ be the Kostant form of $U(\mathfrak{b}^-)$ generated by $\{Y_i^{(k)}, \binom{H_i}{k}\}$, where $\binom{H_i}{k}=H_i(H_i-1)\cdots(H_i-k+1)/k!$. Then $U_\mathbb{Z}(\mathfrak{b}^-)\subset U_\mathbb{Z}$.

For $i=1,\dots,n$, fix a highest weight vector $\bar v_{\omega_i}\in V(\omega_i)_{\omega_i}$ and let $V_\mathbb{Z}(\omega_i) := U_\mathbb{Z}\bar v_{\omega_i}$ be the corresponding $U_\mathbb{Z}$-stable $\mathbb{Z}$-lattice. For $\lambda=\sum a_i\omega_i\in P^+$ denote by $\bar v_\lambda$ the vector $\bar v_{\omega_1}^{\otimes a_1}\otimes\cdots\otimes\bar v_{\omega_n}^{\otimes a_n}$, and let $V_\mathbb{Z}(\lambda)$ be the lattice

$$U_\mathbb{Z}\bar b_\lambda := U_\mathbb{Z}(\bar v_{\omega_1}^{\otimes a_1}\otimes\cdots\otimes\bar v_{\omega_n}^{\otimes a_n}) \hookrightarrow V_\mathbb{Z}(\omega_1)^{\otimes a_1}\otimes\cdots\otimes V_\mathbb{Z}(\omega_n)^{\otimes a_n}.$$

The dual module $\mathrm{Hom}_\mathbb{Z}(V_\mathbb{Z}(\lambda),\mathbb{Z})$ is denoted by $V_\mathbb{Z}(\lambda)^*$. Setting $R_\mathbb{Z} := \bigoplus_{\lambda\in P^+} V_\mathbb{Z}(\lambda)^*$, we have a natural pairing

$$\Phi : U_\mathbb{Z}(\mathfrak{b}^-)\times R_\mathbb{Z}\to\mathbb{Z},\quad \Phi(u,\xi) := \xi(u.\bar v_\lambda)\quad \text{for } u\in U_\mathbb{Z}(\mathfrak{b}^-),\ \xi\in V_\mathbb{Z}(\lambda)^*.$$

Denote by $U(\mathfrak{b}^-)^*$ the Hopf dual of $U(\mathfrak{b}^-)$ (under the standard Hopf algebra structure of $U(\mathfrak{b}^-)$), i.e., $U(\mathfrak{b}^-)^*$ is the subspace of linear forms $f\in\mathrm{Hom}(U(\mathfrak{b}^-),\mathbb{C})$ for which there exists a two-sided ideal $I\subset U(\mathfrak{b}^-)$ of finite codimension such that $f(I)=0$. Recall that $U(\mathfrak{b}^-)^*$ is again a Hopf algebra. Let $U_\mathbb{Z}(\mathfrak{b}^-)^*$ be the $\mathbb{Z}$-submodule of forms $f\in U(\mathfrak{b}^-)^*$ such that $f(U_\mathbb{Z}(\mathfrak{b}^-))\subset\mathbb{Z}$. Note that $U_\mathbb{Z}(\mathfrak{b}^-)^*$ is a $\mathbb{Z}$-subalgebra of $U(\mathfrak{b}^-)^*$.

The following proposition can be viewed as a certain ‘half’ of the Peter–Weyl-type theorem. One may also consider it as an algebraic analog of the result of Bernstein, Gelfand and Gelfand that the ring $\mathbb{C}[G/U]$ of regular functions on G/U is isomorphic to $\bigoplus_{\lambda\in P^+} V(\lambda)^*$, where G is the corresponding simply connected complex algebraic group with Borel subgroup B and unipotent radical $R_u(B) = U$ having $\mathfrak{n}^+$ as its Lie algebra.

Proposition 1. *Φ is a non-degenerate pairing, identifying $R_{\mathbb{Z}}$ as a subalgebra of $U_{\mathbb{Z}}(\mathfrak{b}^-)^*$:*

$$\xi^\lambda \cdot \xi^\mu(u) := (\xi^\lambda \otimes \xi^\mu)|_{V_{\mathbb{Z}}(\lambda+\mu)}(u.\bar{v}_{\lambda+\mu})$$

for $\xi^\lambda \in V_{\mathbb{Z}}(\lambda)^*$, $\xi^\mu \in V_{\mathbb{Z}}(\mu)^*$ and $u \in U(\mathfrak{b}^-)$.

Proof. Denote by Q the root lattice and set $Q^+ = \mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_n$. The Kostant form $U^0_{\mathbb{Z}}$ of the enveloping algebra $U(\mathfrak{h})$ has as basis the monomials $\binom{H_1}{k_1}\cdots\binom{H_n}{k_n}$, $k_i \in \mathbb{N}$. Fix a $\mathbb{Z}$-basis $\mathtt{B}$ of $U^-_{\mathbb{Z}}$ such that for any $\beta\in Q^+$ the elements in $\mathtt{B}_\beta := U^-_{\mathbb{Z},-\beta} \cap \mathtt{B}$ form a basis of the weight space $U^-_{\mathbb{Z},-\beta}$.

Suppose now that $u \in U_{\mathbb{Z}}(\mathfrak{b}^-)$ is such that $\Phi(u, R_{\mathbb{Z}}) = 0$. We can write $u = \sum_{\beta\in Q^+}\sum_{b\in \mathtt{B}_\beta} bh_b$, where $h_b \in U^0_{\mathbb{Z}}$. To prove $u=0$, it is sufficient to show that for any $u \neq 0$, $u\bar{v}_\lambda \neq 0$ for some $\lambda\in P^+$. It is well known that if β is fixed, then for $\lambda \gg 0$ (i.e., $a_i \gg 0$ for all $i = 1,\dots,n$), the vectors $b\bar{v}_\lambda$, $b\in \mathtt{B}_\beta$, form a basis of the weight space $V_{\mathbb{Z}}(\lambda)_{\lambda-\beta}$. So if we choose λ big enough, $u\bar{v}_\lambda = 0$ implies $\lambda(h_b) = 0$ for all b and all $\lambda \gg 0$. But this is possible only if $h_b = 0$.

Consider $f = \sum \xi^\lambda$. Among the $\xi^\lambda \neq 0$ fix ξ^{λ_0} such that λ_0 is maximal in the lexicographic ordering, i.e., if $\lambda_0=\sum a_i\omega_i$ and $\lambda=\sum b_i\omega_i$ is such that $\xi^\lambda \neq 0$, then there exists a $j \le n$ such that $a_i = b_i$ for $i < j$ and $a_j > b_j$. Set $H_{\lambda_0} := \prod_{i=n}^{n}\binom{H_i}{a_i}$. Note that $H_{\lambda_0}\bar{v}_{\lambda_0} = \bar{v}_{\lambda_0}$ and $H_{\lambda_0}\bar{v}_\lambda = 0$ for all λ such that $\xi^\lambda \neq 0$, $\lambda \neq \lambda_0$. Since $V_{\mathbb{Z}}(\lambda_0) = U^-_{\mathbb{Z}}\bar{v}_{\lambda_0}$, we can find $u \in U^-_{\mathbb{Z}}$ such that $\xi^{\lambda_0}(u\bar{v}_{\lambda_0}) \neq 0$. It follows that $\Phi(uH_{\lambda_0}, f) = \sum \xi^\lambda(uH_{\lambda_0}\bar{v}_\lambda) = \xi^{\lambda_0}(u\bar{v}_{\lambda_0}) \neq 0$. This proved that Φ is non-degenerate.

To see that $R_{\mathbb{Z}}$ form a subalgebra of $U_{\mathbb{Z}}(\mathfrak{b}^-)$, note that the co-product Δ induces a natural $U_{\mathbb{Z}}(\mathfrak{b}^-)$-module structure on $V_{\mathbb{Z}}(\lambda)\otimes V_{\mathbb{Z}}(\mu)$. By the definition of the product we have $\xi^\lambda \cdot \xi^\mu(u) = \xi^\lambda \otimes \xi^\mu(\Delta(u))$. For $\Delta(u) = \sum u_1 \otimes u_2$ we have $\xi^\lambda \otimes \xi^\mu(\Delta(u)) = \sum \xi^\lambda(u_1\bar{v}_\lambda)\cdot \xi^\mu(u_2\bar{v}_\mu)$.

Now the map $u\bar{v}_{\lambda+\mu} \mapsto u(\bar{v}_\lambda \otimes \bar{v}_\mu)$, $u \in U_{\mathbb{Z}}(\mathfrak{b}^-)$, induces an isomorphism between $V_{\mathbb{Z}}(\lambda+\mu)$ and the $U_{\mathbb{Z}}$-submodule $U_{\mathbb{Z}}(\mathfrak{b}^-)(\bar{v}_\lambda\otimes\bar{v}_\mu)$ of $V_{\mathbb{Z}}(\lambda)\otimes V_{\mathbb{Z}}(\mu)$. The restriction map induces a map $\mathrm{res}: V_{\mathbb{Z}}(\lambda)^* \otimes V_{\mathbb{Z}}(\mu)^* \to V_{\mathbb{Z}}(\lambda+\mu)^*$. It follows that $\xi^\lambda\cdot\xi^\mu(u) = \mathrm{res}(\xi^\lambda\otimes\xi^\mu)(u_1\bar{v}_{\lambda+\mu})$. $\square$

Remark 1. By using the Peter–Weyl theorem, we get an isomorphism $\mathbb{C}[G/U] \simeq \bigoplus_{\lambda\in P^+} V(\lambda)^*$. Let B^- be the opposite Borel subgroup (Lie $B^- = \mathfrak{b}^-$). Since B^- is open and dense in G/U, we have an inclusion $\mathbb{C}[G/U] \hookrightarrow \mathbb{C}[B^-]$. But by [3, Part I, Sections 7.10 and 7.18], we have $\mathbb{C}[B^-] \simeq U(\mathfrak{b}^-)^*$, and hence we have $\bigoplus_{\lambda\in P^+} V(\lambda)^* \hookrightarrow U(\mathfrak{b}^-)^*$. So this gives an alternative geometric derivation of the above proposition over $\mathbb{C}$.

We have a similar construction for the quantum group $U_q := U_q(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$. Let $d_1, \dots, d_n \geq 1$ be minimal integers such that $(d_i c_{i,j})$ is a symmetric matrix, and let $\tilde{\mathbb{Z}}$ be the ring obtained from $\mathbb{Z}$ by adjoining all roots of unities.

We fix a positive integer ℓ. If ℓ is odd, then let ϕ be the ℓ-cyclotomic polynomial, and if ℓ is even, then let ϕ be the 2ℓ-cyclotomic polynomial. Let $A = \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials and fix a homomorphism $A/(\phi) \hookrightarrow \tilde{\mathbb{Z}}$, where (ϕ) is the ideal in A generated by ϕ. Denote by v the image of q in $\tilde{\mathbb{Z}}$.

We denote the generators of the quantum group U_q over $\mathbb{C}(q)$ by $E_i, F_i, K_i^{\pm}$, and let U_A be the Lusztig form of U_q over A generated by the divided powers $E_i^{(m)} := E_i^m/[m]_i!$ and $F_i^{(m)} := F_i^m/[m]_i!$ and the $K_i^{\pm 1}$ [9, Section 1]. Recall that the Gaussian numbers $[m]_i$ are defined by $[m]_i := (q^{d_i m} - q^{-d_i m})/(q^{d_i} - q^{-d_i})$, and $[m]_i! := [1]_i \cdots [m]_i$.

We denote by U_A^+, U_A^-, U_A^0 the subalgebras of U_A generated by the $E_i^{(m)}$, the $F_i^{(m)}$, and the $\{K_i^{\pm}, \begin{bmatrix} K_i;0 \\ m \end{bmatrix}\}$ respectively for $1 \leq i \leq n$ and $m \in \mathbb{N}$. Recall that the latter is defined by

$$\begin{bmatrix} K_i; c \\ m \end{bmatrix} := \prod_{s=1}^{m} \frac{K_i q^{(c-s+1)d_i} - K_i^{-1} q^{(-c+s-1)d_i}}{q^{sd_i} - q^{-sd_i}} \quad \text{for } c \in \mathbb{Z} \text{ and } m \in \mathbb{N}$$

and U_A^0 has as a basis the monomials of the form $\prod_{i=1}^n \left(\begin{bmatrix} K_i;0 \\ m_i \end{bmatrix} K_i^{e_i} \right)$, where the m_i are non-negative integers and $e_1, \dots, e_n \in \{0, 1\}$ [9, Theorem 6.7(c)]. Let $U_A(\mathfrak{b}^-)$ be the subalgebra generated by the $F_i^{(m)}, K_i^{\pm}$ and the $\begin{bmatrix} K_i;0 \\ m \end{bmatrix}$. The following statements can be found in [8] or [1], or can be easily deduced from [9, Section 6.4]

Lemma 1.

(a) $$\begin{bmatrix} K_i; 0 \\ m \end{bmatrix} \begin{bmatrix} K_i; -m \\ t \end{bmatrix} = \begin{bmatrix} m+t \\ m \end{bmatrix}_i \begin{bmatrix} K_i; 0 \\ m+t \end{bmatrix},$$

(b) $$\begin{bmatrix} K_i; c \\ t' \end{bmatrix} = \sum_{j=0}^{t} \begin{bmatrix} t \\ j \end{bmatrix}_i q^{d_i(tt'-cj)} K_i^{-j} \begin{bmatrix} K_i; c-t \\ t'-j \end{bmatrix} \textit{ for any } t \leq t',$$

(c) $$\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \sum_{s=0}^{t} q^{d_i c(t-s)} \begin{bmatrix} c \\ s \end{bmatrix}_i K_i^{-s} \begin{bmatrix} K_i; 0 \\ t-s \end{bmatrix} \textit{ for } c > 0,$$

(d) $$\begin{bmatrix} K_i; -c \\ t \end{bmatrix} = \sum_{s=0}^{t} (-1)^s q^{d_i c(t-s)} \begin{bmatrix} c+s-1 \\ s \end{bmatrix}_i K_i^{s} \begin{bmatrix} K_i; 0 \\ t-s \end{bmatrix} \textit{ for } c > 0,$$

(e) $$\begin{bmatrix} K_i; 0 \\ x \end{bmatrix} \begin{bmatrix} K_i; 0 \\ y \end{bmatrix} = \sum_{j=0}^{x} \begin{bmatrix} x \\ j \end{bmatrix}_i \begin{bmatrix} x+y-j \\ x \end{bmatrix}_i q^{d_i xy} K_i^{-j} \begin{bmatrix} K_i; 0 \\ x+y-j \end{bmatrix},$$

where

$$\begin{bmatrix} t \\ j \end{bmatrix}_i := \prod_{s=1}^{j} \frac{q^{(t-s+1)d_i} - q^{(-t+s-1)d_i}}{q^{sd_i} - q^{-sd_i}} \quad \textit{for } t \in \mathbb{Z} \textit{ and } j \in \mathbb{N}.$$

We denote by U_v, U_v^+, U_v^0, U_v^- and $U_v(\mathfrak{b}^-)$ the algebras over $\tilde{\mathbb{Z}}$ obtained from the corresponding forms defined over A by base change $A \to A/(\phi) \hookrightarrow \tilde{\mathbb{Z}}$. For $\lambda \in P^+$ let $V_q(\lambda)$ be the irreducible representation of U_q over $\mathbb{C}(q)$ with highest weight λ. As in the classical case, we fix for $i=1,\dots,n$ a highest weight vector $v_{\omega_i} \in V_q(\omega_i)_{\omega_i}$, and let $V_A(\omega_i) = U_A v_{\omega_i}$ be the corresponding A-lattice. For $\lambda = \sum a_i\omega_i \in P^+$ denote by v_λ the vector $v_{\omega_1}^{\otimes a_1} \otimes \cdots \otimes v_{\omega_n}^{\otimes a_n}$, and let $V_A(\lambda)$ be the lattice

$$U_A v_\lambda := U_A(v_{\omega_1}^{\otimes a_1} \otimes \cdots \otimes v_{\omega_n}^{\otimes a_n}) \hookrightarrow V_A(\omega_1)^{\otimes a_1} \otimes \cdots \otimes V_A(\omega_n)^{\otimes a_n}.$$

Then $U_A v_\lambda$ is indeed A-free. We denote by $V_v(\lambda)$ the corresponding representation $V_A(\lambda) \otimes_A \tilde{\mathbb{Z}}$ of U_v. Recall that K_i acts on a weight vector $v_\mu \in V_v(\lambda)_\mu$ by multiplication with $v^{d_i\mu(H_i)}$. As in the classical case, let R_v denote the direct sum $\bigoplus_{\lambda\in P^+} V_v(\lambda)^*$, where $V_v(\lambda)^* := \mathrm{Hom}_{\tilde{\mathbb{Z}}}(V_v(\lambda), \tilde{\mathbb{Z}})$.

Let $\ell_i \in \mathbb{N}$ be minimal such that $d_i\ell_i \equiv 0 \bmod \ell$ (recall: $d_i \in \{1,2,3\}$). Then v^{d_i} is a primitive ℓ_ith root of unity if ℓ is odd and a primitive $2\ell_i$th root of unity if ℓ is even. Note that in either case $K_i^{2\ell_i} = 1$ in U_v (cf. [8, Lemma 4.4(a)]), as can be easily seen from the following relation in U_A^0:

$$\begin{bmatrix} K_i;0 \\ \ell_i \end{bmatrix} \prod_{j=1}^{\ell_i} (q^{d_ij} - q^{-d_ij}) = \prod_{j=1}^{\ell_i} (K_i q^{d_i(-j+1)} - K_i^{-1} q^{d_i(j-1)}).$$

If ℓ is odd, then $K_i^{\ell_i} v_\mu = v_\mu$ for all $\mu \in P$ and all weight vectors v_μ, in particular, $K_i^{\ell_i}$ are in the center of $U_v(\mathfrak{b}^-)$. Denote by J' the ideal of $U_v(\mathfrak{b}^-) \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ generated by $(K_i^{\ell_i} - 1)$, $i = 1,\dots,n$, and let J be the ideal $J' \cap U_v(\mathfrak{b}^-)$ in $U_v(\mathfrak{b}^-)$. Observe that $U_v(\mathfrak{b}^-)$ embeds inside $U_v(\mathfrak{b}^-) \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ since $U_v(\mathfrak{b}^-)$ is $\tilde{\mathbb{Z}}$-free.

Define the pairing $\Phi_v : U_v(\mathfrak{b}^-) \times R_v \to \tilde{\mathbb{Z}}$ by $(u, \xi^\lambda) \mapsto \xi^\lambda(uv_\lambda)$ for $u \in U_v(\mathfrak{b}^-)$ and $\xi^\lambda \in V_v(\lambda)^*$.

Proposition 2.

(a) *If ℓ is odd, then the pairing Φ_v has radical precisely equal to $(J,0)$. The induced pairing $\Phi_v' : U_v(\mathfrak{b}^-)/J \times R_v \to \tilde{\mathbb{Z}}$ is hence non-degenerate.*

(b) *If ℓ is even, then the pairing Φ_v is non-degenerate.*

(c) *Then induced map $\psi_v : R_v \to U_v(\mathfrak{b}^-)^*$ is injective, and the image is a subalgebra of $U_v(\mathfrak{b}^-)^*$, where the multiplication of $\xi^\lambda \in V_v(\lambda)^*$ and $\xi^\mu \in V_v(\mu)^*$ is given by*

$$(\xi^\lambda \cdot \xi^\mu)(u) := (\xi^\lambda \otimes \xi^\mu)\,|_{V_v(\lambda+\mu)}(u \cdot v_{\lambda+\mu}) \quad \textit{for } u \in U_v(\mathfrak{b}^-).$$

Proof. Let $v_\lambda \in V_v(\lambda)$ be the fixed highest weight vector. Recall that $\begin{bmatrix} K_i;0 \\ m \end{bmatrix} v_\lambda = \begin{bmatrix} \lambda(H_i) \\ m \end{bmatrix}_i v_\lambda$, in particular, $\begin{bmatrix} K_i;0 \\ m \end{bmatrix} v_\lambda = 0$ if $m > \lambda(H_i)$. The same argument as above in the classical case shows that the map $\psi_v : R_v \to U_v(\mathfrak{b}^-)^*$ is injective.

Suppose now $u \in U_v(\mathfrak{b}^-)$ is such that $\Phi_v(u, R_v) = 0$. We can find linearly independent $u_1,\dots,u_t \in U_v^-$ and some $h_1,\dots,h_t \in U_v^0$ such that $u = \sum_{i=1}^t u_ih_i$. To say that u is in the radical of the pairing is equivalent to saying that $uv_\lambda = 0$ for all highest weight vectors $v_\lambda \in V_v(\lambda)$, $\lambda \in P^+$. Since u_i are linearly independent, the vectors u_iv_λ are linearly independent for $\lambda \gg 0$. So $uv_\lambda = 0$ for all $\lambda \gg 0$ is equivalent to $h_iv_\lambda = 0$ for all

$i=1,\dots,t$ and all $\lambda \gg 0$. In the next lemma we determine a basis of $U_v^0 \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ to find those $h \in U_v^0 \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ satisfying this property. □

Lemma 2. *The complex algebra $U_v^0 \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ has as basis the monomials $\prod_{i=1}^n \left(\begin{bmatrix} K_i;0 \\ \ell_i m_i \end{bmatrix} K_i^{e_i} \right)$, where $m_i \in \mathbb{N}$ and $0 \le e_i < 2\ell_i$.*

Proof. If $\ell_i = 1$ (which can of course only happen if $\ell = 2, d_i = 2$ or $\ell = 3, d_i = 3$ or $\ell = 1$), then the corresponding i-part of the monomial is exactly of the form $\begin{bmatrix} K_i;0 \\ m_i \end{bmatrix}$ or $\begin{bmatrix} K_i;0 \\ m_i \end{bmatrix} K_i$, as in the basis of U_A^0 mentioned before. So, without loss of generality, we may assume $\ell_i > 1$.

We show that the monomials listed in the proposition form a subalgebra. Recall that [10, Lemma 34.1.2] $\begin{bmatrix} x\ell_i \\ j \end{bmatrix}_i = 0$ unless ℓ_i divides j, and $\begin{bmatrix} x\ell_i \\ j\ell_i \end{bmatrix}_i = \binom{x}{j}$, where $\binom{x}{j}$ is the ordinary binomial coefficient. Further, $v^{d_i \ell_i^2 xy} = v^{(r\ell)\ell_i xy}$ for some $r \in \mathbb{N}$, hence it is equal to ± 1. By specializing the relation in Lemma 1(e) at $q = v$, we get

$$\begin{bmatrix} K_i;0 \\ x\ell_i \end{bmatrix} \begin{bmatrix} K_i;0 \\ y\ell_i \end{bmatrix} = \sum_{j=0}^{x} \binom{x}{j} \binom{x+y-j}{x} (\pm 1) K_i^{-j\ell_i} \begin{bmatrix} K_i;0 \\ (x+y-j)\ell_i \end{bmatrix},$$

which proves that the monomials of this type span a subalgebra of U_v^0. In U_A^0 we have in addition the relation: $(0 < r < \ell_i,\ m \ge 0)$

$$\begin{bmatrix} K_i;0 \\ m\ell_i + r \end{bmatrix} \prod_{s=1+m\ell_i}^{r+m\ell_i} (q^{sd_i} - q^{-sd_i}) = \begin{bmatrix} K_i;0 \\ m\ell_i \end{bmatrix} \prod_{s=1+m\ell_i}^{r+m\ell_i} (K_i q^{d_i(-s+1)} - K_i^{-1} q^{d_i(s-1)}).$$

If we specialize at $q = v$, then $v^{md_i\ell_i} = v^{-md_i\ell_i} = \pm 1$. Since this term occurs on both sides, we can cancel it and get

$$\begin{bmatrix} K_i;0 \\ m\ell_i + r \end{bmatrix} \prod_{s=1}^{r} (v^{sd_i} - v^{-sd_i}) = \begin{bmatrix} K_i;0 \\ m\ell_i \end{bmatrix} \prod_{s=1}^{r} (K_i v^{d_i(-s+1)} - K_i^{-1} v^{d_i(s-1)}).$$

Note that $\prod_{s=1}^r (v^{sd_i} - v^{-sd_i}) \neq 0$. Since $K^{2\ell_i} = 1$, this implies that we can express $\begin{bmatrix} K_i;0 \\ m\ell_i + r \end{bmatrix}$ over $\mathbb{C}$ as a product of $\begin{bmatrix} K_i;0 \\ m\ell_i \end{bmatrix}$ with a linear combination of $1, K_i, \dots, K_i^{2\ell_i - 1}$. The linear independence of the monomials follows from the description of the basis for U_A^0 above. □

Proof of Proposition 2 (*continuation*)**.** Let $h \in U_v^0$ be such that $hv_\lambda = 0$ for all $\lambda \gg 0$. We write h (viewed as an element of $U_v^0 \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$) as a linear combination

$$h = \sum_{\boldsymbol{m} \in \mathbb{N}^n} \left(\prod_{i=1}^n \begin{bmatrix} K_i;0 \\ \ell_i m_i \end{bmatrix} \right) \left(\sum_{\substack{\boldsymbol{e} \in \mathbb{N}^n \\ 0 \le e_i < 2\ell_i}} b_{\boldsymbol{m},\boldsymbol{e}} K_1^{e_1} \cdots K_n^{e_n} \right),$$

where $\boldsymbol{m} := (m_1, \dots, m_n)$ and similarly $\boldsymbol{e}$.

If $v_\lambda \in V_v(\lambda)$ is a highest weight vector of weight $\lambda = \sum_{i=1}^n (a_i\ell_i + r_i)\omega_i$ with $0 \le r_i < \ell_i$, then, by [10, Lemma 34.1.2],

$$\begin{bmatrix} K_i;0 \\ \ell_i m_i \end{bmatrix} v_\lambda = \begin{bmatrix} a_i\ell_i + r_i \\ \ell_i m_i \end{bmatrix}_i v_\lambda = \pm \begin{pmatrix} a_i \\ m_i \end{pmatrix} v_\lambda.$$

From this it follows easily that $hv_\lambda = 0$ for all $\lambda \gg 0$ is equivalent to the condition $\sum_{\boldsymbol{e}} b_{\boldsymbol{m},\boldsymbol{e}} K_1^{e_1}\cdots K_n^{e_n} v_\lambda = 0$ for all $\boldsymbol{m}$ and all $\lambda \gg 0$.

Suppose now we have such an element $h = \sum_{\boldsymbol{e}} b_{\boldsymbol{e}} K_1^{e_1}\cdots K_n^{e_n} \neq 0$ and $hv_\lambda = 0$ for all $\lambda \gg 0$. Since $K_i^{2\ell_i} = 1$, this is equivalent to saying that $hv_\lambda = 0$ for all $\lambda = \sum_{i=1}^n a_i\omega_i$ such that $0 \le a_i < 2\ell_i$.

The $\mathbb{C}$-subalgebra K of $U_v^0 \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ generated by the K_i can be viewed as the group algebra of the group $\prod_{i=1}^n \mathbb{Z}/2\ell_i\mathbb{Z}$. If ℓ is odd, then v^{d_i} is a primitive ℓ_ith root of unity. The one-dimensional representations provided by the action of the K_i's on the highest weight vectors in $V_v(\lambda)$, $\lambda = \sum_{i=1}^n a_i\omega_i$, $0 \le a_i < 2\ell_i$, hence does not give a complete list of all irreducible representations of K. The intersection of the kernels of these representations is the subalgebra generated by $(K_i^{\ell_i} - 1)$.

If ℓ is even, then v^{d_i} is a primitive $2\ell_i$th root of unity. The one-dimensional representations provided by the action of the K_i on the highest weight vectors in $V_v(\lambda)$, $\lambda = \sum_{i=1}^n a_i\omega_i$, $0 \le a_i < 2\ell_i$, hence give a complete list of all irreducible representations, so $h = 0$.

The description of the multiplication can be proved as in the classical case. □

2. The Frobenius maps

We recall in this section the definition of the quantum Frobenius maps Fr and Fr′ defined by Lusztig on U_v^- respectively $U_{\tilde{\mathbb{Z}}}^- := U_{\mathbb{Z}}^- \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$. Fix a positive integer ℓ. To simplify the arguments we assume that ℓ is odd, and if $\mathfrak{g}$ has simple factors of type $\mathtt{G}_2$, then we assume ℓ to be coprime to 3 in addition. Note that these conditions imply $\ell_i = \ell$ for all i; we will make some remarks at the end of this section concerning the cases $\ell = 2, 3$.

Lusztig has constructed two algebra homomorphisms (Theorems 35.1.7 and 35.1.8 in [10]): $\mathrm{Fr} : U_v^- \to U_{\tilde{\mathbb{Z}}}^-$ (respectively $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^- \to U_v^-$) which are defined on the generators by

$$\mathrm{Fr}(F_i^{(k)}) := \begin{cases} 0 & \text{if } \ell k \\ Y_i^{(k/\ell)} & \text{if } \ell | k \end{cases} \quad \left(\text{respectively } \mathrm{Fr}'(Y_i^{(k)}) := F_i^{(\ell k)}\right).$$

The composition Fr o Fr′ is obviously the identity map on $U_{\tilde{\mathbb{Z}}}^-$. One can of course similarly define $\mathrm{Fr} : U_v^+ \to U_{\tilde{\mathbb{Z}}}^+$ and $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^+ \to U_v^+$. The map Fr can be extended to an algebra homomorphism $\mathrm{Fr} : U_v \to U_{\tilde{\mathbb{Z}}}$ (see [10, Theorem 35.1.9] or [9, Theorems 8.10 and 8.11 and Corollary 8.14]):

Proposition 3. *The map defined by* $u \mapsto \mathrm{Fr}(u)$ *for* $u \in U_v^-$ *or* $u \in U_v^+$ *and*

$$K_i \mapsto 1, \quad \begin{bmatrix} K_i;0 \\ m \end{bmatrix} \mapsto \begin{cases} 0 & \text{if } \ell \nmid m \\ \binom{H_i}{m/\ell} & \text{if } \ell | m \end{cases} \quad i = 1,\dots,n,$$

extends the Frobenius maps for U_v^- *and* U_v^+ *to a surjective* $\tilde{\mathbb{Z}}$*-algebra homomorphism* $\mathrm{Fr} : U_v \to U_{\tilde{\mathbb{Z}}}$. *Moreover*, Fr *is a Hopf algebra homomorphism.*

The map Fr' *cannot* be extended to a homomorphism defined on $U_{\tilde{\mathbb{Z}}}$. Though, we can extend it to a homomorphism defined on $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$, the price for the extension is that the range $U_v(\mathfrak{b}^-)$ is to be replaced by $U_v(\mathfrak{b}^-)/J$. Here J is the ideal defined in the last section (Proposition 2).

Lemma 3. *The map defined by* $u \mapsto \mathrm{Fr}'(u)$ *for* $u \in U_{\tilde{\mathbb{Z}}}^-$ *and* $\binom{H_i}{m} \mapsto \begin{bmatrix} K_i;0 \\ \ell m \end{bmatrix}$ *extends* Fr' *to a* $\tilde{\mathbb{Z}}$*-algebra homomorphism* (*again denoted by*) $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-) \to U_v(\mathfrak{b}^-)/J$.

We refer to Fr' *as the Frobenius splitting homomorphism.*

Proof. Recall that $v^\ell = 1$, $\begin{bmatrix} x\ell \\ j \end{bmatrix} = 0$ unless ℓ divides j and $\begin{bmatrix} x\ell \\ j\ell \end{bmatrix} = \binom{x}{j}$. Further, $K_i^\ell = 1$ in $U_v(\mathfrak{b}^-)/J$. Hence Lemma 1(e) implies that

$$\begin{aligned}
\mathrm{Fr}'\left(\binom{H_i}{x}\right)\mathrm{Fr}'\left(\binom{H_i}{y}\right) &= \begin{bmatrix} K_i;0 \\ x\ell \end{bmatrix}\begin{bmatrix} K_i;0 \\ y\ell \end{bmatrix} \\
&= \sum_{j=0}^{x} \binom{x}{j}\binom{x+y-j}{x}\begin{bmatrix} K_i;0 \\ \ell(x+y-j) \end{bmatrix} \\
&= \mathrm{Fr}'\left(\sum_{j=0}^{x} \binom{x}{j}\binom{x+y-j}{x}\binom{H_i}{x+y-j}\right) \\
&= \mathrm{Fr}'\left(\binom{H_i}{x}\binom{H_i}{y}\right).
\end{aligned}$$

Now, for $y \geq 0$, we have

$$\begin{aligned}
\mathrm{Fr}'\left(\binom{H_i + y}{x}\right) &= \mathrm{Fr}'\left(\sum_{s=0}^{x} \binom{y}{s}\binom{H_i}{x-s}\right) \\
&= \sum_{s=0}^{x} \begin{bmatrix} y\ell \\ s\ell \end{bmatrix}_i \begin{bmatrix} K_i;0 \\ \ell(x-s) \end{bmatrix} = \begin{bmatrix} K_i;\ell y \\ \ell x \end{bmatrix},
\end{aligned}$$

because the other terms in the expression (Lemma 1(e)) for $\begin{bmatrix} K_i;\ell y \\ \ell x \end{bmatrix}$ vanish. Similarly, for $y > 0$, we get

$$\mathrm{Fr}'\left(\binom{H_i - y}{x}\right) = \mathrm{Fr}'\left(\sum_{s=0}^{x} (-1)^s \binom{y+s-1}{s}\binom{H_i}{x-s}\right)$$

$$= \sum_{s=0}^{x} (-1)^s \binom{y+s-1}{s} \begin{bmatrix} K_i;0 \\ \ell(x-s) \end{bmatrix}$$

$$= \sum_{s=0}^{x} (-1)^{s\ell} \begin{bmatrix} y\ell+s\ell-1 \\ s\ell \end{bmatrix}_i \begin{bmatrix} K_i;0 \\ \ell(x-s) \end{bmatrix}.$$

To prove the last equality, note that $(-1)^{s\ell} = (-1)^s$, and (see [10, Lemma 34.1.2]) $\begin{bmatrix} y\ell+s\ell-1 \\ s\ell \end{bmatrix}_i = \binom{y+s-1}{s} \begin{bmatrix} \ell-1 \\ 0 \end{bmatrix}_i = \binom{y+s-1}{s}$. Suppose $s' = s\ell + r$ with $0 < r < \ell$. Note that $\begin{bmatrix} y\ell+s'-1 \\ s' \end{bmatrix}_i = \pm \binom{y+s}{s} \begin{bmatrix} r-1 \\ r \end{bmatrix} = 0$, so Lemma 1 gives (for $y > 0$)

$$\mathrm{Fr}'\left(\binom{H_i - y}{x}\right) = \sum_{s'=0}^{x\ell} (-1)^{s'} v^{d_i y\ell(x\ell-s')} \begin{bmatrix} y\ell+s'-1 \\ s' \end{bmatrix}_i \begin{bmatrix} K_i;0 \\ x\ell-s' \end{bmatrix} = \begin{bmatrix} K_i;-\ell y \\ x\ell \end{bmatrix}.$$

From this we conclude that (cf. [9, Section 6.5])

$$\mathrm{Fr}'\left(\binom{H_i}{x}\right) \mathrm{Fr}'(Y_j^{(y)}) = \begin{bmatrix} K_i;0 \\ x\ell \end{bmatrix} F_j^{(y\ell)} = F_j^{(y\ell)} \begin{bmatrix} K_i;-y\ell c_{i,j} \\ x\ell \end{bmatrix}$$
$$= \mathrm{Fr}'(Y_j^{(y)}) \mathrm{Fr}'\left(\binom{H_i - y c_{i,j}}{x}\right),$$

which shows that the map respects the defining relations between the generators of $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$. □

Remark 2. The assumption that ℓ is coprime to 3 if $\mathfrak{g}$ admits simple factors of type $\mathtt{G}_2$ is not necessary for Proposition 3 and Lemma 3. Actually, the construction makes sense for arbitrary ℓ, but we have to redefine the maps; for details see [10, Chapter 35]. In the following we mainly concentrate on the remarks on $\ell = 2, 3$, but, with the appropriate adaptions (similarly to those in [7]), the constructions hold also in the general case.

As before, let ℓ_i be minimal such that $d_i \ell_i \equiv 0 \bmod \ell$, and denote by $C^\#$ the matrix $(c_{i,j}\ell_j/\ell_i)$. This is the Cartan matrix of the root system having the roots $\alpha_i^\# := \ell_i \alpha_i$ as simple roots and $H_i^\# := H_i/\ell_i$ as co-roots. Its weight lattice is the subset $P^\# := \{\lambda \in P \mid \lambda(H_i) \in \ell_i\mathbb{Z}\ \forall i\}$ of P. Note if $\mu \in P^\# \subset P$ and v_μ is a weight vector in a U_v^0-representation, then

$$\begin{bmatrix} K_i;k\ell_i \\ m\ell_i \end{bmatrix} v_\mu = \begin{bmatrix} \mu(H_i)+k\ell_i \\ m\ell_i \end{bmatrix}_i v_\mu = \begin{bmatrix} \ell_i\mu(H_i^\#)+k\ell_i \\ m\ell_i \end{bmatrix}_i v_\mu = \binom{\mu(H_i^\#)+k}{m} v_\mu.$$

Denote by $\mathfrak{g}^\#$ the corresponding Lie algebra and let $U^\#$ be its enveloping algebra. We use the notation $X_i^\#$, $Y_i^\#$ and $H_i^\#$ for the generators. If $\mathfrak{g}$ is simply laced or ℓ is a prime >3, then $C^\# = C$. But if $\ell = 3$, then $C^\#$ is obtained from C by transposing the 2×2 submatrices corresponding to simple factors of type $\mathtt{G}_2$. If $\ell = 2$, then the same has to be applied for simple factors of type $\mathtt{F}_4$, $\mathtt{B}_n$ and $\mathtt{C}_n$. The Frobenius homomorphisms

$\mathrm{Fr} : U_v^- \to U_{\tilde{\mathbb{Z}}}^{\#-}$ (respectively $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^{\#-} \to U_v^-$) are defined by

$$\mathrm{Fr}(F_i^{(k)}) := \begin{cases} 0 & \text{if } \ell_i \nmid k \\ Y_i^{\#(k/\ell_i)} & \text{if } \ell_i | k \end{cases} \quad (\text{respectively } \mathrm{Fr}'(Y_i^{\#(k)}) := F_i^{(\ell_i k)}).$$

If $\ell = 3$, then we extend the Frobenius map to a homomorphism $\mathrm{Fr} : U_v(\mathfrak{b}^-) \to U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^{\#-})$ by setting $\mathrm{Fr}(K_i)=1$, $\mathrm{Fr}\left(\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} H_i^\# \\ m/\ell_i \end{smallmatrix}\right]$ if ℓ_i divides m and $\mathrm{Fr}\left(\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]\right)=0$ otherwise. Similarly, one can extend Fr' to a homomorphism $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^{\#-}) \to U_v(\mathfrak{b}^-)/J$ by setting $\mathrm{Fr}\left(\left[\begin{smallmatrix} H_i^\# \\ m \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} K_i;0 \\ m\ell_i \end{smallmatrix}\right]$. The details of the proof are left to the reader.

The definitions of $\mathrm{Fr} : U_v^- \to U_{\tilde{\mathbb{Z}}}^{\#-}$ and $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^{\#-} \to U_v^-$ given above make sense for arbitrary positive integer ℓ. To avoid problems with the definition of the extensions for $\ell=2$, we assume that $\ell=2d$, where d is the smallest common multiple of $d_1,\dots,d_n$. Since $\ell_i = \ell/d_i = 2(d/d_i)$, we know that all the ℓ_i are even. Denote by $(U_v^0)_{ev}$ the subalgebra of U_v^0 generated by $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]$ and K_i^m, m even, and $K_i \left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right]$ for m odd, and let $(U_v^-)_{ev}$ be the subalgebra of U_v^- spanned by the monomials of weight -2β, $\beta \in Q^+$. Let $U_v(\mathfrak{b}^-)_{ev}$ be the subalgebra of $U_v(\mathfrak{b}^-)$ generated by $(U_v^-)_{ev}$ and $(U_v^0)_{ev}$. Note that $\mathrm{Fr}'(Y_i^{\#(k)}) = F_i^{(\ell_i k)} \in U_v(\mathfrak{b}^-)_{ev}$ because the ℓ_i are even.

Using Lemma 1, it is easy to verify that $U_v(\mathfrak{b}^-)_{ev}$ is spanned by the elements of the form $u\prod_{i=1}^n \left(\left[\begin{smallmatrix} K_i;0 \\ m_i \end{smallmatrix}\right] K_i^{e_i}\right)$, where $u \in (U_v^-)_{ev}$, $m_i \in \mathbb{N}$ and $e_i \in \{0,1\}$ with m_i+e_i even. The elements $(K_i^{\ell_i} - 1)$ are in the center of the even subalgebra. As in the odd case, let J' be the ideal of $U_v(\mathfrak{b}^-)_{ev} \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$ generated by the elements $(K_i^{\ell_i} - 1)$, $i=1,\dots,n$, and let J be the (two sided) ideal $J' \cap U_v(\mathfrak{b}^-)_{ev}$.

Denote by $V_v(\lambda)_{ev}$ the direct sum $\bigoplus_\mu V_v(\lambda)_\mu$ of all weight spaces corresponding to the weights of the form $\mu=\lambda-2\beta$, $\beta \in Q^+$, and set $R_{v,\,ev} := \bigoplus_{\lambda\in 2P^+}(V_v(\lambda)_{ev})^*$. Proposition 2 can then be reformulated as: The pairing $\Phi_v : U_v(\mathfrak{b}^-)_{ev} \times R_{v,\,ev} \to \tilde{\mathbb{Z}}$ defined by $(u,\xi^\lambda) \to \xi^\lambda(uv_\lambda)$ for $u \in U_v(\mathfrak{b}^-)_{ev}$ and $\xi^\lambda \in (V_v(\lambda)_{ev})^*$, has as radical precisely $(J,0)$, and hence the induced pairing $U_v(\mathfrak{b}^-)_{ev}/J \times R_{v,\,ev} \to \tilde{\mathbb{Z}}$ is non-degenerate. In particular, the induced map $\psi_v : R_{v,\,ev} \to (U_v(\mathfrak{b}^-)_{ev})^*$ is injective, and the image is a subalgebra of $(U_v(\mathfrak{b}^-)_{ev})^*$. The Frobenius maps can also be extended correspondingly: the map defined by $u \mapsto \mathrm{Fr}(u)$ for $u \in U_{v,\,ev}^-$, $K_i^2 \mapsto 1$, $K_i\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right] \mapsto 0$ for m odd, $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right] \mapsto 0$ if m is even and $\ell_i m$, and $\left[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}\right] \mapsto \left(\begin{smallmatrix} H_i^\# \\ m/\ell_i \end{smallmatrix}\right)$ if m is even and $\ell_i|m$, extends Fr to an algebra homomorphism $\mathrm{Fr} : U_v(\mathfrak{b}^-)_{ev} \to U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^{\#-})$. Similarly, the map defined by $u \mapsto \mathrm{Fr}'(u)$ for $u \in U_{\tilde{\mathbb{Z}}}^{\#-}$ and $\left(\begin{smallmatrix} H_i^\# \\ m \end{smallmatrix}\right) \mapsto \left[\begin{smallmatrix} K_i;0 \\ \ell_i m \end{smallmatrix}\right]$ extends Fr' to an algebra homomorphism $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^{\#-}) \to U_v(\mathfrak{b}^-)_{ev}/J$. The proofs are very similar to the proofs above, and hence the details are left to the reader.

3. The dual maps Fr^* and Fr'^*

We assume again that ℓ is an odd integer and moreover coprime to 3 if $\mathfrak{g}$ has simple factors of type $\mathtt{G}_2$. We make some remarks concerning the general case at the end of the section. The ideal J (see Proposition 2) is in the kernel of Fr, so we get an induced map Fr^* between the Hopf dual $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)^*$ of $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$ and the Hopf dual $U_v(\mathfrak{b}^-)^*$ of $U_v(\mathfrak{b}^-)$.

The $U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$-module $V_v(\lambda) \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$, $\lambda \in P^+$, is in general not a simple module. Denote by $L_v(\lambda)$ its simple quotient. By Lustzig [8, Proposition 7.2], E_i, F_i and $K_i - 1$ operate trivially on $L_v(\ell\lambda)$ for $\lambda \in P^+$. Further, as in [8, Proposition 7.5], the Frobenius splittings $\mathrm{Fr}' : U^-_{\tilde{\mathbb{Z}}} \to U^-_v$ and $\mathrm{Fr}' : U^+_{\tilde{\mathbb{Z}}} \to U^+_v$ can be glued together to a surjective homomorphism (in fact an isomorphism) $F : U \simeq U_{\tilde{\mathbb{Z}}} \otimes_{\tilde{\mathbb{Z}}} \mathbb{C} \to U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}/\langle E_i, F_i, K_{i-1}\rangle$, and $L_v(\ell\lambda)$ becomes via F a simple U-module $V(\lambda)$ of highest weight λ. We can also view $L_v(\ell\lambda)$ the other way around: We start with the irreducible U-module $V(\lambda)$ and make it into a $U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$-module $V(\lambda)^{\mathrm{Fr}}$ via the Frobenius homomorphism $\mathrm{Fr} : U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C} \to U$ (cf. Proposition 3). Then, Fr being surjective, $V(\lambda)^{\mathrm{Fr}}$ is an irreducible $U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$-module. It is easy to see that $V(\lambda)^{\mathrm{Fr}}$ is, in fact, isomorphic with $L_v(\ell\lambda)$. For each fundamental weight ω_i $(1 \le i \le n)$, choose an isomorphism $\varphi_i : V(\omega_i)^{\mathrm{Fr}} \simeq L_v(\ell\omega_i)$ such that $\bar{v}_{\omega_i} \in V(\omega_i)$ corresponds to $v_{\ell\omega_i} \in L_v(\ell\omega_i)$ (cf. Section 1 for the notation $\bar{v}_{\omega_i}$ and v_{ω_i}). Since Fr is a Hopf algebra homomorphism, the isomorphisms φ_i give rise to a $U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$-module isomorphism $\varphi_\lambda : V(\lambda)^{\mathrm{Fr}} \simeq L_v(\ell\lambda)$ (for all $\lambda \in P^+$) so that $\bar{v}_\lambda \in V(\lambda)$ corresponds to $v_{\ell\lambda} \in L_v(\ell\lambda)$. In the sequel, we fix such an isomorphism φ_λ for each $\lambda \in P^+$. For any $\lambda \in P^+$ let $L_v(\lambda)_{\tilde{\mathbb{Z}}}$ be the U_v-submodule of $L_v(\lambda)$ generated by v_λ.

We thus get a 'natural' U_v-module isomorphism $L_v(\ell\lambda)_{\tilde{\mathbb{Z}}} \to V_{\tilde{\mathbb{Z}}}(\lambda)^{\mathrm{Fr}}$ and hence the dual map $(V_{\tilde{\mathbb{Z}}}(\lambda)^{\mathrm{Fr}})^* \to (L_v(\ell\lambda)_{\tilde{\mathbb{Z}}})^*$, where $V_{\tilde{\mathbb{Z}}}(\lambda) := V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$.

Define the map $\mathrm{Fr}^\vee : R_{\tilde{\mathbb{Z}}} \to R_v$, as the direct sum of the composite maps $V_{\tilde{\mathbb{Z}}}(\lambda)^* \to (L_v(\ell\lambda)_{\tilde{\mathbb{Z}}})^* \to V_v(\ell\lambda)^*$, where the last map is the dual of the quotient map $V_v(\ell\lambda) \to L_v(\ell\lambda)_{\tilde{\mathbb{Z}}}$, and $R_{\tilde{\mathbb{Z}}} := R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$.

Proposition 4. *The map* $\mathrm{Fr}^\vee$ *is nothing but the restriction of* Fr^* *to* $R_{\tilde{\mathbb{Z}}}$ *under the identification of* $R_{\tilde{\mathbb{Z}}}$ *(resp.* R_v*) as a subalgebra of* $U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)^*$ *(resp.* $U_v(\mathfrak{b}^-)^*$*) induced by the pairing* Φ *(resp.* Φ_v*), cf. Propositions* 1 *and* 2.

Equivalently, for any $X \in U_v(\mathfrak{b}^-)$ *and* $\xi \in R_{\tilde{\mathbb{Z}}}$ *we have*

$$\Phi(\mathrm{Fr}\, X, \xi) = \Phi_v(X, \mathrm{Fr}^\vee \xi). \tag{1}$$

Proof. Equivalence of the two assertions is easy and the identity (1) follows readily from the definition of $\mathrm{Fr}^\vee$. $\square$

From now on, we will denote (by abuse of notation) $\mathrm{Fr}^\vee$ by Fr^* itself.

Similarly the algebra homomorphism Fr' gives rise to the dual map $\mathrm{Fr}'^* : (U_v(\mathfrak{b}^-)/J)^* \to U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)^*$. As above, one proves that the dual map Fr'^* induces in fact a map $R_v \to R_{\tilde{\mathbb{Z}}}$.

To describe this map more explicitly, let $\lambda \in P^+$ be a dominant weight. For the Weyl module $V_v(\ell\lambda)$ for U_v denote by $V_v(\ell\lambda)^{1/\ell}$ the direct sum $\bigoplus_{\mu \in \ell P} V_v(\ell\lambda)_\mu$ of all weight spaces corresponding to the weights in ℓP. If $\mu = \ell\mu_1$ is a weight in ℓP, then so is the weight $\mu \pm n\ell\alpha_i = \ell(\mu_1 \pm n\alpha_i)$. It follows that $V_v(\ell\lambda)^{1/\ell}$ is stable under the action of all the $F_i^{(n\ell)}$ and $E_i^{(n\ell)}$.

We make $V_v(\ell\lambda)^{1/\ell}$ into a $U_{\tilde{\mathbb{Z}}}^-$-module respectively $U_{\tilde{\mathbb{Z}}}^+$-module via the homomorphism Fr' (i.e. by letting $X_i^{(m)}$ act as $E_i^{(\ell n)}$ and $Y_i^{(m)}$ act as $F_i^{(\ell n)}$). A simple calculation (see for example [7] for details) shows that if we let $\binom{H_i}{m}$ act as $\left[\begin{smallmatrix} K_i;0 \\ m\ell \end{smallmatrix}\right]$, then this defines a $U_{\tilde{\mathbb{Z}}}$-module structure on $V_v(\ell\lambda)^{1/\ell}$, and the submodule generated by the highest weight vector $v_{\ell\lambda}$ is isomorphic to $V_{\tilde{\mathbb{Z}}}(\lambda)$. Again we choose an isomorphism so that $v_{\ell\lambda}$ corresponds to $\bar{v}_\lambda$.

Similar to Proposition 4, we obtain:

Proposition 5. *The restriction of the dual map* Fr'^* *to* $V_v(\ell\lambda)^*$ *is the dual map of the inclusion* $V_{\tilde{\mathbb{Z}}}(\lambda) \hookrightarrow V_v(\ell\lambda)$, *and* $\mathrm{Fr}'^*|_{V_v(\mu)^*} = 0$ *for* $\mu \notin \ell P^+$.

Remark 3. Recall that we cannot extend Fr' to an algebra homomorphism on the full enveloping algebra, so $V_v(\ell\lambda)$ is not naturally endowed with a structure as a $U_{\tilde{\mathbb{Z}}}$-module. The inclusion $V_{\tilde{\mathbb{Z}}}(\lambda) \hookrightarrow V_v(\ell\lambda)$ hence does not give rise to a $U_{\tilde{\mathbb{Z}}}$-equivariant map $V_v(\ell\lambda)^* \to V_{\tilde{\mathbb{Z}}}(\lambda)^*$. But, using the Frobenius maps $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^- \to U_v^-$ and $\mathrm{Fr}' : U_{\tilde{\mathbb{Z}}}^+ \to U_v^+$, we can make $V_v(\ell\lambda)$ into a $U_{\tilde{\mathbb{Z}}}^-$-respectively $U_{\tilde{\mathbb{Z}}}^+$-module, and, by the definition of the inclusion $V_{\tilde{\mathbb{Z}}}(\lambda) \hookrightarrow V_v(\ell\lambda)^{1/\ell} \hookrightarrow V_v(\ell\lambda)$, the map $V_{\tilde{\mathbb{Z}}}(\lambda) \hookrightarrow V_v(\ell\lambda)$ is equivariant with respect to the action of $U_{\tilde{\mathbb{Z}}}^+$ and $U_{\tilde{\mathbb{Z}}}^-$, and hence so is the dual map.

Remark 4. The composition

$$U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-) \xrightarrow{\mathrm{Fr}'} U_v(\mathfrak{b}^-)/J \xrightarrow{\mathrm{Fr}} U_{\tilde{\mathbb{Z}}}(\mathfrak{b}^-)$$

is the identity map, and hence so is $R_{\tilde{\mathbb{Z}}} \xrightarrow{\mathrm{Fr}^*} R_v \xrightarrow{\mathrm{Fr}'^*} R_{\tilde{\mathbb{Z}}}$.

Remark 5. If $\ell = 2d$, then Fr^* induces a map $R_{\tilde{\mathbb{Z}}} \to R_{v,\,ev}$, which is the direct sum of the duals of the quotient maps $V_v(\lambda) \to L_v(\lambda)_{\tilde{\mathbb{Z}}} = V_{\tilde{\mathbb{Z}}}(\lambda^\#)$, for $\lambda^\# \in P^\#$, and the restriction of the dual map Fr'^* to $V_v(\lambda)^*$ is the dual map of the inclusion $V_{\tilde{\mathbb{Z}}}(\lambda^\#) \hookrightarrow V_v(\lambda)$, and $\mathrm{Fr}'^*|_{V_v(\mu)^*} = 0$ for $\mu^\# \notin P^\#$. To see this, let $\lambda^\# \in P^\# \subset P$ be a dominant weight, we write just λ for the weight if we view it as a U_v-weight. We make $V_{\tilde{\mathbb{Z}}}(\lambda^\#)$ as above into a U_v^-- and U_v^+-module by using the Frobenius map Fr, and we let $[\begin{smallmatrix} K_i;0 \\ m \end{smallmatrix}]$ act on $V_{\tilde{\mathbb{Z}}}(\lambda^\#)$ as $(\begin{smallmatrix} H_i^\# \\ m/\ell_i \end{smallmatrix})$ if m is divisible by ℓ_i, and as 0 if $\ell_i m$.

Then as above, the three actions glue together to give a U_v-module structure on $V_{\tilde{\mathbb{Z}}}(\lambda^\#)$ such that $u \in J$ acts trivially. Thus $V_{\tilde{\mathbb{Z}}}(\lambda^\#)$ becomes in this way a highest weight module for U_v of heighest weight λ. Now $V(\lambda^\#)$ is a simple module for $U(\mathfrak{g}^\#)$ and hence for $U_v \otimes_{\tilde{\mathbb{Z}}} \mathbb{C}$. So, as above, we can view Fr^* as the dual map of the quotient map $V_v(\lambda) \to L_v(\lambda) = V_{\tilde{\mathbb{Z}}}(\lambda^\#)$.

Actually, with the appropriate adaptions (for details see [10, Chapter 35]) one can reformulate the results for arbitrary ℓ.

4. Base change

In the following we assume that $\ell = p$ is in fact an odd prime, and further $p > 3$ if $\mathfrak{g}$ has factors of type $\mathtt{G}_2$. Let k be an algebraically closed field of char. p. We consider k as a $\tilde{\mathbb{Z}}$-module by extending the canonical map $\mathbb{Z} \to k$ to a ring homomorphism $\tilde{\mathbb{Z}} \to k$ (where the first map is given by the projection $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ and the inclusion $\mathbb{Z}/p\mathbb{Z} \subset k$, and the extension $\tilde{\mathbb{Z}} \to k$ follows from the 'Going-up theorem' [12, Theorem 9.3]). We denote by $U_k, U_{v,k}$ the corresponding algebras obtained by the base change.

Note that the image of v in k is 1. Let J_p be the ideal in $U_{v,k}$ generated by the central elements $(K_i - 1)$, $i = 1, \dots, n$; then the quotient $U_{v,k}/J_p$ is naturally isomorphic to U_k. Further, let $\mu \in P^+$ be a dominant weight and $V_{u,k}(\mu)$ be the corresponding Weyl module for $U_{v,k}$. Since all the $(K_i - 1)$ operate trivially on $V_{v,k}(\mu)$, this becomes in a natural way the Weyl module $V_k(\mu)$ for U_k. Let $L_k(\mu)$ be the U_k-module $L_v(\mu)_{\tilde{\mathbb{Z}}} \otimes_{\tilde{\mathbb{Z}}} k$.

We are now left with only one algebra, namely U_k. The module $L_k(p\lambda)$ is, as a U_K-module the same as the U_k-module $V_k(\lambda)^{\mathrm{Fr}}$, where as in Section 3, $V_k(\lambda)^{\mathrm{Fr}}$ is the same k-vector space as $V_k(\lambda)$, but its U_k-module structure has been twisted via the Frobenius map $\mathrm{Fr} : U_k \to U_k$ given by $F_i^{(m)} \mapsto F_i^{(m/p)}$: $E_i^{(m)} \mapsto E_i^{(m/p)}$. If m is divisible by p and 0 otherwise.

We are going to twist the k-vector space structure of $V_k(\lambda)^{\mathrm{Fr}}$. Let $\phi : k \mapsto k$ be the ring homomorphism given by the inverse of the pth power map, i.e., $z \mapsto z^{1/p}$, and denote by $V_k(\lambda)^{(1)}$ the k-vector space (and U_k-module) having as underlying abelian group the same as $V_k(\lambda)$, but where the scalar multiplication has been twisted by ϕ: $a * v := \phi(a)v$, and where U_k acts as on $V_k(\lambda)^{\mathrm{Fr}}$. (Note that the operation of U_k is linear also with respect to the twisted scalar multiplication.) The U_k-module $V_k(\lambda)^{(1)}$ can be seen explicitly as a quotient of $V_k(p\lambda)$ as follows: The map $V_k(\lambda)^{(1)} \to S^p V_k(\lambda)$, defined by $v \mapsto v^p$ (which is linear because of the twisted scalar multiplication), induces an isomorphism onto the image of the canonical map $V_k(p\lambda) \to S^p V_k(\lambda)$ which sends the highest weight vector $v_{p\lambda} \in V_k(p\lambda)$ to the highest weight vector $v_\lambda^p \in S^p V_k(\lambda)$.

Let G be the semisimple simply connected algebraic group over k corresponding to the Lie algebra $\mathfrak{g}$ and let $B \subset G$ be the Borel subgroup corresponding to the Lie algebra $\mathfrak{b}$. Let $\mathscr{L}_\lambda$ be the line bundle on $X := G/B$ corresponding to a weight $-\lambda$. Recall that for λ dominant, we have $H^0(X, \mathscr{L}_\lambda)$, as G-module, isomorphic to $V_k(\lambda)^*$. It follows from the considerations above that the dual map $\mathrm{Fr}^*: H^0(X, \mathscr{L}_\lambda)^{(1)} \to H^0(X, \mathscr{L}_{p\lambda})$ is just the pth power map sending a section $s \in H^0(X, \mathscr{L}_\lambda)$ to $s^p \in H^0(\mathscr{L}_{p\lambda})$ (again, recall that this map is linear with respect to the twisted scalar multiplication).

The inclusion $V_K(\lambda) \hookrightarrow V_k(p\lambda)$ respectively $\mathrm{Fr}'^*: H^0(X, \mathscr{L}_{p\lambda}) \to H^0(X, \mathscr{L}_\lambda)$, the associated dual map, does not have such an equivariant interpretation, but Remark 4 implies that Fr'^* is a section to Fr^*. Observe that Fr'^* restricted to $H^0(X, \mathscr{L}_\lambda)$ is zero if $\lambda \notin pP^+$.

Theorem 1. *The dual map* $\mathrm{Fr}^*: H^0(X, \mathscr{L}_\lambda) \to H^0(X, \mathscr{L}_\lambda)$ *is the map* $s \mapsto s^p$ *sending a section to its pth power*, *and* $\mathrm{Fr}'^*: H^0(X, \mathscr{L}_{p\lambda}) \to H^0(X, \mathscr{L}_\lambda)$ *provides a splitting of*

this map. For any $S \in H^0(X, \mathscr{L}_{j\lambda})$ *and* $f \in H^0(X, \mathscr{L}_{m\lambda})$, *the Frobenius map satisfies the following properties*:
(a) $\mathrm{Fr}'^*(s^p f) = s\,\mathrm{Fr}'^*(f)$, *and*
(b) $\mathrm{Fr}'^*(X_i^{(pq)} f) = X_i^{(q)}\,\mathrm{Fr}'^*(f)$ *for all* $1 \le i \le n$, *and* $q \in \mathbb{N}$.

Remark 6. For notational convenience assume that $\lambda \notin pP^+$. The property (a) implies that Fr'^* induces a graded Frobenius endomorphism of the graded algebra $S := \bigoplus_{m \ge 0} H^0(X, \mathscr{L}_{m\lambda})$, more specifically, Fr'^* maps the homogeneous elements of degree not divisible by p to zero and if f is of degree qp then $\mathrm{Fr}'^*(f)$ is of degree q, the map is additive: $\mathrm{Fr}'^*(s_1 + s_2) = \mathrm{Fr}'^*(s_1) + \mathrm{Fr}'^*(s_2)$, and $\mathrm{Fr}'^*(s_1^p s_2) = s_1 \mathrm{Fr}'^*(s_2)$. The second property implies that Fr'^* is the *canonical* splitting, see [11]. In particular, Fr'^* maps B-modules to B-modules.

Proof. It remains to prove that the two properties (a) and (b) hold. For notational convenience assume that $\lambda \notin pP^+$. If m is not divisible by p, then in both the equalities all the terms on the right and left are zero, so the properties hold in this case trivially. Suppose now that m is divisible by p, say $m = pq$. Again, both the properties hold trivially if f is a weight vector of a weight not divisible by p, so in the following we may assume that f is a weight vector corresponding to a weight divisible by p. The element f is hence an element of $(V_k(pq\lambda)^{1/p})^*$. Recall that the embedding $\imath : V_k(q\lambda) \hookrightarrow V_k(pq\lambda)^{1/p}$ satisfies $\imath(X_i^{(s)} v) = X_i^{(ps)} \imath(v)$, so the corresponding property holds also for the dual map $\mathrm{Fr}'^* : H^0(X, \mathscr{L}_{pq\lambda}) \to H^0(X, \mathscr{L}_{q\lambda})$. This implies the second property in the theorem above.

Abbreviate the module $L_v(\lambda)_{\tilde{\mathbb{Z}}}$ by $L_v(\lambda)$. To prove the first property, consider the following diagram of Weyl modules (for $U_{\tilde{\mathbb{Z}}}$ respectively U_v) defined over $\tilde{\mathbb{Z}}$: There are two inclusions of $U_{\tilde{\mathbb{Z}}}$-modules using the Frobenius map: $V_{\tilde{\mathbb{Z}}}((q+j)\lambda) \hookrightarrow V_v(p(q+j))\lambda)^{1/p}$, and the other inclusion is $V_{\tilde{\mathbb{Z}}}(j\lambda) \otimes V_{\tilde{\mathbb{Z}}}(q\lambda) \hookrightarrow L_v(pj\lambda) \otimes V_{\tilde{\mathbb{Z}}}(pq\lambda)^{1/p}$, using the fact that $L_v(pj\lambda) = V_{\tilde{\mathbb{Z}}}(j\lambda)^{\mathrm{Fr}}$ as U_v-module. Note that $E_i^{(mp)}$ acts on $V_{\tilde{\mathbb{Z}}}(j\lambda)$ as $X_i^{(m)}$, so $\mathrm{Fr}'(X_i^{(m)})$ acts on $L_v(pj\lambda) = V_{\tilde{\mathbb{Z}}}(j\lambda)$ as $X_i^{(m)}$. Then we have two inclusions of $U_{\tilde{\mathbb{Z}}}^-$-respectively $U_{\tilde{\mathbb{Z}}}^+$-modules which act on the U_v-modules via Fr': The inclusions are $V_v(p(q+j)\lambda)^{1/p} \hookrightarrow V_v(p(q+j)\lambda)$ and $L_v(pj\lambda) \otimes V_{\tilde{\mathbb{Z}}}(pq\lambda)^{1/p} \hookrightarrow L_v(pj\lambda) \otimes V_{\tilde{\mathbb{Z}}}(pq\lambda)$. In addition we have two maps between Weyl modules: $V_{\tilde{\mathbb{Z}}}((j+q)\lambda) \to V_{\tilde{\mathbb{Z}}}(j\lambda) \otimes V_{\tilde{\mathbb{Z}}}(q\lambda)$ and $V_v(p(j+q)\lambda) \to V_v(pj\lambda) \otimes V_v(pq\lambda)$:

$$
\begin{array}{ccccccc}
 & & V_v(p(q+j)\lambda)^{1/p} & \hookrightarrow & V_v(p(q+j)\lambda) & \longrightarrow & V_v(pj\lambda) \otimes V_v(pq\lambda) \\
 & \nearrow & & & & & \downarrow \\
V_{\tilde{\mathbb{Z}}}((j+q)\lambda) & & & & & & \\
 & \searrow & & & & & \\
 & & V_{\tilde{\mathbb{Z}}}(j\lambda) \otimes V_{\tilde{\mathbb{Z}}}(q\lambda) & \hookrightarrow & L_v(pj\lambda) \otimes V_v(pq\lambda)^{1/p} & \hookrightarrow & L_v(pj\lambda) \otimes V_v(pq\lambda)
\end{array}
$$

The vertical map is the identity on the second factor and the projection on the first. All these maps are equivariant with respect to the $U_{\tilde{\mathbb{Z}}}^{\pm}$-actions on these spaces, they all map the highest weight vector (resp. the tensor product of highest weight vectors) to a highest weight vector (resp. the tensor product of highest weight vectors). It follows that the diagram is commutative and provides two different ways to construct a map $V_{\tilde{\mathbb{Z}}}((j+q)\lambda) \to L_v(pj\lambda) \otimes V_v(pq\lambda)$.

Over the field k, the dual of the bottom row is the map $H^0(X, \mathscr{L}_{j\lambda}) \otimes H^0(X, \mathscr{L}_{pq\lambda}) \to H^0(X, \mathscr{L}_{(q+j)\lambda})$ defined by $s \otimes f \mapsto s\,\mathrm{Fr}'^*(f)$ for sections $s \in H^0(X, \mathscr{L}_{j\lambda})$ and $f \in H^0(X, \mathscr{L}_{pq\lambda})$.

The dual of the top row provides a decomposition of this map in the following way: $s \otimes f \in H^0(X, \mathscr{L}_{j\lambda}) \otimes H^0(X, \mathscr{L}_{pq\lambda})$ is first mapped to $s^p \otimes f \in H^0(X, \mathscr{L}_{pj\lambda}) \otimes H^0(X, \mathscr{L}_{pq\lambda})$, then to the product $s^p f \in H^0(X, \mathscr{L}_{p(j+q)\lambda})$, and then to $\mathrm{Fr}'^*(s^p f) \in H^0(X, \mathscr{L}_{(j+q)\lambda})$. Since the two maps are the same, it follows $\mathrm{Fr}'^*(s^p f) = s\mathrm{Fr}'^*(f)$. This proves (b). □

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