Frobenius splitting in characteristic zero and the quantum Frobenius map

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Abstract

The aim of the article is to show that Lusztig’s Frobenius map (for quantum groups at roots of unity) can be, after dualizing, viewed as a characteristic zero lift of the geometric Frobenius splitting of $G/B$ (in char $p > 0$) introduced by Mehta and Ramanathan. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

In the representation theory of a semisimple algebraic group $G$, the Schubert varieties $X(w)$, $w$ an element of the Weyl group $W$ of $G$, play a prominent rôle. An important breakthrough in the analysis of the geometry of these subvarieties of the flag variety $G/B$ was the introduction by Mehta–Ramanathan of the notion of a Frobenius split variety and compatibly split subvarieties (for varieties defined over a field of char. $p > 0$). They proved [13] that $G/B$ (more generally any $G/P$ for a parabolic subgroup $P$) is a Frobenius split variety such that all of the Schubert subvarieties are compatibly split, in particular, one immediately obtains the Kodaira–Kempf...
vanishing theorem. Using this tool, it was shown for example that Schubert varieties are normal, Cohen–Macaulay and have rational singularities. Moreover, they are projectively normal, projectively Cohen–Macaulay, and are defined by quadratic relations in any embedding given by an ample homogeneous line bundle on $G/B$ (cf. [14–16]). Of course, as is well known, the normality of Schubert varieties is equivalent to the validity of the Demazure’s character formula. Further, the Frobenius splitting was used by Mathieu to give a uniform proof that the category of finite-dimensional $G$-representations (over char. $p > 0$) admitting a good filtration is stable under tensor product and more generally under the restriction to the semisimple part of a Levi subgroup (cf. [2,11]).

Earlier, a different way to analyze the geometry of Schubert varieties was suggested by Seshadri and his school. They proposed to construct a standard monomial theory for the homogeneous coordinate ring of an embedding $G/B$, and also naturally by standard monomial theory, see [4,6,7].

The aim of this article is to systematically begin unifying these two approaches. For the Borel subgroup $B$ of $G$, let $b$ be its Lie algebra over the complex numbers $\mathbb{C}$ and $b^-$ be the opposite Borel subalgebra. We first establish a duality between the algebra $U_{\mathbb{Z}}(b^-)$ (resp. its quantum analog $U_q(b^-)$), and the direct sum of the dual modules of all Weyl modules $\bigoplus_{\lambda \in P^-} V_\lambda(\lambda)^*$ (resp. its quantum analog $\bigoplus_{\lambda \in P^-} V_\lambda(\lambda)^*$) (cf. Propositions 1 and 2), where $v$ as earlier is an $\ell$th root of unity, $U_{\mathbb{Z}}(b^-)$ is the Kostant’s $\mathbb{Z}$-lattice of the enveloping algebra $U(b^-)$, $U_q(b^-)$ is the Lusztig’s $\mathbb{Z}$-lattice of the quantized algebra $U_q(b^-)$ and $\mathbb{Z}$ is the ring obtained from $\mathbb{Z}$ by adjoining all the roots of unities. Now Lusztig defined for $\ell$ an odd integer ($\ell$ coprime to 3 if $G_2$ is a factor of $G$) a certain Frobenius homomorphism $Fr : U_q(b^-) \to U_{\mathbb{Z}}(b)$ and also a certain splitting of it on the ‘$n^-$-part’ (which we shall refer to as Frobenius splitting homomorphism) $Fr' : U_{\mathbb{Z}}(n^-) \to U_q(n^-)$, where $U_{\mathbb{Z}}(b^-) := U_{\mathbb{Z}}(b^-) \otimes_{\mathbb{Z}} \mathbb{Z}$, $n^-$ is the nilradical of $b^-$, and $U_{\mathbb{Z}}(n^-), U_q(n^-)$ have meaning similar to that of the corresponding $b^-$. We extend the definition of $Fr'$ to $U_{\mathbb{Z}}(b^-)$ (cf. Lemma 3). By using the duality mentioned above, we get maps $Fr^* : \bigoplus_{\lambda \in P^-} V_{\mathbb{Z}}(\lambda)^* \to \bigoplus_{\lambda \in P^-} V_\lambda(\lambda)^*$ respectively $Fr' : \bigoplus_{\lambda \in P^-} V_{\mathbb{Z}}(\lambda)^* \to \bigoplus_{\lambda \in P^-} V_\lambda(\lambda)^*$ (cf. Propositions 4 and 5).

Next we assume that $\ell$ is a prime, and show that, after a base change with an algebraically closed field of char. $p = \ell$, the map $Fr^*$ has a natural interpretation as the $p$th power map on the space of sections $H^0(G/B, \mathcal{I}_\lambda)$, and $Fr'^*$ can naturally be interpreted as a splitting of this map. This is our principal result of the paper (cf. Theorem 1). So on the level of quantum groups, the map $Fr'^*$ can be considered as a
char. zero lift of the Frobenius splitting of \(G/B\) in char. \(p\), and it is exactly this map which has been used in [7] to define the ‘\(t\)h root’ of certain sections.

In a subsequent paper we will use Lusztig’s \(F_r\) (resp. \(F_r^\ast\)) to define the Frobenius map (resp. Frobenius splitting map) at ‘higher cohomology’ level as well and show that \(F_r^\ast\) after base change provides a canonical (in the sense of Mathieu) splitting of \(G/B\) compatibly splitting all of the Schubert subvarieties. In particular, this will provide a purely algebraic proof (via the quantum groups at roots of unity) of results of Mehta–Ramanathan mentioned above and also various results on the geometry of Schubert varieties mentioned in the first paragraph.

1. A pairing and its quantum analogue

For a complex semisimple Lie algebra \(\mathfrak{g}\), associated to a Cartan matrix \(C = (c_{ij})_{1 \leq i, j \leq n}\), fix a Borel subalgebra \(\mathfrak{b}\) and a Cartan subalgebra \(\mathfrak{h} \subset \mathfrak{b}\). Let \(\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+\) be the associated triangular decomposition and set \(b^- = n^- \oplus \mathfrak{h}\). We write \(U, U^+, U^-\) for the enveloping algebras \(U(\mathfrak{g}), U(\mathfrak{n}^+), U(\mathfrak{n}^-)\), respectively.

Let \(\omega_1, \ldots, \omega_n\) be the set of fundamental weights in the weight lattice \(P\) of \(\mathfrak{g}\). Denote by \(P^+\) the set of non-negative linear combinations \(\mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_n\), \(\mathbb{N}\) being the set of non-negative integers. We write \(V(\lambda)\) for the complex irreducible representation of highest weight \(\lambda \in P^+\). As an \(\mathfrak{h}\)-module, \(V(\lambda)\) decomposes into the direct sum \(\bigoplus_{\lambda \in P^+} V(\lambda)_{\lambda}\) of weight spaces. Here \(V(\lambda)_{\lambda} = \{v \in V(\lambda) \mid hv = \mu(h)v \ \forall h \in \mathfrak{h}\}\).

Let \(a_1, \ldots, a_n\) be the set of simple roots (corresponding to the Borel subalgebra \(\mathfrak{b}\)). For a simple root \(\alpha_i\) choose \(X_i \in \mathfrak{g}_{\alpha_i}\), \(Y_i \in \mathfrak{g}_{-\alpha_i}\) and \(H_i \in \mathfrak{h}\) such that \([H_i, X_j] = a_{ij}X_j, [H_i, Y_j] = -a_{ij}Y_j, [X_i, Y_j] = \delta_{i,j}H_i\). We denote by \(U_Z\) the Kostant form of \(U\) over the ring of integers \(\mathbb{Z}\), generated by the divided powers \(X_i^{(k)} := X_i^k / k!\) and \(Y_i^{(k)} := Y_i^k / k!\), \(k \geq 0\). Let \(U_Z^+, U_Z^-\) be the corresponding Kostant forms of \(U^+\) and \(U^-\), generated respectively by \(\{X_i^{(k)}\}\) and \(\{Y_i^{(k)}\}\), and let \(U_Z(b^-)\) be the Kostant form of \(U(b^-)\) generated by \(\{Y_i^{(k)}(H_i^k)\}\), where \((H_i^k) = H_i(H_i - 1) \cdots (H_i - k + 1) / k!\). Then \(U_Z(b^-) \subset U_Z\).

For \(i = 1, \ldots, n\), fix a highest weight vector \(v_{\omega_i} \in V(\omega_i)_{\omega_i}\) and let \(V_Z(\omega_i) := U_Z v_{\omega_i}\) be the corresponding \(U_Z\)-stable \(\mathbb{Z}\)-lattice. For \(\lambda = \sum a_i \omega_i \in P^+\) denote by \(v_\lambda\) the vector \(v_{\omega_1} \otimes \cdots \otimes v_{\omega_n}\), and let \(V_Z(\lambda)\) be the lattice

\[U_Z v_\lambda := U_Z(v_{\omega_1} \otimes \cdots \otimes v_{\omega_n}) \subset V_Z(\omega_1) \otimes \cdots \otimes V_Z(\omega_n)\]

The dual module \(\text{Hom}_\mathbb{Z}(V_Z(\lambda), \mathbb{Z})\) is denoted by \(V_Z(\lambda)^*\). Setting \(R_Z := \bigoplus_{\lambda \in P^+} V_Z(\lambda)^*\), we have a natural pairing

\[\Phi : U_Z(b^-) \times R_Z \rightarrow \mathbb{Z}, \quad \Phi(u, \xi) := \xi(u, v_\lambda) \quad \text{for} \ u \in U_Z(b^-), \ \xi \in V_Z(\lambda)^*.\]

Denote by \(U(b^-)^*\) the Hopf dual of \(U(b^-)\) (under the standard Hopf algebra structure of \(U(b^-)\)), i.e., \(U(b^-)^*\) is the subspace of linear forms \(f \in \text{Hom}(U(b^-), \mathbb{C})\) for which there exists a two-sided ideal \(I \subset U(b^-)\) of finite codimension such that \(f(I) = 0\). Recall that \(U(b^-)^*\) is again a Hopf algebra. Let \(U_Z(b^-)^*\) be the \(\mathbb{Z}\)-submodule of forms \(f \in U(b^-)^*\) such that \(f(U_Z(b^-)) \subset \mathbb{Z}\). Note that \(U_Z(b^-)^*\) is a \(\mathbb{Z}\)-subalgebra of \(U(b^-)^*\).
The following proposition can be viewed as a certain ‘half’ of the Peter–Weyl-type theorem. One may also consider it as an algebraic analog of the result of Bernstein, Gelfand and Gelfand that the ring $\mathbb{C}[G/U]$ of regular functions on $G/U$ is isomorphic to $\bigoplus_{\lambda \in P^+} V(\lambda)^*$, where $G$ is the corresponding simply connected complex algebraic group with Borel subgroup $B$ and unipotent radical $R_u(B) = U$ having $n^+$ as its Lie algebra.

**Proposition 1.** $\Phi$ is a non-degenerate pairing, identifying $R_Z$ as a subalgebra of $U_Z(b^-)^*$:

$$\xi^\lambda \cdot \xi^\mu(u) := (\xi^\lambda \otimes \xi^\mu)|_{V(\lambda + \mu)}(u \cdot \tilde{v}_{\lambda + \mu})$$

for $\xi^\lambda \in V_2(\lambda)^*$, $\xi^\mu \in V_2(\mu)^*$ and $u \in U(b^-)$.

**Proof.** Denote by $Q$ the root lattice and set $Q^+ = \mathbb{N}z_1 + \cdots + \mathbb{N}z_n$. The Kostant form $U^0_Z$ of the enveloping algebra $U(h)$ has as basis the monomials $\left( \frac{H_i}{k_i} \right) \cdots \left( \frac{H_n}{k_n} \right)$, $k_i \in \mathbb{N}$. Fix a $Z$-basis $B$ of $U^0_Z$ such that for any $\beta \in Q^+$ the elements in $B^\beta$, form a basis of the weight space $V_Z(\lambda)_\beta$. So if we choose $\lambda$ big enough, $u \tilde{v}_\lambda = 0$ implies $\lambda(h_b) = 0$ for all $b$ and all $\lambda \gg 0$. But this is possible only if $h_b = 0$.

Consider $f = \sum \xi^j$. Among the $\xi^j \neq 0$ fix $\xi^{i_0}$ such that $\lambda_0$ is maximal in the lexicographic ordering, i.e., if $\lambda_0 = \sum a_i z_i$ and $\lambda = \sum b_i z_i$ is such that $\xi^j \neq 0$, then there exists a $j \leq n$ such that $a_i = b_i$ for $i < j$ and $a_j > b_j$. Set $H_{\lambda_0} := \prod_{i=0}^n \left( \frac{H_i}{a_i} \right)$. Note that $H_{\lambda_0} \tilde{v}_{\lambda_0} = \tilde{v}_{\lambda_0}$ and $H_{\lambda_0} \tilde{v}_j = 0$ for all $\lambda$ such that $\xi^{j_0} \neq 0$, $\lambda \neq \lambda_0$. Since $V_Z(\lambda_0) = U^0_Z \tilde{v}_{\lambda_0}$, we can find $u \in U^0_Z$ such that $\xi^{i_0}(u \tilde{v}_{\lambda_0}) \neq 0$. It follows that $\Phi(u \tilde{v}_{\lambda_0}, f) = \sum_{i=0}^n \xi^i(u \tilde{v}_{\lambda_0}) = \xi^{i_0}(u \tilde{v}_{\lambda_0}) \neq 0$. This proved that $\Phi$ is non-degenerate.

To see that $R_Z$ form a subalgebra of $U_Z(b^-)$, note that the co-product $\Delta$ induces a natural $U_Z(b^-)$-module structure on $V_Z(\lambda) \otimes V_Z(\mu)$. By the definition of the product we have $\xi^\lambda \cdot \xi^\mu(u) = \xi^\lambda \otimes \xi^\mu(\Delta(u))$. For $\Delta(u) = \sum u_1 \otimes u_2$ we have $\xi^\lambda \otimes \xi^\mu(\Delta(u)) = \sum \xi^\lambda(u_1 \tilde{v}_{\lambda}) \cdot \xi^\mu(u_2 \tilde{v}_{\mu})$.

Now the map $u \tilde{v}_{\lambda + \mu} \mapsto u(\tilde{v}_{\lambda} \otimes \tilde{v}_{\mu})$, $u \in U(b^-)$, induces an isomorphism between $V_Z(\lambda + \mu)$ and the $U_Z$-submodule $U_Z(b^-)(\tilde{v}_{\lambda} \otimes \tilde{v}_{\mu})$ of $V_Z(\lambda) \otimes V_Z(\mu)$. The restriction map induces a map res: $V_Z(\lambda)^* \otimes V_Z(\mu)^* \mapsto V_Z(\lambda + \mu)^*$. It follows that $\xi^\lambda \cdot \xi^\mu(u) = \text{res}(\xi^\lambda \otimes \xi^\mu)(u \tilde{v}_{\lambda + \mu})$. □

**Remark 1.** By using the Peter–Weyl theorem, we get an isomorphism $\mathbb{C}[G/U] \simeq \bigoplus_{\lambda \in P^+} V(\lambda)^*$. Let $B^-$ be the opposite Borel subgroup (Lie $B^- = b^-$). Since $B^-$ is open and dense in $G/U$, we have an inclusion $\mathbb{C}[G/U] \hookrightarrow \mathbb{C}[B^-]$. But by [3, Part I, Sections 7.10 and 7.18], we have $\mathbb{C}[B^-] \simeq U(b^-)^*$, and hence we have $\bigoplus_{\lambda \in P^+} V(\lambda)^* \hookrightarrow U(b^-)^*$. So this gives an alternative geometric derivation of the above proposition over $\mathbb{C}$. 
We have a similar construction for the quantum group $U_q := U_q(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$. Let $d_1, \ldots, d_n \geq 1$ be minimal integers such that $(d_i c_i, j_i)$ is a symmetric matrix, and let $\overline{\mathbb{Z}}$ be the ring obtained from $\mathbb{Z}$ by adjoining all roots of unities.

We fix a positive integer $\ell$. If $\ell$ is odd, then let $\phi$ be the $\ell$-cyclotomic polynomial, and if $\ell$ is even, then let $\phi$ be the $2\ell$-cyclotomic polynomial. Let $A = \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials and fix a homomorphism $A(\phi) \rightarrow \overline{\mathbb{Z}}$, where $(\phi)$ is the ideal in $A$ generated by $\phi$. Denote by $\mathfrak{v}$ the image of $q$ in $\overline{\mathbb{Z}}$.

We denote the generators of the quantum group $U_q$ over $\mathbb{C}(q)$ by $E_i, F_i, K_i^\pm$, and let $U_A$ be the Lusztig form of $U_q$ over $A$ generated by the divided powers $E_i^{(m)} := E_i^m/[m]!$ and $F_i^{(m)} := F_i^m/[m]!$ and the $K_i^{\pm 1}$ [9, Section 1]. Recall that the Gaussian numbers $[m]_i$ are defined by $[m]_i := (q^{d_i i m} - q^{-d_i i m})/(q^{d_i i} - q^{-d_i i})$, and $[m]_i ! := [1]_i \cdots [m]_i$.

We denote by $U^+_A, U^-_A, U_A^0$ the subalgebras of $U_A$ generated by the $E_i^{(m)}, F_i^{(m)}$, and the $\{K_i^+, K_i, K_i^-, \ [m]_i \}$ respectively for $1 \leq i \leq n$ and $m \in \mathbb{N}$. Recall that the latter is defined by

$$
\begin{bmatrix}
K_i; c \\
m
\end{bmatrix} := \prod_{s=1}^{m} \frac{K_i q^{(c-s+1)d_i} - K_i^{-1} q^{(-c+s-1)d_i}}{q^{d_i} - q^{-d_i}} \quad \text{for } c \in \mathbb{Z} \text{ and } m \in \mathbb{N}
$$

and $U_A^0$ has as a basis the monomials of the form $\prod_{i=1}^{n} \left( \frac{[K_i; 0]}{[m]_i} K_i^{0} \right)$, where the $m_i$ are non-negative integers and $e_1, \ldots, e_n \in \{0, 1\}$ [9, Theorem 6.7(c)]. Let $U_A(b^-)$ be the subalgebra generated by the $F_i^{(m)}, K_i^{-}$ and the $\frac{[K_i; 0]}{[m]_i}$. The following statements can be found in [8] or [1], or can be easily deduced from [9, Section 6.4]

**Lemma 1.**

(a) \[
\begin{bmatrix}
K_i; 0 \\
m
\end{bmatrix} \begin{bmatrix}
K_i; -m \\
t
\end{bmatrix} = \begin{bmatrix}
m + t \\
m
\end{bmatrix} \begin{bmatrix}
K_i; 0 \\
m + t
\end{bmatrix},
\]

(b) \[
\begin{bmatrix}
K_i; c \\
\ell
\end{bmatrix} = \sum_{j=0}^{t} \begin{bmatrix}
t \\
j
\end{bmatrix} q^{d_i (c - j)} K_i^{-j} \begin{bmatrix}
K_i; c - t \\
t - j
\end{bmatrix} \quad \text{for any } t \leq \ell,
\]

(c) \[
\begin{bmatrix}
K_i; c \\
t
\end{bmatrix} = \sum_{s=0}^{c} q^{d_i (c-s)} \begin{bmatrix}
c \\
s
\end{bmatrix} K_i^{s} \begin{bmatrix}
K_i; 0 \\
t - s
\end{bmatrix} \quad \text{for } c > 0,
\]

(d) \[
\begin{bmatrix}
K_i; c \\
x
\end{bmatrix} \begin{bmatrix}
K_i; 0 \\
y
\end{bmatrix} = \sum_{j=0}^{x} \begin{bmatrix}
x \\
j
\end{bmatrix} \begin{bmatrix}
x + y - j \\
x
\end{bmatrix} q^{d_i x y} K_i^{-j} \begin{bmatrix}
K_i; 0 \\
x + y - j
\end{bmatrix},
\]

where

$$
\begin{bmatrix}
f \\
j
\end{bmatrix} := \prod_{s=1}^{j} \frac{q^{(f-s+1)d_i} - q^{(f-t+s-1)d_i}}{q^{d_i} - q^{-d_i}} \quad \text{for } t \in \mathbb{Z} \text{ and } j \in \mathbb{N}.
$$
We denote by $U_v, U_v^0, U_v^1, U_v^-$ and $U_v(b^-)$ the algebras over $\hat{\mathbb{Z}}$ obtained from the corresponding forms defined over $A$ by base change $A \rightarrow A/(\phi) \rightarrow \hat{\mathbb{Z}}$. For $\lambda \in P^+$ let $V_\lambda(\lambda)$ be the irreducible representation of $U_q$ over $\mathbb{C}(q)$ with highest weight $\lambda$. As in the classical case, we fix for $i = 1, \ldots, n$ a highest weight vector $v_{oi} \in V_q(\alpha_i)_{o_i}$, and let $V_A(o_i) := U_A v_{oi}$ be the corresponding $A$-lattice. For $\lambda = \sum a_i \alpha_i \in P^+$ denote by $v_\lambda$ the vector $v_{o_1}^{a_1} \otimes \cdots \otimes v_{o_n}^{a_n}$, and let $V_A(\lambda)$ be the lattice

$$U_A v_\lambda := U_A (v_{o_1}^{a_1} \otimes \cdots \otimes v_{o_n}^{a_n}) \rightarrow V_A(o_1)^{\otimes a_1} \otimes \cdots \otimes V_A(o_n)^{\otimes a_n}.$$ 

Then $U_A v_\lambda$ is indeed $A$-free. We denote by $V_\lambda(\lambda)$ the corresponding representation $V_A(\lambda) \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}$ of $U_A$. Recall that $K_i$ acts on a weight vector $v_\mu \in V_\mu(\lambda)_i$ by multiplication with $v_\mu^{\phi_i}(H_i)$. As in the classical case, let $R_v$ denote the direct sum $\bigoplus_{j \in P^+} V_j(\lambda)^*$, where $V_j(\lambda)^* := \text{Hom}_{\mathbb{C}}(V_j(\lambda), \hat{\mathbb{Z}})$.

Let $\ell_i \in \mathbb{N}$ be minimal such that $d_i \ell_i \equiv 0 \mod \ell$ (recall: $d_i \in \{1, 2, 3\}$). Then $v_\ell$ is a primitive $\ell$th root of unity if $\ell$ is odd and a primitive $2\ell$th root of unity if $\ell$ is even. Note that in either case $K_i^{d_i/2} = 1$ in $U_v$ (cf. [8, Lemma 4.4(a)]), as can be easily seen from the following relation in $U_A^0$:

$$\left[K_i, 0 \right]_{\ell_i} \prod_{j=1}^{\ell_i} (q^{d_i/2} - q^{-d_i/2}) = \prod_{j=1}^{\ell_i} (K_i q^{d_i(-j+1)} - K_i^{-1} q^{d_i(j-1)}).$$

If $\ell$ is odd, then $K_i^{d_j/2} v_\mu = v_\mu$ for all $\mu \in P$ and all weight vectors $v_\mu$, in particular, $K_i^{d_j/2}$ are in the center of $U_v(b^-)$. Denote by $J'$ the ideal of $U_v(b^-) \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{C}}$ generated by $(K_i^{d_j/2} - 1)$, $i = 1, \ldots, n$, and let $J$ be the ideal $J' \cap U_v(b^-)$ in $U_v(b^-)$. Observe that $U_v(b^-)$ embeds inside $U_v(b^-) \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{C}}$ since $U_v(b^-)$ is $\hat{\mathbb{Z}}$-free.

Define the pairing $\psi : U_v(b^-) \times R_v \rightarrow \hat{\mathbb{Z}}$ by $(u, \xi^\lambda) \mapsto \xi^\lambda(uv_\lambda)$ for $u \in U_v(b^-)$ and $\xi^\lambda \in V_\lambda(\lambda)^*$. 

**Proposition 2.**

(a) If $\ell$ is odd, then the pairing $\Phi_\epsilon : U_v(b^-) \times R_v \rightarrow \hat{\mathbb{Z}}$ is non-degenerate.

(b) If $\ell$ is even, then the pairing $\Phi_\epsilon$ is non-degenerate.

(c) Then induced map $\psi_\epsilon : R_v \rightarrow U_v(b^-)^*$ is injective, and the image is a subalgebra of $U_v(b^-)^*$, where the multiplication of $\xi^\lambda \in V_\lambda(\lambda)^*$ and $\xi^\mu \in V_\mu(\mu)^*$ is given by $(\xi^\lambda \cdot \xi^\mu)(u) := (\xi^\lambda \otimes \xi^\mu)|_{V(\lambda + \mu)}(u \cdot v_{\lambda + \mu})$ for $u \in U_v(b^-)$.

**Proof.** Let $v_\lambda \in V_\lambda(\lambda)$ be the fixed highest weight vector. Recall that $\left[K_i, 0 \right]_{m} v_\lambda = \left[\lambda(H_i) \right]_{m} v_\lambda$, in particular, $\left[K_i, 0 \right]_{m} v_\lambda = 0$ if $m > \lambda(H_i)$. The same argument as above in the classical case shows that the map $\psi_\epsilon : R_v \rightarrow U_v(b^-)^*$ is injective.

Suppose now $u \in U_v(b^-)$ is such that $\Phi_\epsilon(u, R_v) = 0$. We can find linearly independent $u_1, \ldots, u_t \in U_v^-$ and some $h_1, \ldots, h_t \in U_v^0$ such that $u = \sum_{i=1}^{t} u_i h_i$. To say that $u$ is in the radical of the pairing is equivalent to saying that $uv_\lambda = 0$ for all highest weight vectors $v_\lambda \in V_\lambda(\lambda)$, $\lambda \in P^+$. Since $u_i$ are linearly independent, the vectors $u_i v_\lambda$ are linearly independent for $\lambda \gg 0$. So $uv_\lambda = 0$ for all $\lambda \gg 0$ is equivalent to $h_i v_\lambda = 0$ for all
for

The linear independence of the monomials follows from the description of the basis

If we specialize at

those

\( h \)

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which proves that the monomials of this type span a subalgebra of \( U^0 \).

In \( U^0 \) we have in addition the relation: (0 < \( r < \ell_i \), \( m \geq 0 \))

If we specialize at \( q = v \), then \( v^{md_i} = v^{-md_i} = \pm 1 \). Since this term occurs on both sides, we can cancel it and get

Note that \( \prod_{s=1}^{r} (v^{md_i} - v^{-md_i}) \neq 0 \). Since \( K^{2\ell_i} = 1 \), this implies that we can express

the monomials listed in the proposition form a subalgebra. Recall that

We show that the monomials listed in the proposition form a subalgebra. Recall that

\[ \left[ \frac{K_i}{m} \right] = 0 \text{ unless } \ell_i \text{ divides } j, \quad \text{and} \quad \left[ \frac{x_j}{y_j} \right] = \binom{x}{y}, \]

where \( \binom{x}{y} \) is the ordinary binomial coefficient. Further,

\[ v^{md_i} = v^{md_i} \text{ for some } r \in \mathbb{N}, \]

hence it is equal to \( \pm 1 \). By specializing the relation in Lemma 1(e) at \( q = v \), we get

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which proves that the monomials of this type span a subalgebra of \( U^0 \).

In \( U^0 \) we have in addition the relation: (0 < \( r < \ell_i \), \( m \geq 0 \))

If we specialize at \( q = v \), then \( v^{md_i} = v^{-md_i} = \pm 1 \). Since this term occurs on both sides, we can cancel it and get

\[ \left[ \frac{K_i}{m} \right] = 0 \text{ unless } \ell_i \text{ divides } j, \quad \text{and} \quad \left[ \frac{x_j}{y_j} \right] = \binom{x}{y}, \]

where \( \binom{x}{y} \) is the ordinary binomial coefficient. Further,

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hence it is equal to \( \pm 1 \). By specializing the relation in Lemma 1(e) at \( q = v \), we get

which proves that the monomials of this type span a subalgebra of \( U^0 \).

In \( U^0 \) we have in addition the relation: (0 < \( r < \ell_i \), \( m \geq 0 \))
If \( v_\lambda \in V(\lambda) \) is a highest weight vector of weight \( \lambda = \sum_{i=1}^{n} (a_i \ell_i + r_i) \omega_i \) with \( 0 \leq r_i < \ell_i \), then, by [10, Lemma 34.1.2],

\[
\left[ K_i; 0 \right] \left[ \begin{array}{c} a_i \ell_i + r_i \\ \ell_i m_i \end{array} \right] v_\lambda = \left[ \begin{array}{c} a_i m_i \\ m_i \end{array} \right] v_\lambda.
\]

From this it follows easily that \( hv_\lambda = 0 \) for all \( h \gg 0 \) is equivalent to the condition \( \sum_{i} \frac{b_i}{e_i} K_i^{e_i} \cdot K_n^{e_n} v_\lambda = 0 \) for all \( m \) and all \( \lambda \gg 0 \).

Suppose now we have such an element \( h = \sum_{i} b_i K_i^{e_i} \cdot K_n^{e_n} v_\lambda \neq 0 \) and \( hv_\lambda = 0 \) for all \( \lambda \gg 0 \). Since \( K_2^{2\ell} = 1 \), this is equivalent to saying that \( hv_\lambda = 0 \) for all \( \lambda = \sum_{i=1}^{n} a_i \omega_i \) such that \( 0 \leq a_i < 2\ell_i \).

The \( \mathbb{C} \)-subalgebra \( K \) of \( U_0 \otimes \mathbb{C} \) generated by the \( K_i \) can be viewed as the group algebra of the group \( \mathbb{Z}^{\oplus \mathbb{Z}} \). If \( k \) is odd, then \( e_i^{\ell_i} \) is a primitive \( \ell_i \)-th root of unity. The one-dimensional representations provided by the action of the \( K_i \)'s on the highest weight vectors in \( V(\lambda) \), \( \lambda = \sum_{i=1}^{n} a_i \omega_i \), \( 0 \leq a_i < 2\ell_i \), hence does not give a complete list of all irreducible representations of \( K \). The intersection of the kernels of these representations is the subalgebra generated by \( (K_i^{\ell_i} - 1) \).

If \( k \) is even, then \( e_i^{\ell_i} \) is a primitive \( 2\ell_i \)-th root of unity. The one-dimensional representations provided by the action of the \( K_i \) on the highest weight vectors in \( V(\lambda) \), \( \lambda = \sum_{i=1}^{n} a_i \omega_i \), \( 0 \leq a_i < 2\ell_i \), hence give a complete list of all irreducible representations, so \( h = 0 \).

The description of the multiplication can be proved as in the classical case.

2. The Frobenius maps

We recall in this section the definition of the quantum Frobenius maps \( Fr \) and \( Fr' \) defined by Lusztig on \( U^- \) respectively \( U^- := U^- \otimes \mathbb{Z} \). Fix a positive integer \( \ell \). To simplify the arguments we assume that \( \ell \) is odd, and if \( q \) has simple factors of type \( G_2 \), then we assume \( \ell \) to be coprime to 3 in addition. Note that these conditions imply \( \ell_i = \ell \) for all \( i \); we will make some remarks at the end of this section concerning the cases \( \ell = 2, 3 \).

Lusztig has constructed two algebra homomorphisms (Theorems 35.1.7 and 35.1.8 in [10]): \( Fr : U^- \rightarrow U^- \) (respectively \( Fr' : U^- \rightarrow U^- \)) which are defined on the generators by

\[
Fr(F_i^{(k)}) := \begin{cases} 
0 & \text{if } \ell \not| k \\
Y_i^{(k/\ell)} & \text{if } \ell | k
\end{cases}
\]

(respectively \( Fr'(Y_i^{(k)}) := F_i^{(k)} \)).

The composition \( Fr \circ Fr' \) is obviously the identity map on \( U^- \). One can of course similarly define \( Fr : U^+ \rightarrow U^+ \) and \( Fr' : U^+ \rightarrow U^+ \). The map \( Fr \) can be extended to an algebra homomorphism \( Fr : U_v \rightarrow U_v \) (see [10, Theorem 35.1.9] or [9, Theorems 8.10 and 8.11 and Corollary 8.14]):
Proposition 3. The map defined by \( u \mapsto Fr(u) \) for \( u \in U_v^- \) or \( u \in U_v^+ \) and
\[
K_i \mapsto 1, \quad \begin{bmatrix} K_i; 0 \\ \ell / m \end{bmatrix} \mapsto \begin{cases} 0 & \text{if } \ell \not| \, m \\ \left( H_i \right)_{\ell / m} & \text{if } \ell | \, m \end{cases} \quad i = 1, \ldots, n,
\]
extends the Frobenius maps for \( U_v^- \) and \( U_v^+ \) to a surjective \( \mathbb{Z} \)-algebra homomorphism \( Fr : U_v \to U_\mathbb{Z} \). Moreover, \( Fr \) is a Hopf algebra homomorphism.

The map \( Fr' \) cannot be extended to a homomorphism defined on \( U_\mathbb{Z} \). Though, we can extend it to a homomorphism defined on \( U_\mathbb{Z}(b^-) \), the price for the extension is that the range \( U_\mathbb{Z}(b^-) \) is to be replaced by \( U_\mathbb{Z}(b^-)/J \). Here \( J \) is the ideal defined in the last section (Proposition 2).

Lemma 3. The map defined by \( u \mapsto Fr'(u) \) for \( u \in U_\mathbb{Z}^- \) and \( \left( \frac{H_i}{\ell / m} \right) \mapsto \begin{bmatrix} K_i; 0 \\ \ell / m \end{bmatrix} \) extends \( Fr' \) to a \( \mathbb{Z} \)-algebra homomorphism (again denoted by) \( Fr' : U_\mathbb{Z}(b^-) \to U_\mathbb{Z}(b^-)/J \).

We refer to \( Fr' \) as the Frobenius splitting homomorphism.

Proof. Recall that \( v' = 1, \begin{bmatrix} x \ \\ j \end{bmatrix} = 0 \) unless \( \ell \) divides \( j \) and \( \begin{bmatrix} x \ \\ j \end{bmatrix} = \begin{bmatrix} x' \ \\ j' \end{bmatrix} \). Further, \( K'_i = 1 \) in \( U_\mathbb{Z}(b^-)/J \). Hence Lemma 1(e) implies that
\[
Fr' \left( \begin{bmatrix} H_i \\ x \end{bmatrix} \right) Fr' \left( \begin{bmatrix} H_i \\ y \end{bmatrix} \right) = \begin{bmatrix} K_i; 0 \\ x' \end{bmatrix} \begin{bmatrix} K_i; 0 \\ y' \end{bmatrix} \\
= \sum_{j=0}^{x} \binom{x}{j} \left( x + y - j \right) \left( \frac{K_i; 0}{\ell (x + y - j)} \right) \\
= Fr' \left( \sum_{j=0}^{x} \binom{x}{j} \left( x + y - j \right) \left( \frac{H_i}{x + y - j} \right) \right) \\
= Fr' \left( \begin{bmatrix} H_i \\ x \end{bmatrix} \begin{bmatrix} H_i \\ y \end{bmatrix} \right).
\]
Now, for \( y \geq 0 \), we have
\[
Fr' \left( \begin{bmatrix} H_i + y \\ x \end{bmatrix} \right) = Fr' \left( \sum_{s=0}^{x} \binom{y}{s} \begin{bmatrix} H_i \\ x - s \end{bmatrix} \right) \\
= \sum_{s=0}^{x} \binom{y'}{s'} \begin{bmatrix} K_i; 0 \\ \ell (x - s) \end{bmatrix} = \begin{bmatrix} K_i; \ell y' \\ \ell x \end{bmatrix},
\]
because the other terms in the expression (Lemma 1(e)) for \( \begin{bmatrix} K_i; \ell y' \\ \ell x \end{bmatrix} \) vanish. Similarly, for \( y > 0 \), we get
\[
Fr' \left( \begin{bmatrix} H_i - y \\ x \end{bmatrix} \right) = Fr' \left( \sum_{s=0}^{x} (-1)^s \binom{y + s - 1}{s} \begin{bmatrix} H_i \\ x - s \end{bmatrix} \right)
\]
Denote by \( (\cdot) \). From this we conclude that (cf. [9, Section 6.3]),

\[
\sum_{x=0}^{\ell-1} (-1)^x \begin{pmatrix} y+s-1 \\ s \end{pmatrix} \begin{bmatrix} K_i; 0 \\ \ell(x-s) \end{bmatrix}
\]

\[
= \sum_{x=0}^{\ell} (-1)^s' \begin{pmatrix} y{\ell} + s' - 1 \\ s' \end{pmatrix} \begin{bmatrix} K_i; 0 \\ \ell(x-s) \end{bmatrix}.
\]

To prove the last equality, note that \((-1)^s' = (-1)^s\), and (see [10, Lemma 34.1.2])

\[
\begin{pmatrix} y{\ell} + s' - 1 \\ s' \end{pmatrix} = \begin{pmatrix} y+s-1 \\ s \end{pmatrix} \begin{pmatrix} \ell-1 \\ 0 \end{pmatrix} = \begin{pmatrix} y+s-1 \\ s \end{pmatrix} \begin{pmatrix} r-1 \\ 0 \end{pmatrix} = \begin{pmatrix} y+s-1 \\ s \end{pmatrix} \begin{pmatrix} r-1 \\ 0 \end{pmatrix}.\]

Suppose \( s' = s' + r \) with \( 0 < r < \ell \). Note that \( \begin{pmatrix} y+s-1 \\ s \end{pmatrix} \begin{pmatrix} s' \end{pmatrix} = \begin{pmatrix} y+s-1 \\ s \end{pmatrix} \begin{pmatrix} r \end{pmatrix} = 0 \), so Lemma 1 gives (for \( y > 0 \))

\[
\text{Fr}' \begin{pmatrix} H_i - y \begin{pmatrix} x \end{pmatrix} \end{pmatrix} = \sum_{x'=0}^{x'} (-1)^{y'} \begin{pmatrix} y{\ell} + s' - 1 \\ s' \end{pmatrix} \begin{bmatrix} K_i; 0 \\ \ell(x'-s') \end{bmatrix} = \begin{bmatrix} K_i; -\ell y \end{pmatrix}.
\]

From this we conclude that (cf. [9, Section 6.5])

\[
\text{Fr}' \begin{pmatrix} H_i \begin{pmatrix} x \end{pmatrix} \end{pmatrix} \text{Fr}' \begin{pmatrix} Y_i \begin{pmatrix} \begin{pmatrix} x \end{pmatrix} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} K_i; 0 \\ \ell(x) \end{pmatrix} \text{Fr}' \begin{pmatrix} Y_i \begin{pmatrix} \begin{pmatrix} x \end{pmatrix} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} K_i; -\ell y \begin{pmatrix} c_i \end{pmatrix} \end{pmatrix}
\]

\[
= \text{Fr}' \begin{pmatrix} Y_i \begin{pmatrix} \begin{pmatrix} x \end{pmatrix} \end{pmatrix} \end{pmatrix} \text{Fr}' \begin{pmatrix} H_i - y \begin{pmatrix} \begin{pmatrix} x \end{pmatrix} \end{pmatrix} \end{pmatrix}.
\]

which shows that the map respects the defining relations between the generators of \( U_{\ell}(b^-) \). \( \square \)

**Remark 2.** The assumption that \( \ell \) is coprime to 3 if \( \mathfrak{g} \) admits simple factors of type \( \mathfrak{g}_2 \) is not necessary for Proposition 3 and Lemma 3. Actually, the construction makes sense for arbitrary \( \ell \), but we have to redefine the maps; for details see [10, Chapter 35]. In the following we mainly concentrate on the remarks on \( \ell = 2, 3 \), but, with the appropriate adaptations (similarly to those in [7]), the constructions hold also in the general case.

As before, let \( \ell_i \) be minimal such that \( d_i \ell_i \equiv 0 \mod \ell_i \), and denote by \( C^{\mathfrak{g}} \) the matrix \((c_i, \ell_i / \ell_i).\) This is the Cartan matrix of the root system having the roots \( x_i := \ell_i \alpha_i \) as simple roots and \( H_i^{\mathfrak{g}} := H_i / \ell_i \) as co-roots. Its weight lattice is the subset \( P^{\mathfrak{g}} := \{ \lambda \in P \mid \lambda(H_i) \in \ell_i \mathbb{Z} \forall i \} \) of \( P \). Note if \( \mu \in P^{\mathfrak{g}} \subset P \) and \( v_\mu \) is a weight vector in a \( U^{\mathfrak{g}} \)-representation, then

\[
\begin{pmatrix} K_i; k \ell_i \\ m \ell_i \end{pmatrix} v_\mu = \begin{pmatrix} \mu(H_i) + k \ell_i \\ m \ell_i \end{pmatrix} v_\mu = \begin{pmatrix} \ell_i \mu(H_i) + k \ell_i \\ m \ell_i \end{pmatrix} v_\mu = \begin{pmatrix} \mu(H_i) + k \\ m \end{pmatrix} v_\mu.
\]

Denote by \( \mathfrak{g}^{\mathfrak{g}} \) the corresponding Lie algebra and let \( U^{\mathfrak{g}} \) be its enveloping algebra. We use the notation \( X_i^{\mathfrak{g}}, Y_i^{\mathfrak{g}} \) and \( H_i^{\mathfrak{g}} \) for the generators. If \( \mathfrak{g} \) is simply laced or \( \ell \) is a prime > 3, then \( C^{\mathfrak{g}} = C \). But if \( \ell = 3 \), then \( C^{\mathfrak{g}} \) is obtained from \( C \) by transposing the \( 2 \times 2 \) submatrices corresponding to simple factors of type \( \mathfrak{g}_2 \). If \( \ell = 2 \), then the same has to be applied for simple factors of type \( \mathfrak{f}_4, \mathfrak{b}_n \) and \( \mathfrak{c}_n \). The Frobenius homomorphisms...
Let $U_v$ since $i$ then we extend the Frobenius map to a homomorphism $Fr: U_v(b^-) \to U_\mathbb{Z}(b^{\#})$ by setting $Fr(k) = \frac{[k,0]/m}{m/e_i}$ for arbitrary positive integer $m$ and $Fr(k) = 0$ otherwise. Similarly, one can extend $Fr'$ to a homomorphism $U_\mathbb{Z}(b^{\#}) \to U_v(b^-)/J$ by setting $Fr(k) = \frac{[k,0]/m}{m/e_i}$. The details of the proof are left to the reader.

The definitions of $Fr: U_v^- \to U_v^{\#}$ and $Fr': U_\mathbb{Z}(b^{\#}) \to U_v(b^-)$ given above make sense for arbitrary positive integer $\ell$. To avoid problems with the definition of the extensions for $\ell = 2$, we assume that $\ell = 2d$, where $d$ is the smallest common multiple of $d_1, \ldots, d_n$. Since $\ell_i = \ell_i/d_i = 2(d/d_i)$, we know that all the $\ell_i$ are even. Denote by $(U_v)^e$ the subalgebra of $U_v$ generated by $K_{0,m}$ and $K^m_0$, $m$ even, and $K_{i,0}$ for $m$ odd, and let $(U_v^{-})_e$ be the subalgebra of $U_v^-$ spanned by the monomials of weight $-2\beta$, $\beta \in Q^+$. Let $U_v(b^-)_e$ be the subalgebra of $U_v(b^-)$ generated by $(U_v^-)_e$ and $(U_v^0)_e$. Note that $Fr(Y_i) = F_i \in U_v(b^-)_e$ because the $\ell_i$ are even.

Using Lemma 1, it is easy to verify that $U_v(b^-)_e$ is spanned by the elements of the form $u \prod_{i=1}^n \left( \frac{K_{i,0}}{m_i} \right) K_{i,0}$, where $u \in (U_v)_e$, $m_i \in \mathbb{N}$ and $e_i \in \{0, 1\}$ with $m_i + e_i$ even. The elements $(K_{i,0}^\ell - 1)$ are in the center of the even subalgebra. As in the odd case, let $J'$ be the ideal of $U_v(b^-)_e \otimes \mathbb{C}$ generated by the elements $(K_{i,0}^\ell - 1)$, $i = 1, \ldots, n$, and let $J$ be the (two sided) ideal $J' \cap U_v^-(b^-)_e$.

Denote by $V_v(\ell)_e$ the direct sum $\bigoplus_u V_v(\ell)_u$ of all weight spaces corresponding to the weights of the form $\mu = \lambda - 2\beta$, $\beta \in Q^+$, and set $R_v(e) := \bigoplus_{e \in 2\mathbb{Z}} (V_v(\ell)_e)^\ast$. Proposition 2 can then be reformulated as: The pairing $\Phi_v: U_v(b^-)_e \times R_v(e) \to \mathbb{Z}$ defined by $(u, \xi^\ell) \to \xi^\ell(uw_v)$ for $u \in U_v(b^-)_e$ and $\xi^\ell \in V_v(\ell)_e$, has as radical precisely $(J, 0)$, and hence the induced pairing $U_v^-(b^\#)_e/J \times R_v(e) \to \mathbb{Z}$ is non-degenerate. In particular, the induced map $\psi_v: R_v(e) \to (U_v(b^-)_e)^\ast$ is injective, and the image is a subalgebra of $(U_v(b^-)_e)^\ast$. The Frobenius maps can also be extended correspondingly: the map defined by $u \mapsto Fr(u)$ for $u \in U_v^-(b^-)$, $K_{i,0}^\ell \to 1$, $K_{0,m} \to 0$ for $m$ odd, $K_{0,m} \to 0$ if $m$ is even and $\ell|m$, and $\frac{K_{0,m}}{m/e_i}$ if $m$ is even and $\ell|m$, extends Fr to an algebra homomorphism $Fr: U_v(b^-)_e \to U_v(b^\#)_e$. Similarly, the map defined by $u \mapsto Fr'(u)$ for $u \in U_v^\#$ and $\frac{K_{0,m}}{m/e_i}$ extends $Fr'$ to an algebra homomorphism $Fr': U_v^\# \to U_v(b^-)_e/J$. The proofs are very similar to the proofs above, and hence the details are left to the reader.

3. The dual maps $Fr^\ast$ and $Fr'^\ast$

We assume again that $\ell$ is an odd integer and moreover coprime to 3 if $g$ has simple factors of type $G_2$. We make some remarks concerning the general case at the end of
the section. The ideal \(J\) (see Proposition 2) is in the kernel of \(\text{Fr}\), so we get an induced map \(\text{Fr}^*\) between the Hopf dual \(U_{\mathbb{Z}}(\mathfrak{b}^-)^*\) of \(U_{\mathbb{Z}}(\mathfrak{b}^-)\) and the Hopf dual \(U_{\mathbb{Z}}(\mathfrak{b}^-)^*\) of \(U_{\mathbb{Z}}(\mathfrak{b}^-)\).

The \(U_{\mathbb{Z}} \otimes \mathbb{C}\) module \(V_{\mathbb{Z}}(\lambda) \otimes \mathbb{C}\), \(\lambda \in \mathbb{P}^+\), is in general not a simple module. Denote by \(L_{\mathbb{Z}}(\lambda)\) its simple quotient. By Lustzig [8, Proposition 7.2], \(E_i\), \(F_i\) and \(K_i - 1\) operate trivially on \(L_{\mathbb{Z}}(\lambda)\) for \(\lambda \in \mathbb{P}^+\). Further, as in [8, Proposition 7.5], the Frobenius splittings \(\text{Fr}^*: U_{\mathbb{Z}}^- \rightarrow U_{\mathbb{Z}}^-\) and \(\text{Fr}': U_{\mathbb{Z}}^+ \rightarrow U_{\mathbb{Z}}^+\) can be glued together to a surjective homomorphism (in fact an isomorphism) \(F: U \simeq U_{\mathbb{Z}} \otimes \mathbb{C} \rightarrow U_{\mathbb{Z}} \otimes \mathbb{C}/(E_iF_i,K_i-1)\), and \(L_{\mathbb{Z}}(\lambda)\) becomes via \(F\) a simple \(U\)-module \(V(\lambda)\) of highest weight \(\lambda\). We can also view \(L_{\mathbb{Z}}(\lambda)\) the other way around: We start with the irreducible \(U\)-module \(V(\lambda)\) and make it into a \(U_{\mathbb{Z}} \otimes \mathbb{C}\) module (by abuse of notation) \(V_{\mathbb{Z}}(\lambda)\) via the Frobenius homomorphism \(\text{Fr}\) : \(U \rightarrow U_{\mathbb{Z}} \otimes \mathbb{C}\). Then, \(\text{Fr}\) being surjective, \(V_{\mathbb{Z}}(\lambda)^{\text{Fr}}\) is an irreducible \(U_{\mathbb{Z}} \otimes \mathbb{C}\) module. It is easy to see that \(V_{\mathbb{Z}}(\lambda)^{\text{Fr}}\) is, in fact, isomorphic with \(L_{\mathbb{Z}}(\lambda)\).

For each fundamental weight \(\omega_i\) \((1 \leq i \leq n)\), choose an isomorphism \(\phi_i : V(\omega_i)^{\text{Fr}} \simeq L_{\mathbb{Z}}(\omega_i)\) such that \(v_{\omega_i} \in V(\omega_i)\) corresponds to \(v_{\omega_i} \in L_{\mathbb{Z}}(\omega_i)\) (cf. Section 1 for the notation \(v_{\omega_i}\) and \(v_{\omega_i}\)). Since \(\text{Fr}\) is a Hopf algebra homomorphism, the isomorphisms \(\phi_i\) give rise to a \(U_{\mathbb{Z}} \otimes \mathbb{C}\) module isomorphism \(\phi_j : V_{\mathbb{Z}}(\lambda)^{\text{Fr}} \simeq L_{\mathbb{Z}}(\lambda)\) (for all \(\lambda \in \mathbb{P}^+\)) so that \(v_{\lambda} \in V(\lambda)\) corresponds to \(v_{\lambda} \in L_{\mathbb{Z}}(\lambda)\). In the sequel, we fix such an isomorphism \(\phi_i\) for each \(\lambda \in \mathbb{P}^+\). For any \(\lambda \in \mathbb{P}^+\) let \(L_{\mathbb{Z}}(\lambda)_{\mathbb{Z}}\) be the \(U_{\mathbb{Z}}\)-submodule of \(L_{\mathbb{Z}}(\lambda)\) generated by \(v_{\lambda}\).

We thus get a ‘natural’ \(U_{\mathbb{Z}}\)-module isomorphism \(L_{\mathbb{Z}}(\lambda)_{\mathbb{Z}} \rightarrow V_{\mathbb{Z}}(\lambda)^{\text{Fr}}\) and hence the dual map \((V_{\mathbb{Z}}(\lambda)^{\text{Fr}})^* \rightarrow (L_{\mathbb{Z}}(\lambda)_{\mathbb{Z}})^*\), where \(V_{\mathbb{Z}}(\lambda)^{\text{Fr}} := V_{\mathbb{Z}}(\lambda) \otimes \mathbb{Z}_{\mathbb{Z}}\).

Define the map \(\text{Fr}^\vee : R_{\mathbb{Z}} \rightarrow R_{\mathbb{Z}}\), as the direct sum of the composite maps \(V_{\mathbb{Z}}(\lambda)^* \rightarrow (L_{\mathbb{Z}}(\lambda)_{\mathbb{Z}})^* \rightarrow V_{\mathbb{Z}}(\lambda)^*\), where the last map is the dual of the quotient map \(V_{\mathbb{Z}}(\lambda) \rightarrow L_{\mathbb{Z}}(\lambda)_{\mathbb{Z}}\), and \(R_{\mathbb{Z}} := R_{\mathbb{Z}} \otimes \mathbb{Z}_{\mathbb{Z}}\).

**Proposition 4.** The map \(\text{Fr}^\vee\) is nothing but the restriction of \(\text{Fr}^*\) to \(R_{\mathbb{Z}}\) under the identification of \(R_{\mathbb{Z}}\) (resp. \(R_{\mathbb{Z}}\)) as a subalgebra of \(U_{\mathbb{Z}}(\mathfrak{b}^-)^*\) (resp. \(U_{\mathbb{Z}}(\mathfrak{b}^-)^*\)) induced by the pairing \(\Phi\) (resp. \(\Phi_i\)), cf. Propositions 1 and 2.

Equivalently, for any \(X \in U_{\mathbb{Z}}(\mathfrak{b}^-)\) and \(\xi \in R_{\mathbb{Z}}\) we have

\[
\Phi(\text{Fr}_X, \xi) = \Phi_X(\text{Fr}^\vee, \xi).
\]

**Proof.** Equivalence of the two assertions is easy and the identity (1) follows readily from the definition of \(\text{Fr}^\vee\). \qed

From now on, we will denote (by abuse of notation) \(\text{Fr}^\vee\) by \(\text{Fr}^*\) itself.

Similarly the algebra homomorphism \(\text{Fr}'\) gives rise to the dual map \(\text{Fr}'^* : (U_{\mathbb{Z}}(\mathfrak{b}^-)/J)^* \rightarrow U_{\mathbb{Z}}(\mathfrak{b}^-)^*\). As above, one proves that the dual map \(\text{Fr}'^*\) induces in fact a map \(R_{\mathbb{Z}} \rightarrow R_{\mathbb{Z}}\).

To describe this map more explicitly, let \(\lambda \in \mathbb{P}^+\) be a dominant weight. For the Weyl module \(V_{\mathbb{Z}}(\lambda)\) for \(U_{\mathbb{Z}}\) denote by \(V_{\mathbb{Z}}(\lambda)^{\mathbb{Z}}\) the direct sum \(\bigoplus_{\mu \in \mathbb{P}} V_{\mathbb{Z}}(\lambda)_{\mu}\) of all weight spaces corresponding to the weights in \(\mathbb{P}\). If \(\mu = \ell \mu_1\) is a weight in \(\mathbb{P}\), then
so is the weight \( \mu \pm n \alpha_i = (\mu_1 \pm n \alpha_i) \). It follows that \( V_\ell(\lambda) \) is stable under the action of all the \( F_i^{(m')} \) and \( E_i^{(m')} \).

We make \( V_\ell(\lambda) \) into a \( U^-_\ell \)-module respectively \( U^+_\ell \)-module via the homomorphism \( Fr' \) (i.e. by letting \( X_i^{(m)} \) act as \( E_i^{(m')} \) and \( Y_i^{(m)} \) act as \( F_i^{(m')} \)). A simple calculation (see for example [7] for details) shows that if we let \( (\frac{h_i}{m}) \) act as \( [K_{m'}] \), then this defines a \( U^-_\ell \)-module structure on \( V_\ell(\lambda) \), and the submodule generated by the highest weight vector \( v_\lambda \) is isomorphic to \( V_\ell(\lambda) \). Again we choose an isomorphism so that \( v_\lambda \) corresponds to \( v_\lambda \).

Similar to Proposition 4, we obtain:

**Proposition 5.** The restriction of the dual map \( Fr'^* \) to \( V_\ell(\lambda)^* \) is the dual map of the inclusion \( V_\ell(\lambda) \hookrightarrow V_\ell(\lambda)^* \), and \( Fr'^*|_{V_\ell(\mu)^*} = 0 \) for \( \mu \not\in \ell P^+ \).

**Remark 3.** Recall that we cannot extend \( Fr' \) to an algebra homomorphism on the full enveloping algebra, so \( V_\ell(\lambda) \) is not naturally endowed with a structure as a \( U^-_\ell \)-module. The inclusion \( V_\ell(\lambda) \hookrightarrow V_\ell(\lambda)^* \) hence does not give rise to a \( U^-_\ell \)-equivariant map \( V_\ell(\lambda)^* \rightarrow V_\ell(\lambda)^* \). But, using the Frobenius maps \( Fr' : U^-_\ell \rightarrow U^-_\ell \) and \( Fr' : U^+_\ell \rightarrow U^+_\ell \), we can make \( V_\ell(\lambda)^* \) into a \( U^-_\ell \)-respectively \( U^+_\ell \)-module, and, by the definition of the inclusion \( V_\ell(\lambda) \hookrightarrow V_\ell(\lambda)^{1/\ell} \rightarrow V_\ell(\lambda) \), the map \( V_\ell(\lambda)^* \rightarrow V_\ell(\lambda)^* \) is equivariant with respect to the action of \( U^+_\ell \) and \( U^-_\ell \), and hence so is the dual map.

**Remark 4.** The composition

\[
U^-_\ell(b^-) \xrightarrow{Fr'} U_\ell(b^-) \xrightarrow{Fr} U^-_\ell(b^-)
\]

is the identity map, and hence so is \( R^-_\ell \xrightarrow{Fr'^*} R_\ell \xrightarrow{Fr'} R^-_\ell \).

**Remark 5.** If \( \ell = 2d \), then \( Fr'^* \) induces a map \( R^-_\ell \rightarrow R_\ell \), which is the direct sum of the duals of the quotient maps \( V_\ell(\lambda) \rightarrow L_\ell(\lambda) = V(\lambda) \), for \( \lambda \in P^0 \), and the restriction of the dual map \( Fr'^* \) to \( V_\ell(\lambda)^* \) is the dual map of the inclusion \( V_\ell(\lambda)^* \hookrightarrow V_\ell(\lambda)^* \), and \( Fr'^*|_{V_\ell(\mu)^*} = 0 \) for \( \mu \not\in P^0 \). To see this, let \( \lambda \in P^0 \subset P \) be a dominant weight, we write just \( \lambda \) for the weight if we view it as a \( U_\ell \)-weight. We make \( V(\lambda) \) as above into a \( U^-_\ell \) and \( U^+_\ell \)-module by using the Frobenius map \( Fr \), and we let \( [^K_{m'}] \) act on \( V(-\lambda) \) as \( (\frac{h_i}{m'}) \) if \( m \) is divisible by \( \ell \), and as \( 0 \) if \( \ell \not|m \).

Then as above, the three actions glue together to give a \( U_\ell \)-module structure on \( V(\lambda) \) such that \( u \in J \) acts trivially. Thus \( V(\lambda) \) becomes in this way a highest weight module for \( U_\ell \) of highest weight \( \lambda \). Now \( V(\lambda) \) is a simple module for \( U(\mathfrak{g}) \) and hence for \( U_\ell \otimes \mathbb{C} \). So, as above, we can view \( Fr'^* \) as the dual map of the quotient map \( V_\ell(\lambda) \rightarrow L_\ell(\lambda) = V(\lambda) \).

Actually, with the appropriate adoptions (for details see [10, Chapter 35]) one can reformulate the results for arbitrary \( \ell \).
4. Base change

In the following we assume that $\ell = p$ is in fact an odd prime, and further $p > 3$ if $\frak{g}$ has factors of type $G_2$. Let $k$ be an algebraically closed field of char. $p$. We consider $k$ as a $\mathbb{Z}$-module by extending the canonical map $\mathbb{Z} \rightarrow k$ to a ring homomorphism $\mathbb{Z} \rightarrow k$ (where the first map is given by the projection $\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ and the inclusion $\mathbb{Z}/p\mathbb{Z} \subset k$, and the extension $\mathbb{Z} \rightarrow k$ follows from the ‘Going-up theorem’ [12, Theorem 9.3]). We denote by $U_k, U_{c,k}$ the corresponding algebras obtained by the base change.

Note that the image of $v$ in $k$ is 1. Let $J_p$ be the ideal in $U_{c,k}$ generated by the central elements $(K_i - 1), \ i = 1, \ldots, n$; then the quotient $U_{c,k}/J_p$ is naturally isomorphic to $U_k$. Further, let $\mu \in P^+$ be a dominant weight and $V_{\nu,k}(\mu)$ be the corresponding Weyl module for $U_{c,k}$. Since all the $(K_i - 1)$ operate trivially on $V_{\nu,k}(\mu)$, this becomes in a natural way the Weyl module $V_k(\mu)$ for $U_k$. Let $L_k(\mu)$ be the $U_k$-module $L_k(\mu)\mathbb{Z} \otimes \mathbb{Z} k$.

We are now left with only one algebra, namely $U_k$. The module $L_k(p\lambda)$ is, as a $U_k$-module the same as the $U_k$-module $V_k(\lambda)^\mathfrak{fr}$, where as in Section 3, $V_k(\lambda)^\mathfrak{fr}$ is the same $k$-vector space as $V_k(\lambda)$, but its $U_k$-module structure has been twisted via the Frobenius map $\mathfrak{fr}: U_k \rightarrow U_k$ given by $F_i^{(m)} \rightarrow F_i^{(m/p)}, E_i^{(m)} \rightarrow E_i^{(m/p)}$. If $m$ is divisible by $p$ and 0 otherwise.

We are going to twist the $k$-vector space structure of $V_k(\lambda)^\mathfrak{fr}$. Let $\phi: k \rightarrow k$ be the ring homomorphism given by the inverse of the $p$th power map, i.e., $z \rightarrow z^{1/p}$, and denote by $V_k(\lambda)^{(1)}$ the $k$-vector space (and $U_k$-module) having as underlying abelian group the same as $V_k(\lambda)$, but where the scalar multiplication has been twisted by $\phi: \alpha \ast v := \phi(\alpha)v$, and where $U_k$ acts as on $V_k(\lambda)^\mathfrak{fr}$. (Note that the operation of $U_k$ is linear also with respect to the twisted scalar multiplication.) The $U_k$-module $V_k(\lambda)^{(1)}$ can be seen explicitly as a quotient of $V_k(p\lambda)$ as follows: The map $V_k(\lambda)^{(1)} \rightarrow S^p V_k(\lambda)$, defined by $v \mapsto v^p$ (which is linear because of the twisted scalar multiplication), induces an isomorphism onto the image of the canonical map $V_k(p\lambda) \rightarrow S^p V_k(\lambda)$ which sends the highest weight vector $v_{p\lambda} \in V_k(p\lambda)$ to the highest weight vector $v_{\lambda}^p \in S^p V_k(\lambda)$.

Let $G$ be the semisimple simply connected algebraic group over $k$ corresponding to the Lie algebra $\frak{g}$ and let $B \subset G$ be the Borel subgroup corresponding to the Lie algebra $\frak{b}$. Let $\mathcal{L}_\lambda$ be the line bundle on $X := G/B$ corresponding to a weight $-\lambda$. Recall that for $\lambda$ dominant, we have $H^0(X, \mathcal{L}_\lambda)$, as $G$-module, isomorphic to $V_k(\lambda)^*$. It follows from the considerations above that the dual map $\mathfrak{fr}^*: H^0(X, \mathcal{L}_\lambda)^{(1)} \rightarrow H^0(X, \mathcal{L}_{p\lambda})$ is just the $p$th power map sending a section $s \in H^0(X, \mathcal{L}_\lambda)$ to $s^p \in H^0(X, \mathcal{L}_{p\lambda})$ (again, recall that this map is linear with respect to the twisted scalar multiplication).

The inclusion $V_k(\lambda) \hookrightarrow V_k(p\lambda)$ respectively $\mathfrak{fr}^*: H^0(X, \mathcal{L}_\lambda) \rightarrow H^0(X, \mathcal{L}_{\lambda})$, the associated dual map, does not have such an equivariant interpretation, but Remark 4 implies that $\mathfrak{fr}^*$ is a section to $\mathfrak{fr}^*$. Observe that $\mathfrak{fr}^*$ restricted to $H^0(X, \mathcal{L}_\lambda)$ is zero if $\lambda \notin pP^+$.

**Theorem 1.** The dual map $\mathfrak{fr}^*: H^0(X, \mathcal{L}_\lambda) \rightarrow H^0(X, \mathcal{L}_{\lambda})$ is the map $s \mapsto s^p$ sending a section to its $p$th power, and $\mathfrak{fr}^*: H^0(X, \mathcal{L}_{p\lambda}) \rightarrow H^0(X, \mathcal{L}_{\lambda})$ provides a splitting of
this map. For any $S \in H^0(X, \mathcal{L}_{j\ell})$ and $f \in H^0(X, \mathcal{L}_{m\ell})$, the Frobenius map satisfies the following properties:

(a) $\text{Fr}^*(s^p f) = s \text{Fr}^*(f)$, and
(b) $\text{Fr}^*(X^{i,p}) f = X_i^{(q)} \text{Fr}^*(f)$ for all $1 \leq i \leq n$, and $q \in \mathbb{N}$.

**Remark 6.** For notational convenience assume that $\lambda \notin pP^+$. The property (a) implies that $\text{Fr}^*$ induces a graded Frobenius endomorphism of the graded algebra $S := \bigoplus_{m \geq 0} H^0(X, \mathcal{L}_{m\ell})$, more specifically, $\text{Fr}^*$ maps the homogeneous elements of degree not divisible by $p$ to zero and if $f$ is of degree $qp$ then $\text{Fr}^*(f)$ is of degree $q$. The map is additive: $\text{Fr}^*(s_1 + s_2) = \text{Fr}^*(s_1) + \text{Fr}^*(s_2)$, and $\text{Fr}^*(s_1^p s_2) = s_1^{sp} \text{Fr}^*(s_2)$. The second property implies that $\text{Fr}^*$ is the canonical splitting, see [11]. In particular, $\text{Fr}^*$ maps $B$-modules to $B$-modules.

**Proof.** It remains to prove that the two properties (a) and (b) hold. For notational convenience assume that $\lambda \notin pP^+$. If $m$ is not divisible by $p$, then in both the equalities all the terms on the right and left are zero, so the properties hold in this case trivially. Suppose now that $m$ is divisible by $p$, say $m = pq$. Again, both the properties hold trivially if $f$ is a weight vector of a weight not divisible by $p$, so in the following we may assume that $f$ is a weight vector corresponding to a weight divisible by $p$. The element $f$ is hence an element of $(V_k(pq\lambda))^{1/p}$. Recall that the embedding $i : V_k(q\lambda) \hookrightarrow V_k(pq\lambda)^{1/p}$ satisfies $i(X^{(i)}_1 v) = X^{(p)}_1 (i(v))$, so the corresponding property holds also for the dual map $\text{Fr}^* : H^0(X, \mathcal{L}_{pq\ell}) \to H^0(X, \mathcal{L}_{q\ell})$. This implies the second property in the theorem above.

Abbreviate the module $L_v(\lambda)_{\mathfrak{g}}$ by $L_v(\lambda)$. To prove the first property, consider the following diagram of Weyl modules (for $U_{\mathfrak{g}}$ respectively $U_v$) defined over $\mathbb{Z}$:

There are two inclusions of $U_{\mathfrak{g}}$-modules using the Frobenius map: $V_{\mathfrak{g}}((q + j)\lambda) \hookrightarrow V_v((q + j)\lambda)$ and the other inclusion is $V_{\mathfrak{g}}((q + j)\lambda) \hookrightarrow L_v(pj\lambda) \otimes V_{\mathfrak{g}}(pq\lambda)^{1/p}$, using the fact that $L_v(pj\lambda) = V_{\mathfrak{g}}(pq\lambda)^{1/p}$ as $U_v$-module. Note that $E_i^{(m,p)}$ acts on $V_{\mathfrak{g}}(j\lambda)$ as $X_i^{(m)}$, so $\text{Fr}^* \cdot (X_i^{(m)})$ acts on $L_v(pj\lambda)$ as $X_i^{(m)}$. Then we have two inclusions of $U_v$-modules which act on the $U_v$-modules via $\text{Fr}^*$. The inclusions are $V_v((q + j)\lambda)^{1/p} \hookrightarrow V_v((q + j)\lambda)$ and $L_v(pj\lambda) \otimes V_{\mathfrak{g}}(pq\lambda)^{1/p} \hookrightarrow L_v(pj\lambda) \otimes V_{\mathfrak{g}}(pq\lambda)$.

In addition we have two maps between Weyl modules: $V_{\mathfrak{g}}((q + j)\lambda) \to V_v((q + j)\lambda)$ and $V_v((q + j)\lambda) \otimes V_v(q\lambda) \to V_v(pj\lambda) \otimes V_v(q\lambda)$:

The vertical map is the identity on the second factor and the projection on the first. All these maps are equivariant with respect to the $U_{\mathfrak{g}}$-actions on these spaces, they all map the highest weight vector (resp. the tensor product of highest weight vectors) to...
a highest weight vector (resp. the tensor product of highest weight vectors). It follows that the diagram is commutative and provides two different ways to construct a map $V_{Z}((j + q)\lambda) \to L_{\nu}(pj\lambda) \otimes V_{\nu}(pq\lambda)$.

Over the field $k$, the dual of the bottom row is the map $H^{0}(X, L_{pj\lambda}) \otimes H^{0}(X, L_{pq\lambda}) \to H^{0}(X, L_{(q + j)\lambda})$ defined by $s \otimes f \mapsto s\text{Fr}^{*}(f)$ for sections $s \in H^{0}(X, L_{j\lambda})$ and $f \in H^{0}(X, L_{p\lambda})$.

The dual of the top row provides a decomposition of this map in the following way: $s \otimes f \in H^{0}(X, L_{j\lambda}) \otimes H^{0}(X, L_{pq\lambda})$ is first mapped to $s^p \otimes f \in H^{0}(X, L_{pj\lambda}) \otimes H^{0}(X, L_{pq\lambda})$, then to the product $s^p f \in H^{0}(X, L_{(j + q)\lambda})$, and then to $\text{Fr}^{*}(s^p f) \in H^{0}(X, L_{(j + q)\lambda})$. Since the two maps are the same, it follows $\text{Fr}^{*}(s^p f) = s\text{Fr}^{*}(f)$. This proves (b).

References