Positivity of the Cup Product in Cohomology of Flag Varieties
Associated to Kac-Moody Groups

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Let $G$ be a semisimple complex algebraic group with a Borel subgroup $B$ and the associated Weyl group $W$. Then the cohomology with integral coefficients $H^*(G/B)$ of the flag variety $G/B$ has the Schubert basis $[e^a]_w$ (cf. §7). Now, as it is well known, the cup product in this basis has nonnegative coefficients. The aim of this note is to extend this "nonnegativity" result to the flag variety of an arbitrary (not necessarily even symmetrizable) Kac-Moody group $G$ (cf. Theorem 8). To our knowledge, this nonnegativity result was not known for any Kac-Moody group beyond the (finite dimensional) semisimple group. The main difficulty in extending the proof from the finite dimensional case to an arbitrary Kac-Moody case lies in the fact that (unlike the finite dimensional case) there is no algebraic group which acts transitively on the flag variety of an infinite dimensional Kac-Moody group $G$.

In our attempt to prove this result, we obtained Theorem 4, which is more general.

By a variety, we mean a reduced (but not necessarily irreducible) separated scheme of finite type over the complex numbers $C$.

(1) Assumption

We take $X$ to be a complete (e.g., projective) variety over $C$, and $H$ to be a unipotent (in particular, connected) complex affine algebraic group which acts algebraically on $X$ with finitely many orbits.

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(2) Topology of $X$

Let $\{C_p\}_{p \in \mathbb{Z}}$ be the set of all the (distinct) $H$-orbits in $X$. Then each $C_p$ is a (Zariski) locally-closed subset of $X$ (cf. [H, Proposition 8.3]) and, moreover, $H$ being unipotent, each $C_p$ is isomorphic (as a variety) with the affine space $A^d$ for some nonnegative integer $d_p$.

Let $\bar{C}_p$ be the closure of $C_p$. Then, $\bar{C}_p$ being a complete irreducible variety of dim $d_p$, the singular homology $H_{2d_p}(\bar{C}_p, \mathbb{Z})$ with integral coefficients is freely generated by the fundamental class $[\bar{C}_p]$. Let $\text{Cl}(\bar{C}_p)$ denote its image in $H_{2d_p}(X, \mathbb{Z})$.

Using connected solvable group actions and Borel's fixed point theorem as an aid to relate the Chow group and the singular homology of a variety is, of course, quite old. For a modern account, see [Pa], where the authors prove Lemma (3) below under the additional assumption of smoothness. For our purpose, it is essential to avoid this hypothesis, and hence we have included an indication of its proof.

(3) Lemma. The total homology $H_i(X, \mathbb{Z})$ is freely generated (over $\mathbb{Z}$) by the homology classes $[\text{Cl}(\bar{C}_p)]_{p \in \mathbb{Z}}$ under the assumptions (1) on $X$. In particular, $H_{2d+1}(X, \mathbb{Z}) = 0$ for any $n \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. □

Proof. Define the filtration of $X$:

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$$

by $H$-stable closed subcones, where

$$X_i := \bigcup_{d \leq i} C_d.$$

Now the lemma follows by induction on $i$ and the long exact homology sequence associated to the pair $(X_{i+1}, X_i)$ (observe that each $X_i$ satisfies the assumption (1) as well.)

Let $(x^\sigma)_{\sigma \in \mathbb{Z}}$ be the dual $\mathbb{Z}$-basis of the singular cohomology $H^*(X, \mathbb{Z}) \cong \text{Hom}_\mathbb{Z}(H_*(X, \mathbb{Z}), \mathbb{Z})$, i.e., $x^\sigma([\bar{C}_p]) = \delta_{\sigma, p}$ for any $\sigma, p \in \mathbb{Z}$.

Express the cup product for any $\varphi_1, \varphi_2$:

$$x^{\varphi_1} \cdot x^{\varphi_2} = \sum_{\sigma \in \mathbb{Z}} n_{\sigma_1, \sigma_2}^{\sigma} x^{\sigma},$$

for some unique integers $n_{\sigma_1, \sigma_2}^{\sigma}$. With this notation, we have the following theorem.

(4) Theorem. Under assumption (3),

$$n_{\sigma_1, \sigma_2}^{\sigma} \geq 0 \quad \text{for any } \sigma, \sigma_1, \sigma_2 \in \mathbb{Z}. \quad □$$
As a preparation, we first prove the following.

(5) **Proposition.** Let a unipotent algebraic group $H$ act algebraically on a variety $X$ not necessarily with finitely many orbits, and let $Y \subset X$ be any closed irreducible subvariety of $X$ of dim $d$. Then there exists a cycle $D = \sum n_i [D_i]$, satisfying the following three conditions:

1. $D$ is an effective cycle, i.e., each $n_i \geq 0$.
2. Each $D_i$ is an irreducible $H$-stable closed subvariety of $X$ of dim $d$.
3. $D$ is rationally equivalent to the cycle $[Y]$. $\square$

**Proof.** We first consider the case $H = G_a \subset A^1$. Embed $G_a \subset P^1 = A^1(\mathbb{C})$. The morphism $\theta: G_a \times X \rightarrow G_a \times X$ given by $(t, x) = (t, t \cdot x)$ is a $G_a$-equivariant isomorphism, where $G_a$ acts on the domain of $\theta$ by $t \cdot (t_1, x) = (t + t_1, x)$, and it acts on the range of $\theta$ by $t \cdot (t_1, x) = (t + t_1, t \cdot x)$. Let $Z$ be the Zariski closure of the image $\theta(G_a \times Y)$ inside the product variety $P^1 \times X$, and let $\pi: P^1 \times X \rightarrow P^1$ be the projection on the first factor. We denote by $\pi_2^{-1}(0)$ the restriction of $\pi$ to $Z$. Then, as is easy to see,

$$\pi_2^{-1}(0) = Y.$$

The action $\circ$ of $G_a$ on $G_a \times X$ clearly extends to an action of $G_a$ on $P^1 \times X$. Since $\infty \in P^1$ is fixed and $Z$ is stable under $G_a$, and moreover $\pi_2$ is $G_a$-equivariant, we get that the fibre $\tilde{D} \coloneqq \pi_2^{-1}(\infty)$ is $G_a$-stable. Moreover, since $\dim Z = d + 1$, and each irreducible component $D_i$ of $\tilde{D}$ is of dim at least $d$ (cf. [S], Chapter I, §6.3, Theorem 7b), in particular, $\dim D_i = d$ for each $i$. Further, $\tilde{D}$ being $G_a$-stable, each $D_i$ is $G_a$-stable. Now taking the cycle $D$ determined by the scheme theoretic fibre $\pi_2^{-1}(\infty)$, we get the proposition for $H = G_a$.

The general case can be obtained, by induction on dim $H$, from the above case by considering a filtration of $H$ by closed subgroups $H_i$:

$$H = H_1 \supset H_2 \supset \cdots \supset H_d = \{1\},$$

such that each $H_{i+1}$ is normal in $H_i$ and $H_i/H_{i+1} \cong S_i$ as algebraic groups. Consider the $H_i$-varieties $H_i \times_{H_{i+1}} X$ and $(H_i/H_{i+1}) \times X$, where $H_i$ acts on the first variety via the left multiplication on the first factor and on the second variety via the diagonal action. Then there is an $H_i$-equivariant isomorphism $\phi_i: H_i \times_{H_{i+1}} X \rightarrow (H_i/H_{i+1}) \times X$ taking $(h, x) \mapsto (h_0 h^{-1}, h x)$, where $\lim_{h \rightarrow \infty} h x$, $\text{mod}(H_{i+1}, H_i)$. Now if we start with $Y$ being $H_{i+1}$-stable, the same argument as above (replacing $\circ$ by $\phi$) allows us to construct $D$ satisfying (1)-(3) with each $D_i$ being $H_i$-stable. This completes the induction and here proves the proposition. $\blacksquare$
(6) Proof of Theorem (4)

Let $A_1(X) = \mathbb{Q}_{x \in X} A_0(x)$ denote the total Chow group of $X$ (cf. [F, Chapter 1]). Then the proper morphism $\Delta: X \to X \times X, x \mapsto (x, x)$, induces a group homomorphism $\Delta_*: A_1(X) \to A_1(X \times X)$. Let $(\overline{C}_x) \subset A_0(X \times X)$ denote the element corresponding to the irreducible subvariety $\overline{C}_x \subset X$ of dim $d_x$. Let $C_x$ be the (irreducible) subvariety $\Delta(\overline{C}_x)$ of $X \times X$.

The group $H \times H$ acts on $X \times X$ factorwise and, moreover, the $H \times H$ orbits are precisely $\{C_x \times C_y\}_{x, y \in X}$. In particular, the complete variety $X \times X$ (cf. [M, §1.9]) under the action of $H \times H$ satisfies assumption (1). In particular, applying Proposition (5) to the $H \times H$ variety $X \times X$, we get (as elements of $A_0(X \times X)$)

$$[Y] = \sum m^p_{x, y} (C_x \times C_y),$$

for some nonnegative integers $m^p_{x, y}$. Consider the natural map (cf. [F, §19.1])

$$C = C_{x \times x}: A_0(X \times X) \to H_{1,0}(X \times X, \mathbb{Z}),$$

which takes a cycle $\sum C_i(V_i) \to \sum C_i(V_i)$, where $C_i(V_i)$ denotes the image of the fundamental homology class $\mu(V_i)$ of $V_i$ in $H_i(X \times X, \mathbb{Z})$.

Taking the image of (*) under $C$, we get

$$C([Y]) = \sum m^p_{x, y} C(C_x \times C_y).$$

From the above, we easily obtain that $m^p_{x, y} = m^p_{y, x}$, and hence the theorem follows.

(7) Flag varieties associated to Kac-Moody groups

We refer the reader to [K, §1], the notation of which we shall follow freely. In particular, $G$ is the Kac-Moody group (associated to any, not necessarily symmetric, Kac-Moody Lie algebra $g$) with standard maximal torus $T$, $U \subset G$ is the pro-potent pro-algebraic group as in [K], and $B := T \times U \subset G$ is the standard Borel subgroup. Let $W := \text{N}(T)/T$ be the Weyl group associated to $G$, where $\text{N}(T)$ is the normalizer of $T$ in $G$. Then $W$ is a Coxeter group. We have the Bruhat decomposition

$$X := G/B = \bigcup_{w \in W} (LwB)/B,$$

where the union is disjoint. For any $n \geq 0$, set

$$X_n = \bigcup_{w \in W_n} (LwB)/B,$$
where \( f_{\omega} \) is the length of \( \omega \). Then each \( X_n \) acquires the structure of a projective variety, called the 'stable' variety structure on \( X_n \). Under this structure, the inclusion \( i_n : X_n \rightarrow X_{n+1} \) is a closed embedding. In particular, \( X \) becomes a projective ind-variety. We endow \( X \) with the inductive limit Zariski and Hausdorff topologies and refer to them as the Zariski and Hausdorff topologies on \( X \), respectively. Each \( UwB/B \) is a locally closed subset of \( X \) under both the topologies and, moreover, under the induced variety structure, it is isomorphic with the affine space \( \mathbb{A}^{\dim \omega} \). Define the Schubert variety (for any \( \omega \in W \))

\[
X_\omega := \overline{UwB/B} \subset X,
\]

where the 'bar' denotes the closure under the Zariski topology. Then \( X_\omega \) is an irreducible projective variety of dim \( = f_{\omega} \) (under the subvariety structure inherited from any \( X_n \), \( n \geq 0 \)).

Now we consider \( X \) with the Hausdorff topology. By an argument similar to that of the proof of Lemma (3), we see that the elements \( \{ \Phi \} \in \mathcal{H}_{\omega \omega}(X, \mathbb{Z}) \) form a \( \mathbb{Z} \)-basis of \( \mathcal{H}(X, \mathbb{Z}) \). Let \( \{ \Phi \} \) be the dual basis of \( \mathcal{H}(X, \mathbb{Z}) \). Let \( \mathcal{H}(X, \mathbb{Z}) \) be the dual basis of \( \mathcal{H}(X, \mathbb{Z}) \); i.e.,

\[
\epsilon(\Phi) \Phi \Phi = \delta_{\omega \omega} \quad \text{for} \quad \omega, \omega \in W.
\]

Write

\[
\epsilon(\Phi) \Phi = \sum_{\omega \in W} n_{\omega} \Phi \Phi,
\]

for some unique \( n_{\omega} \in \mathbb{Z} \). We call the numbers \( n_{\omega} \) the cup product coefficients.

As a consequence of Theorem (4), we obtain the following.

(8) Theorem. With the notation and assumptions as in the above section (7), we have

\[
n_{\omega} \in \mathbb{Z}_+ \quad \text{for all} \quad u, v, w \in W.
\]

Proof. For any \( n \geq 0 \), as is easy to see, \( \{ \Phi^{(n)} \} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{H}(X_n, \mathbb{Z}) \), where \( i_n : X_n \rightarrow X \) is the inclusion and \( \mathcal{H}(X_n, \mathbb{Z}) \) is the induced map in cohomology. Further, \( \mathcal{H}(X_n, \mathbb{Z}) \) is the \( \mathcal{H}(X_n, \mathbb{Z}) \) for \( i_n \).

Now fix \( u, v, w \) and choose \( n \geq \max(f(u), f(v), f(w)) \). Then the theorem follows immediately by applying Theorem (4) to the projective variety \( X_n \) and using the following lemma.

The following lemma is well known and follows from [SI, §1.11, Lemma 2].

(9) Lemma. For any \( n \geq 0 \), there exists a (finite dimensional) unipotent algebraic group \( U_n \), which is a quotient group of \( U \), such that the action of \( U \) on \( X_n \) (given by the left
multiplication factors through the action of \( U \) on \( X \), to give an algebraic action of \( U \) on \( X \). In particular, the \( U \)-variety \( X \) satisfies assumption (1).

\( \square \)

Remarks. (a) There is a 'combinatorial' formula for the cup product coefficients given by Kostant and Kumar (cf. [KX, Corollary 5.13(a)]) in terms of the 'Nil Hecke ring.' More specifically, the 'matrix D.' A simpler expression for the matrix D is given by Billey in the finite case and extended to the arbitrary Kac-Moody case by Kumar; cf. [B, Theorem 3 and the Appendix]. But the formula involves summation of certain positive and negative terms. We have not been able to deduce the nonnegativity of the cup product coefficients \( n_{\alpha} \) from this formula. It may be mentioned that even in the case when \( G \) is a finite dimensional semisimple algebraic group, to our knowledge, there are no combinatorial proofs for the nonnegativity of the cup product coefficients.

(b) Theorem (4) can easily be generalized to the case when \( X \) is an arbitrary (not necessarily complete) variety with an action of a unipotent group with finitely many orbits, provided we replace the singular cohomology \( H^*(X, \mathbb{Z}) \) by the singular cohomology with compact supports \( H^*_c(X, \mathbb{Z}) \), and replace the singular homology \( H_*(X, \mathbb{Z}) \) by the Borel-Moore homology. The necessary prerequisites for evaluating cycle classes in the Borel-Moore homology can be found in [F, Chapter 19]. The precise formulation of this generalization (and its proof) is straightforward and hence is left to the interested reader.

(c) As mentioned by Dale Peterson, the nonnegativity of the cup product coefficients, in the case when \( G \) is an affine Kac-Moody group, gives rise to a certain 'nonnegativity result' for the quantum cohomology of the finite dimensional flag varieties.

(d) In Theorem (4), both the hypotheses—that \( H \) is unipotent, and \( H \) has finitely many orbits in \( X \)—are essential. This can easily be seen by considering the blow-up of \( \mathbb{P}^1 \) at a point.

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References


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