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Positivity of the Cup Product in Cohomology of Flag Varieties Associated to Kac-Moody Groups

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Let G be a semisimple complex algebraic group with a Borel subgroup B and the associated Weyl group W. Then the cohomology with integral coefficients $H^*(G/B)$ of the flag variety G/B has the Schubert basis $\{\epsilon^w\}_{w \in W}$ (cf. §7). Now, as it is well known, the cup product in this basis has nonnegative coefficients. The aim of this note is to extend this 'nonnegativity' result to the flag variety of an arbitrary (not necessarily even symmetrizable) Kac-Moody group G (cf. Theorem 8). To our knowledge, this nonnegativity result was not known for any Kac-Moody group beyond the (finite dimensional) semisimple group. The main difficulty in extending the proof from the finite dimensional case to an arbitrary Kac-Moody case lies in the fact that (unlike the finite dimensional case) there is no algebraic group which acts transitively on the flag variety of an infinite dimensional Kac-Moody group G.

In our attempt to prove this result, we obtained Theorem (4), which is more general.

By a variety, we mean a reduced (but not necessarily irreducible) separated scheme of finite type over the complex numbers $\mathbb{C}$.

(1) Assumption

We take X to be a complete (e.g., projective) variety over $\mathbb{C}$, and H to be a unipotent (in particular, connected) complex affine algebraic group which acts algebraically on X with finitely many orbits.

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(2) Topology of X

Let $\{C_\sigma\}_{\sigma\in S}$ be the set of all the (distinct) H-orbits in X. Then each C_σ is a (Zariski) locally-closed subset of X (cf. [H, Proposition 8.3]) and, moreover, H being unipotent, each C_σ is isomorphic (as a variety) with the affine space $\mathbb{A}^{d_\sigma}$ for some nonnegative integer d_σ.

Let $\bar{C}_\sigma$ be the closure of C_σ. Then, $\bar{C}_\sigma$ being a complete irreducible variety of dim d_σ, the singular homology $H_{2d_\sigma}(\bar{C}_\sigma, \mathbb{Z})$ with integral coefficients is freely generated by the fundamental class $\mu(\bar{C}_\sigma)$. Let $\mathrm{Cl}(\bar{C}_\sigma)$ denote its image in $H_{2d_\sigma}(X,\mathbb{Z})$.

Using connected solvable group actions and Borel's fixed point theorem as an aid to relate the Chow group and the singular homology of a variety is, of course, quite old. For a modern account, see [Fu], where the authors prove Lemma (3) below under the additional assumption of smoothness. For our purpose, it is essential to avoid this hypothesis, and hence we have included an indication of its proof.

(3) Lemma. The total homology $H_*(X,\mathbb{Z})$ is freely generated (over $\mathbb{Z}$) by the homology classes $\{\mathrm{Cl}\,(\bar{C}_\sigma)\}_{\sigma\in S}$ (under the assumptions (1) on X). In particular, $H_{2n+1}(X,\mathbb{Z}) = 0$ for any $n \in \mathbb{Z}_+ := \{0,1,2,\ldots\}$. □

Proof. Define the filtration of X:

$$X_0 \subset X_1 \subset \cdots \subset X_N = X$$

by H-stable closed subsets, where

$$X_i := \bigcup_{d_\sigma \le i} C_\sigma.$$

Now the lemma follows by induction on i and the long exact homology sequence associated to the pair (X_{i+1}, X_i). (Observe that each X_i satisfies the assumption (1) as well.) ■

Let $\{x^\sigma\}_{\sigma\in S}$ be the dual $\mathbb{Z}$-basis of the singular cohomology

$H^*(X,\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_*(X,\mathbb{Z}),\mathbb{Z})$, i.e., $x^\sigma(\mathrm{Cl}\,(\bar{C}_\mu)) = \delta_{\sigma,\mu}$, for any $\sigma, \mu \in S$.

Express the cup product for any σ_1, σ_2:

$$x^{\sigma_1}\cdot x^{\sigma_2} = \sum_{\sigma\in S} n^{\sigma}_{\sigma_1,\sigma_2} x^\sigma,$$

for some (unique) integers $n^{\sigma}_{\sigma_1,\sigma_2}$. With this notation, we have the following theorem.

(4) Theorem. Under assumption (1),

$$n^{\sigma}_{\sigma_1,\sigma_2} \ge 0 \quad \text{for any } \sigma, \sigma_1, \sigma_2 \in S.$$ □

As a preparation, we first prove the following.

(5) Proposition. Let a unipotent algebraic group H act algebraically on a variety X (not necessarily with finitely many orbits), and let $Y \subset X$ be any closed irreducible subvariety (of dim d). Then there exists a cycle $D = \sum_i n_i[D_i]$, satisfying the following three conditions:

(1) D is an effective cycle, i.e., each $n_i \geq 0$.

(2) Each D_i is an irreducible H-stable closed subvariety of X of dim d.

(3) D is rationally equivalent to the cycle $[Y]$. □

Proof. We first consider the case $H = G_a \simeq \mathbb{A}^1$: Embed $G_a \subset \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. The morphism θ: $G_a \times X \to G_a \times X$ given by $(t, x) = (t, t \cdot x)$ is a G_a-equivariant isomorphism, where G_a acts on the domain of θ by $t \cdot (t_1, x) = (t + t_1, x)$, and it acts on the range of θ by $t \odot (t_1, x) = (t + t_1, t \cdot x)$. Let Z be the Zariski closure of the image $\theta(G_a \times Y)$ inside the product variety $\mathbb{P}^1 \times X$, and let $\pi : \mathbb{P}^1 \times X \to \mathbb{P}^1$ be the projection on the first factor. We denote by π_Z the restriction of π to Z. Then, as is easy to see,

$$\pi_Z^{-1}(0) = Y.$$

The action $\odot$ of G_a on $G_a \times X$ clearly extends to an action of G_a on $\mathbb{P}^1 \times X$. Since $\infty \in \mathbb{P}^1$ is fixed and Z is stable under G_a, and moreover π_Z is G_a-equivariant, we get that the fibre $\tilde{D} := \pi_Z^{-1}(\infty)$ is G_a-stable. Moreover, since $\dim Z = d + 1$, and each irreducible component D_i of $\tilde{D}$ is of dim at least d (cf. [S, Chapter I, §6.3, Theorem 7]), in particular, $\dim D_i = d$ for each i. Further, $\tilde{D}$ being G_a-stable, each D_i is G_a-stable. Now taking the cycle D determined by the scheme theoretic fibre $\pi_Z^{-1}(\infty)$, we get the proposition for $H = G_a$.

The general case can be obtained, by induction on $\dim H$, from the above case by considering a filtration of H by closed subgroups H_j:

$$H = H_1 \supset H_2 \supset \cdots \supset H_\ell = (e),$$

such that each H_{j+1} is normal in H_j and $H_j/H_{j+1} \approx G_a$ as algebraic groups. Consider the H_j-varieties $H_j \times_{H_{j+1}} X$ and $(H_j/H_{j+1}) \times X$, where H_j acts on the first variety via the left multiplication on the first factor and on the second variety via the diagonal action. Then there is an H_j-equivariant isomorphism σ_j: $H_j \times_{H_j+1} X \to (H_j/H_{j+1}) \times X$ taking $(h, x) \mapsto (h \bmod H_{j+1}, hx)$. Now if we start with Y being H_{j+1}-stable, the same argument as above (replacing σ by σ_j) allows us to construct D satisfying (1)–(3) with each D_i being H_j-stable. This completes the induction and here proves the proposition. ■

(6) Proof of Theorem (4)

Let $A_*(X) = \oplus_{d\geq 0} A_d(X)$ denote the total Chow group of X (cf. [F, Chapter 1]). Then the proper morphism $\Delta\colon X \to X\times X$, $x \mapsto (x,x)$, induces a group homomorphism $\Delta_*\colon A_*(X) \to A_*(X\times X)$. Let $[\bar{C}_\sigma] \in A_{d_\sigma}(X)$ denote the element corresponding to the irreducible subvariety $\bar{C}_\sigma \subset X$ of dim d_σ. Let Y_σ be the (irreducible) subvariety $\Delta(\bar{C}_\sigma)$ of $X \times X$.

The group $H \times H$ acts on $X \times X$ factorwise and, moreover, the $H \times H$ orbits are precisely $\{C_{\sigma_1} \times C_{\sigma_2}\}_{\sigma_1,\sigma_2\in S}$. In particular, the complete variety $X \times X$ (cf. [M, §I.9]) under the action of $H\times H$ satisfies assumption (1). In particular, applying Proposition (5) to the $H \times H$ variety $X \times X$, we get (as elements of $A_{d_\sigma}(X \times X)$)

$$[Y_\sigma] = \sum m^\sigma_{\sigma_1,\sigma_2}[\bar{C}_{\sigma_1} \times \bar{C}_{\sigma_2}], \qquad (*)$$

for some nonnegative integers $m^\sigma_{\sigma_1,\sigma_2}$. Consider the natural map (cf. [F, §19.1])

$$\mathrm{Cl} = \mathrm{Cl}_{X\times X}\colon\ A_*(X \times X) \to H_{2*}(X \times X, \mathbb{Z}),$$

which takes a cycle $\sum n_i[V_i] \mapsto \sum n_i\ \mathrm{Cl}(V_i)$, where $\mathrm{Cl}(V_i)$ denotes the image of the fundamental homology class $\mu(V_i)$ of V_i in $H_*(X \times X, \mathbb{Z})$.

Taking the image of (*) under Cl, we get

$$\mathrm{Cl}[Y_\sigma] = \sum m^\sigma_{\sigma_1,\sigma_2}\ \mathrm{Cl}(\bar{C}_{\sigma_1}) \otimes \mathrm{Cl}(\bar{C}_{\sigma_2}).$$

From the above, we easily obtain that $m^\sigma_{\sigma_1,\sigma_2} = n^\sigma_{\sigma_1,\sigma_2}$, and hence the theorem follows. ■

(7) Flag varieties associated to Kac-Moody groups

We refer the reader to [K, §1], the notation of which we shall follow freely. In particular, G is the Kac-Moody group (associated to any, not necessarily symmetrizable, Kac-Moody Lie algebra $\mathfrak{g}$) with standard maximal torus T, $U \subset G$ is the prounipotent proalgebraic group as in [K], and $B := T \ltimes U \subset G$ is the standard Borel subgroup. Let $W := N(T)/T$ be the Weyl group associated to G, where N(T) is the normalizer of T in G. Then W is a Coxeter group. We have the Bruhat decomposition

$$\mathcal{X} := G/B = \bigcup_{w\in W} (UwB/B),$$

where the union is disjoint. For any $n \geq 0$, set

$$\mathcal{X}_n = \bigcup_{\substack{w\in W\\ \ell(w)\leq n}} (UwB/B),$$

where $\ell(w)$ is the length of w. Then each $\mathcal{X}_n$ acquires the structure of a projective variety, called the 'stable' variety structure on $\mathcal{X}_n$. Under this structure, the inclusion $i_n\colon \mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$ is a closed embedding. In particular, $\mathcal{X}$ becomes a projective ind-variety. We endow $\mathcal{X}$ with the inductive limit Zariski and Hausdorff topologies and refer to them as the Zariski and Hausdorff topologies on $\mathcal{X}$, respectively. Each UwB/B is a locally closed subset of $\mathcal{X}$ under both the topologies and, moreover, under the induced variety structure, it is isomorphic with the affine space $\mathbb{A}^{\ell(w)}$. Define the Schubert variety (for any $w \in W$)

$$\mathcal{X}_w := \overline{UwB/B} \subset \mathcal{X},$$

where the 'bar' denotes the closure under the Zariski topology. Then $\mathcal{X}_w$ is an irreducible projective variety of dim $= \ell(w)$ (under the subvariety structure inherited from any $\mathcal{X}_n$, $n \geq \ell(w)$).

Now we consider $\mathcal{X}$ with the Hausdorff topology. By an argument similar to that of the proof of Lemma (3), we see that the elements $\{\mathrm{Cl}\,(\mathcal{X}_w) \in H_{2\ell(w)}(\mathcal{X}, \mathbb{Z})\}_{w\in W}$ form a $\mathbb{Z}$-basis of $H_*(\mathcal{X}, \mathbb{Z})$. Let $\{\varepsilon^w\}_{w\in W}$ be the dual basis of $H^*(\mathcal{X}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_*(\mathcal{X}, \mathbb{Z}), \mathbb{Z})$; i.e.,

$$\varepsilon^v(\mathrm{Cl}\,(\mathcal{X}_w)) = \delta_{v,w} \quad \text{for } v, w \in W.$$

Write

$$\varepsilon^u \varepsilon^v = \sum_{w\in W} n^w_{u,v} \varepsilon^w,$$

for some (unique) $n^w_{u,v} \in \mathbb{Z}$. We call the numbers $n^w_{u,v}$ the *cup product coefficients.*

As a consequence of Theorem (4), we obtain the following.

(8) Theorem. With the notation and assumptions as in the above section (7), we have

$$n^w_{u,v} \in \mathbb{Z}_+ \text{ for all } u, v, w \in W. \qquad \square$$

Proof. For any $n \geq 0$, as is easy to see, $\{i_n^*(\varepsilon^\theta)\}_{\ell(\theta)\leq n}$ is a $\mathbb{Z}$-basis of $H^*(\mathcal{X}_n, \mathbb{Z})$, where $i_n\colon \mathcal{X}_n \hookrightarrow \mathcal{X}$ is the inclusion and $i_n^*\colon H^*(\mathcal{X}, \mathbb{Z}) \to H^*(\mathcal{X}_n, \mathbb{Z})$ is the induced map in cohomology. Further, $i_n^*(\varepsilon^\theta) = 0$, if $\ell(\theta) > n$.

Now fix u, v, w and choose $n \geq \max(\ell(u), \ell(v), \ell(w))$. Then the theorem follows immediately by applying Theorem (4) to the projective variety $\mathcal{X}_n$ and using the following lemma. ■

The following lemma is well known and follows from [Sl, §1.11, Lemma 2].

(9) Lemma. For any $n \geq 0$, there exists a (finite dimensional) unipotent algebraic group U_n, which is a quotient group of U, such that the action of U on $\mathcal{X}_n$ (given by the left multiplication) factors through the action of $\mathcal{U}_n$ on $\mathcal{X}_n$, to give an algebraic action of $\mathcal{U}_n$ on $\mathcal{X}_n$. In particular, the $\mathcal{U}_n$-variety $\mathcal{X}_n$ satisfies assumption (1). $\square$

(10) Remarks. (a) There is a 'combinatorial' formula for the cup product coefficients given by Kostant and Kumar (cf. [KK, Corollary 5.13(a)]) in terms of the 'Nil Hecke ring,' more specifically, the 'matrix D.' (A simpler expression for the matrix D is given by Billey in the finite case and extended to the arbitrary Kac-Moody case by Kumar; cf. [B, Theorem 3 and the Appendix].) But the formula involves summation of certain positive and negative terms. We have not been able to deduce the nonnegativity of the cup product coefficients $n^w_{u,v}$ from this formula. It may be mentioned that even in the case when G is a (finite dimensional) semisimple algebraic group, to our knowledge, there are no combinatorial proofs for the nonnegativity of the cup product coefficients.

(b) Theorem (4) can easily be generalized to the case when X is an arbitrary (not necessarily complete) variety with an action of a unipotent group with finitely many orbits, provided we replace the singular cohomology $H^*(X, \mathbb{Z})$ by the singular cohomology with compact supports $H^*_c(X, \mathbb{Z})$, and replace the singular homology $H_*(X, \mathbb{Z})$ by the Borel-Moore homology. The necessary prerequisites for evaluating cycle classes in the Borel-Moore homology can be found in [F, Chapter 19]. The precise formulation of this generalization (and its proof) is straightforward and hence is left to the interested reader.

(c) As mentioned by Dale Peterson, the nonnegativity of the cup product coefficients, in the case when G is an affine Kac-Moody group, gives rise to a certain 'nonnegativity result' for the quantum cohomology of the finite dimensional flag varieties.

(d) In Theorem (4), both the hypotheses—that H is unipotent, and H has finitely many orbits in X—are essential. This can easily be seen by considering the blow-up of $\mathbb{P}^2$ at a point.

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