Fusion Product of Positive Level Representations and Lie Algebras Homology

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Introduction

Let \( g \) be a finite-dimensional simple Lie algebra (with the associated simply-connected complex simple Lie group \( G \)) and let \( \mathfrak{a} \) be the corresponding affine Kac-Moody Lie algebra. Fix a positive integer \( 
\alpha \), and let \( \mathfrak{a}_+ \) be the root lattice generated by the set of nonnegative weight vectors \( \rho \) of the lie algebra \( \mathfrak{g} \). We recall the definition of the tensor product \( \otimes \) in \( \mathfrak{g}(\mathfrak{g}) \) and of the standard tensor product \( \otimes \) in \( \mathfrak{g}(\mathfrak{g}) \) of the finite-dimensional irreducible modules \( V \) of \( \mathfrak{g} \) (cf. Definition 2.2). Our definition of the product \( \otimes \) is very similar and geometric in nature.

A comprehensive study of tensor products \( \otimes \) with \( g \) is given in [1]. The authors of this manuscript are partially supported by grants from the NSF. We first establish some lemmas and technical results relevant for the fusion product \( \mathfrak{g}(\mathfrak{g}) \) of \( \mathfrak{g} \). We then use these results to prove Theorem 2.1, which states that the fusion product \( \mathfrak{g}(\mathfrak{g}) \) is isomorphic to the Lie algebra \( \mathfrak{g}(\mathfrak{g}) \) of the Lie algebra \( \mathfrak{g} \).
is completely determined by Kostant's "homology result" for the affine Kac-Moody algebra g (proved by Gindikin-Lepowsky).

Validity of the above-mentioned Conjecture 2.5 will immediately imply that the two products $\Delta$ and $\Delta'$ in $R(g)$ are the same. In fact, a much weaker result will imply these equalities (cf. Lemma 4.3). We prove this weaker result for the simple case of $g$ of type $A_n, B_n, C_n, D_n$ and $G_2$ (cf. Theorem 4.4). This provides an alternative (more uniform) proof of a result of Fuchs (cf. Remark 4.2.5); see also Remark 4.4.5. In fact, we are also able to characterize the full homology $H_i(g)$ only for the group $G=SL(2)$.

This is only an announcement of results without proofs.

1. Preliminaries and Notation

Definition 1.1. Let $g$ be a finite-dimensional complex simple Lie algebra. We also fix a fixed subspace $h$ and a Cartan submodule $h \leq \mathfrak{g}$ of $g$. Then the associated affine Kac-Moody Lie algebra is defined by the space

$$\hat{g} \cong g \otimes \mathbb{C}[t, t^{-1}]$$

which is the direct sum of $g$ with the one-dimensional complex vector space $\mathbb{C}$.

Definition 1.2. The Killing form on $g$ is the bilinear form $\kappa : g \times g \to \mathbb{C}$ defined by

$$\kappa(\xi, \eta) = \mbox{tr}(\theta(\xi) \cdot \eta),$$

where $\theta$ is the inner automorphism of $g$ induced by the Cartan subalgebra $h$.

Definition 1.3. The Levi subalgebra $\mbox{Levi}(g)$ of $g$ is the direct sum of the subalgebra $\mathfrak{h}$ of $g$ and the subalgebra of all the elements of $g$ which are in $\mathfrak{h}$.

2. A Certain Complex and Lie Algebra Homology

Definition 2.1. (Definition of a complex). Fix a positive integer $m$ and a finite-dimensional $m$-dimensional complex representation $V = V(\lambda)$ of $g$ with highest weight $\lambda$ in the root lattice of $g$. Recall the parabolic subalgebra of Kac-Moody Lie algebras (cf. [6] and [10], Theorem (3.22))

$$\hat{g} = \mathfrak{g} \oplus \mathbb{C} \cdot 1,$$

where $1$ is the identity element of $g$, and the associated complex, $\hat{g}$, is the direct sum of the space $\mathfrak{g}$ with the complex vector $\mathbb{C}$.

Therefore, we have the following definition:

$$\hat{g}^\wedge = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \otimes \mathbb{C},$$

and the Cartan subalgebra $\mathfrak{h}$ of $\hat{g}$ is the direct sum of the subalgebra $\mathfrak{h}$ of $g$ and the subalgebra of all the elements of $g$ which are in $\mathfrak{h}$.

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and the Cartan subalgebra $\mathfrak{h}$ of $\hat{g}$ is the direct sum of the subalgebra $\mathfrak{h}$ of $g$ and the subalgebra of all the elements of $g$ which are in $\mathfrak{h}$.
Take any \( \mu \in \text{P}^3 \), realize \( V(\mu) \) as a module for \( \mathfrak{g} \) via evaluation at \( 1 \), consider it as a module for \( \mathfrak{g} \) by letting \( \mathfrak{g} \) act trivially on \( V(\mu) \) via the Lie algebra structure by \( V(\mu) \). Then:

\[
\text{Tr}(\mathfrak{g}) \otimes V(\mu) \to \mathfrak{g} \otimes V(\mu) \to V(\mu) 
\]

we get a resolution:

\[
\cdots \to F_2 \otimes V(\mu) \to F_1 \otimes V(\mu) \to F_0 \otimes V(\mu) \to V(\mu) \to 0.
\tag{2.12}
\]

Torsion the complex (2.12) with \( C \) over \( \mathbb{Z}_l \) [19] and using the Hopf algebra principle [cf. [7], Proposition 1.15] we obtain a complex of \( \mu \)-modules and \( \mu \)-module maps:

\[
\cdots \to F_2 \otimes \mathbb{Z}_l \rightarrow F_1 \otimes \mathbb{Z}_l \rightarrow F_0 \otimes \mathbb{Z}_l \rightarrow 0,
\tag{3.32}
\]

where \( F_K \otimes \mathbb{Z}_l \rightarrow \mathbb{Z}_l \otimes V(\mu) \).

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\tag{2.12}
\]

Then for each \( \mu \in \text{P}^3 \), there is a tensor product:

\[
L(V(\mu), \phi) = L(V(\mu), \sigma),
\]

which is again an integrable representation of \( \mathfrak{g} \) with the same central charge \( \ell \), in the following sense:

\[
L(V(\mu), \phi) = \bigoplus_{\sigma} \bigoplus_{\mathbf{c}(\sigma) \in \mathbb{Z}_l} R(V(\mu), \sigma),
\]

where \( \mathbf{c}(\sigma) \) is the dimension of the space of vacua for the Riemann surface \( \mathbb{P}^2 \) with three punctures \( k, l, m \) and the representation \( V(\mu), \phi \) and \( V(k) \) attached to them respectively (cf. [30]).

Let \( \mathcal{G} \) be the affine Kac-Moody group associated to the Lie algebra \( \mathfrak{g} \) and \( \tilde{\mathcal{G}} \) be the \( \mathfrak{g} \)-module associated to the Lie algebra \( \mathfrak{g} \) (cf. [31]). Then \( \tilde{\mathcal{G}} \) is a projective limit. Now, given a finite-dimensional \( \mathfrak{g} \)-module \( V \), we can consider the associated homogeneous vector bundle \( V \otimes \mathcal{O} \) on \( X \) and the corresponding Euler-Poincaré characteristic (which is a virtual \( \mathbb{Z} \)-module):

\[
\chi(X, V) = \sum_{j=0}^{\infty} (-1)^j \dim H_j(X, V).
\]

Recall that \( \chi(X, V) \) is determined in [9]. Corollary 2.15 (and also in [13]). We give a new definition of a fusion product \( \text{F} \) following the following:

Definition 3.3. For any positive integer \( \ell \) and \( \mu \in \text{P}^3 \), define

\[
\text{F}_\mu(\ell) \otimes \text{F}_\mu(\ell) \equiv \chi(X, V).
\]

As virtual \( \mathbb{Z} \)-modules, where the \( \mu \)-module \( V = \text{F}_\mu(\ell) \otimes \text{F}_\mu(\ell) \) and the notation \( \ell \).
Let $R[G]$ be the free Abelian group generated by $\{LV(u) : u \in P]\}$. Then $G\otimes^\alpha R[G]$ gives rise to a product in $R[G]$.

Let $G$ be the (singly-)monochromatic complex algebraic group with Lie algebra $\mathfrak{g}$, and let $R[G]$ be its representation ring. $R[G]$ is the free Abelian group generated by the $G$-modules $L(V) : \alpha \in \mathfrak{g}^*$, which is a ring under the usual tensor product of $G$-modules. Define the $\alpha$-linear map $R: R[G] \rightarrow R[G]$ by $R(V) = V(X^\alpha)$, where $X$ is the homogeneous vector bundle on $X$ associated to the $\mathbb{P}$-module $L(W)$ and $Y^\alpha$ is the dual vector bundle on $X$.

We have the following lemma.

Lemma 3.2. The kernel of $\alpha$ is an ideal of $R[G]$. Moreover, $\alpha$ is a homomorphism with respect to the product $G\otimes^\alpha R[G]$.

In particular, $R[G]$ is an associative (and commutative) algebra under $G\otimes^\alpha$.

4. Comparison of the Two Fusion Products

We denote $R_\beta[G]$ equipped with the fusion product $G\otimes^\beta R[G]$ (see [10, 12]) by $(R[G], G\otimes^\beta)$. Recall that the associativity of $(R[G], G\otimes^\beta)$ follows from the Schur-Weyl duality [10] for $P$ with parameters.

Let $\alpha \in \mathfrak{g}$ be a regular vector, $\alpha \neq 0$.

$L(V,\alpha) = \sum_{x \in G} \text{dim} \left( H_{\text{L}}(V,\alpha,\text{L}(L(V,\alpha),\alpha \in \mathfrak{g}^*)) \right)$.

For any $x \in G$ let $L_x$ be the $4$th fundamental weight corresponding to $x$. We have the following lemma.

Lemma 4.1. The products $G\otimes^\alpha R[G]$ coincide if and only if for all $x, y, v \in P^\alpha$, $L_x(v, \alpha) = 0$.

In particular, if $G\otimes^\alpha R[G]$ is associative then one must have $L_x(v, \alpha) = 0$.

As a consequence of the above lemma, together with some results of [1, Corollary 4.3], [2] and some partial determinations of $L_x(L(V,\alpha),\alpha \in \mathfrak{g}^*)$ for those $\alpha$ such that $(\alpha, P^\alpha) \neq 0$, we obtain the following result.

Theorem 4.2. For any simply-connected group $G$ of type $A_n, B_n, C_n, D_n$ or $G_2$, the products $G\otimes^\alpha R[G]$ coincide, and in particular, for those groups, the $\alpha$-linear map $R: R[G] \rightarrow R[G]$ (cf. Definitions 3.1) is an algebra homomorphism with respect to the product $G\otimes^\alpha R[G]$.

Theorem 4.3. For any vector $(x, \alpha) \neq 0$, for all $x, y, v \in P^\alpha$, if $\alpha$ is integer, $\alpha \in \mathbb{Z}$, then $L_x(v, \alpha) = 0$.

The “in particular” statement of the above theorem is due to Pakings [5, Appendix].

References


