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Fusion Product of Positive Level Representations and Lie Algebra Homology

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Introduction

Let $\mathfrak{g}$ be a (finite-dimensional) complex simple Lie algebra (with the associated simply-connected complex algebraic group G) and let $\tilde{\mathfrak{g}}$ be the corresponding affine Kac-Moody Lie algebra. Fix a positive integer ℓ. Let $R_\ell(G)$ be the free Abelian group generated by the set of integrable highest weight irreducible $\tilde{\mathfrak{g}}$-modules of level (or central charge) ℓ. Then there is a fusion product $\otimes^\bullet$ in $R_\ell(G)$ making it into a commutative and associative algebra. (The associativity of this algebra follows from the so called factorization rule.) The definition of the product $\otimes^\bullet$ is in terms of the dimension of a certain space of vacua (cf. Definition 3.1). We give a new definition of a fusion product denoted $\otimes^F$ in $R_\ell(G)$ in terms of the Euler-Poincaré characteristic of certain homogeneous vector bundles on the generalised affine flag variety X (cf. Definition 3.2). Our definition of the product $\otimes^F$ is very simple and geometric in nature.

A comparison of the two fusion products led us to define a certain chain-complex $\hat{F}$ whose terms are finite-dimensional G-modules (cf. (2.3)). The differentials of this complex are highly non-trivial and are obtained by considering the BGG resolution for the affine Kac-Moody algebra $\tilde{\mathfrak{g}}$. We have made a conjecture on the homology $H_\bullet(\hat{F})$ of this complex (cf. Conjecture 2.3 and Theorem 2.4). The homology $H_\bullet(\hat{F})$ is isomorphic to the Lie algebra homology $H_\bullet(\tilde{\mathfrak{u}}^-, L(V(\nu),\ell)\otimes V(\mu;1))$, where the notation is as in §1 and Definition 2.1. Recall that if we take $\mu = 0$, then $H_\bullet(\tilde{\mathfrak{u}}^-, L(V(\nu),\ell)\otimes V(\mu;1))$

is completely determined by Kostant's "n-homology result" for the affine Kac-Moody algebra $\tilde{\mathfrak{g}}$ (proved by Garland-Lepowsky).

Validity of the above-mentioned Conjecture 2.3 will immediately imply that the two products $\otimes^*$ and $\otimes^F$ in $R_\ell(G)$ are the same. In fact, a much weaker result will imply their equality (cf. Lemma 4.1). We prove this weaker result for all simple $\mathfrak{g}$ of type A_n, B_n, C_n, D_n and G_2 (cf. Theorem 4.2). This provides an alternative (more uniform) proof of a result of Faltings (cf. Remark 4.3(b); see also Remark 4.3(c)). So far, we are able to determine the full homology $H_*(\bar{F})$ only for the group $G = SL(2)$ (cf. Proposition 4.4).

This is only an announcement of results without proofs.

1. Preliminaries and Notation

DEFINITION 1.1. Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. (We also fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$ of $\mathfrak{g}$.) Then the associated *affine Kac-Moody Lie algebra* is by definition the space

$$\tilde{\mathfrak{g}} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$$

together with the Lie bracket (for $X, Y \in \mathfrak{g}$ and $P, Q \in \mathbb{C}[t^{\pm 1}]$)

$$[X \otimes P, Y \otimes Q] = [X, Y] \otimes PQ + (\langle X, Y\rangle \operatorname*{Res}_{t=0}(\frac{dP}{dt}Q))K, \quad \text{and}$$
$$[\tilde{\mathfrak{g}}, K] = 0,$$

where $\langle \cdot, \cdot \rangle$ is the Killing form on $\mathfrak{g}$, normalized so that $\langle \theta, \theta \rangle = 2$ for the highest root θ of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ sits as a Lie subalgebra of $\tilde{\mathfrak{g}}$ as $\mathfrak{g} \otimes t^0$. The Lie algebra $\tilde{\mathfrak{g}}$ admits a distinguished "parabolic" subalgebra

$$\tilde{\mathfrak{p}} := \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K.$$

We also define its "nil-radical" $\tilde{\mathfrak{u}}$ (which is an ideal of $\tilde{\mathfrak{p}}$) by

$$\tilde{\mathfrak{u}} := \mathfrak{g} \otimes t\mathbb{C}[t],$$

and its "Levi component" (which is a Lie subalgebra of $\tilde{\mathfrak{p}}$)

$$\tilde{\mathfrak{p}}^0 := \mathfrak{g} \otimes t^0 \oplus \mathbb{C}K.$$

Clearly (as a vector space)

$$\tilde{\mathfrak{p}} = \tilde{\mathfrak{u}} \oplus \tilde{\mathfrak{p}}^0.$$

Also define $\tilde{\mathfrak{u}}^- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \subset \tilde{\mathfrak{g}}$, and the Cartan subalgebra $\tilde{\mathfrak{h}} = \mathfrak{h} \otimes t^0 \oplus \mathbb{C}K$ of $\tilde{\mathfrak{g}}$. Let W (resp. $\widetilde{W}$) be the Weyl group of $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$). Then $\widetilde{W}$ acts on the dual space $\tilde{\mathfrak{h}}^*$ by linear automorphisms. Let $\rho \in \mathfrak{h}^*$ be half the sum of the positive roots of $\mathfrak{g}$ and define $\tilde{\rho} \in \tilde{\mathfrak{h}}^*$ by $\tilde{\rho}|_{\mathfrak{h}} = \rho$ and

$$\tilde{\rho}(K) = 1 + \langle \rho, \theta^\vee \rangle = \text{dual Coxeter number of } \mathfrak{g},$$

where θ (as earlier) is the highest root of $\mathfrak{g}$ and $\theta^\vee$ is the associated coroot. Define the shifted action of $\widetilde{W}$ on $\tilde{\mathfrak{h}}^*$ by $w * \beta = w(\beta + \tilde{\rho}) - \tilde{\rho}$, for $\beta \in \tilde{\mathfrak{h}}^*$ and $w \in \widetilde{W}$. Fix a positive integer ℓ. Let $P^+ \subset \mathfrak{h}^*$ be the set of dominant integral weights of $\mathfrak{g}$ and let $P_\ell^+ := \{\lambda \in P^+ : \langle \lambda, \theta^\vee \rangle \leqslant \ell\}$ be the fundamental alcove.

Define the loop algebra $\Omega(\mathfrak{g}) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$ with Lie bracket $[X \otimes P, Y \otimes Q] = [X, Y] \otimes PQ$, for $X, Y \in \mathfrak{g}$ and $P, Q \in \mathbb{C}[t^{\pm 1}]$. Then $\tilde{\mathfrak{g}}$ can be viewed as a one-dimensional central extension of $\Omega(\mathfrak{g})$:

$$0 \longrightarrow \mathbb{C}K \longrightarrow \tilde{\mathfrak{g}} \xrightarrow{\pi} \Omega(\mathfrak{g}) \longrightarrow 0, \tag{1.1}$$

where the Lie algebra homomorphism π is defined by $\pi(X \otimes P) = X \otimes P$ and $\pi(K) = 0$.

1.1. Irreducible Representations of $\tilde{\mathfrak{g}}$. Fix an irreducible (finite-dimensional) representation V of $\mathfrak{g}$ and a number $\ell \in \mathbb{C}$ (to be called the *level* or *central charge*). Then we define the associated *generalized Verma module* for $\tilde{\mathfrak{g}}$ as

$$M(V, \ell) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{p}})} I_\ell(V),$$

where the $\tilde{\mathfrak{p}}$-module $I_\ell(V)$ has the same underlying vector space as V on which $\tilde{\mathfrak{u}}$ acts trivially, the central element K acts via the scalar ℓ and the action of $\mathfrak{g} = \mathfrak{g} \otimes t^0$ is via the $\mathfrak{g}$-module structure on V.

In the case when ℓ is a positive integer (in fact, it suffices to assume that $\ell \neq -h$, where h is the dual Coxeter number of $\mathfrak{g}$), $M(V, \ell)$ has a unique irreducible quotient, denoted $L(V, \ell)$.

Remark 1.2. It is easy to see that any vector $v \in M(V, \ell)$ is contained in a finite-dimensional $\mathfrak{g}$-submodule of $M(V, \ell)$. In particular, the same property holds for any vector in $L(V, \ell)$.

DEFINITION 1.3. Consider the Lie subalgebra $\mathfrak{r}^0$ of $\tilde{\mathfrak{g}}$ spanned by $\{Y_\theta \otimes t, \theta^\vee \otimes 1, X_\theta \otimes t^{-1}\}$, where Y_θ (resp. X_θ) is a non-zero root vector of $\mathfrak{g}$ corresponding to the root $-\theta$ (resp. θ) and the coroot $\theta^\vee$ is to be thought of as an element of $\mathfrak{h}$. Then the Lie algebra $\mathfrak{r}^0$ is isomorphic to $sl(2)$.

A $\tilde{\mathfrak{g}}$-module W is said to be *integrable* if every vector $v \in W$ is contained in a finite-dimensional $\mathfrak{g}$-submodule of W and also v is contained in a finite-dimensional $\mathfrak{r}^0$-submodule of W.

Then it follows easily from the $sl(2)$-theory that the irreducible module $L(V, \ell)$ is integrable if and only if ℓ is an integer and $\ell \geqslant \langle \lambda, \theta^\vee \rangle$, where λ is the highest weight of V.

2. A Certain Complex and Lie Algebra Homology

DEFINITION 2.1 (DEFINITION OF A COMPLEX). Fix a positive integer ℓ and a finite-dimensional irreducible representation $V = V(\nu)$ of $\mathfrak{g}$ with highest weight $\nu \in P_\ell^+$. Recall the parabolic BGG resolution for Kac-Moody Lie algebras (cf. [13] and [10, Theorem (3.27)]):

$$\cdots \to F_p \to \cdots \to F_1 \to F_0 \to L(V, \ell) \to 0, \tag{2.1}$$

where $F_p := \bigoplus_{w \in \widetilde{W}', \ell(w) = p} M\big(V((w * \nu_\ell)|_{\mathfrak{h}}), \ell\big)$, $\widetilde{W}'$ denotes the set of those $w \in \widetilde{W}$ such that w is the smallest element in its coset Ww, $\ell(w)$ denotes the length of w, and $\nu_\ell \in \tilde{\mathfrak{h}}^*$ is defined by $\nu_\ell|_{\mathfrak{h}} = \nu$, $\nu_\ell(K) = \ell$. (Observe that $(w * \nu_\ell)|_{\mathfrak{h}} \in P^+$ for any $w \in \widetilde{W}'$.)

Take any $\mu \in P^+$, realize $V(\mu)$ as a module for $\Omega(\mathfrak{g})$ via evaluation at 1, and consider it as a module for $\tilde{\mathfrak{g}}$ (by letting K act trivially on $V(\mu)$) via the Lie algebra homomorphism π (cf. (1.1)). We denote $V(\mu)$ with this $\tilde{\mathfrak{g}}$-module structure by $V(\mu;1)$.

Tensoring (2.1) with $V(\mu;1)$, we get a resolution:

$$\cdots \to F_p \otimes V(\mu;1) \to \cdots \to F_0 \otimes V(\mu;1) \to L(V,\ell) \otimes V(\mu;1) \to 0. \qquad (2.2)$$

Tensoring the complex (2.2) with $\mathbb{C}$ over $U(\tilde{\mathfrak{u}}^-)$ and using the Hopf principle (cf. [7, Proposition 1.7]) we obtain a complex of $\mathfrak{g}$-modules and $\mathfrak{g}$-module maps:

$$\cdots \longrightarrow \hat{F}_p \xrightarrow{\delta_p} \cdots \xrightarrow{\delta_1} \hat{F}_0 \longrightarrow 0, \qquad (2.3)$$

where $\hat{F}_p := \bigoplus_{w \in \widetilde{W}', \ell(w)=p} [V((w * \nu_\ell)|_{\mathfrak{h}}) \otimes V(\mu)]$.

The maps δ_p are quite non-trivial, e.g., the map $\delta_1 : \hat{F}_1 \to \hat{F}_0$ can be explicitly described as below:

First of all, $\hat{F}_1 = V(\nu + m\theta) \otimes V(\mu)$, where $m = \ell + 1 - (\nu, \theta^\vee)$, and of course $\hat{F}_0 = V(\nu) \otimes V(\mu)$.

Lemma 2.2. *The map $\delta_1 : V(\nu + m\theta) \otimes V(\mu) \to V(\nu) \otimes V(\mu)$ is the composite map $\eta \circ (j \otimes I)$ given as follows: (observe that $V(\theta)$ is the adjoint representation of $\mathfrak{g}$)*

$$V(\nu + m\theta) \otimes V(\mu) \underset{j \otimes I}{\hookrightarrow} V(\nu) \otimes \mathfrak{g}^{\otimes m} \otimes V(\mu) \xrightarrow{\eta} V(\nu) \otimes V(\mu),$$

where $j : V(\nu + m\theta) \hookrightarrow V(\nu) \otimes \mathfrak{g}^{\otimes m}$ is the canonical inclusion and $\eta(v \otimes (x_1 \otimes \cdots \otimes x_m) \otimes w) = v \otimes x_m \cdots x_1 w$, for $v \in V(\nu)$, $w \in V(\mu)$ and $x_i \in \mathfrak{g}$.

Observe that, since $M(V((w * \nu_\ell)|_{\mathfrak{h}}), \ell)$ are $(\tilde{\mathfrak{u}}^-)$-free, $H_*(\hat{F})$ is isomorphic to the Lie algebra homology $H_*(\tilde{\mathfrak{u}}^-, L(V(\nu),\ell) \otimes V(\mu;1))$. Moreover, if a $\mathfrak{g}$-module $V(\lambda)$ is a component of $H_0(\hat{F})$, then $\lambda \in P_\ell^+$.

We make the following conjecture.

Conjecture 2.3. *Assume that $\mu \in P_\ell^+$ (and of course $\nu \in P_\ell^+$). Then for any $\lambda \in P_\ell^+$, the $\mathfrak{g}$-module $V(\lambda)$ does not occur as a component of the homology $H_p(\tilde{\mathfrak{u}}^-, L(V(\nu),\ell) \otimes V(\mu;1)) = H_p(\hat{F})$ of the complex (2.3), for any $p \geqslant 1$.*

If we take $\mu = 0$, this conjecture follows immediately from Kostant's result on $\mathfrak{n}$-homology (for affine Kac-Moody Lie algebras, as proved by Garland-Lepowsky [7]).

Assuming the validity of the above conjecture, and using the Hochschild-Serre spectral sequence for the Lie algebra homology, we obtain the following:

Theorem 2.4. *Let $\mathfrak{g}$ be any (finite-dimensional simple) Lie algebra, for which Conjecture 2.3 is true. Decompose the Lie algebra homology (as $\mathfrak{g}$-modules)*

$$H_0(\tilde{\mathfrak{u}}^-, L(V(\nu),\ell) \otimes V(\mu;1)) = \sum_{\theta \in P_\ell^+} m_\theta V(\theta),$$

where $m_\theta = m_\theta(\nu,\mu)$ is the multiplicity of $V(\theta)$ in the left-hand side.

Then for any $i \geqslant 0$ (and any such $\mathfrak{g}$, i.e., for which Conjecture 2.3 is true),

$$H_i(\tilde{\mathfrak{u}}^-, L(V(\nu),\ell) \otimes V(\mu;1)) = \sum_\theta m_\theta \sum_{\substack{w \in \widetilde{W}' \\ \ell(w)=i}} V((w * \theta_\ell)|_{\mathfrak{h}}),$$

as $\mathfrak{g}$-modules.

3. A New Geometric Definition of Fusion Product

Fix a positive integer ℓ. We first recall the definition of fusion product for positive level representations.

Definition 3.1. For any two $\lambda, \mu \in P_\ell^+$, there is a fusion product

$$L(V(\lambda),\ell) \otimes^* L(V(\mu),\ell),$$

which is again an integrable representation of $\tilde{\mathfrak{g}}$ with the same central charge ℓ. It is given as:

$$L(V(\lambda),\ell) \otimes^* L(V(\mu),\ell) := \bigoplus_{\nu \in P_\ell^+} n_{\lambda,\mu}(\nu) L(V(\nu),\ell),$$

where $n_{\lambda,\mu}(\nu)$ is the dimension of the space of vacua for the Riemann sphere $\mathbf{P}^1$ with three punctures 0, 1, ∞ and the representations $V(\lambda)$, $V(\mu)$ and $V(\nu)^*$ attached to them respectively (cf. [15]).

Let $\widetilde{\mathcal{G}}$ be the affine Kac-Moody group associated to the Lie algebra $\tilde{\mathfrak{g}}$ and $\widetilde{\mathcal{P}}$ its parabolic subgroup (corresponding to the Lie subalgebra $\tilde{\mathfrak{p}}$) (cf. [9, §1]). Then $X = \widetilde{\mathcal{G}}/\widetilde{\mathcal{P}}$ is a projective ind-variety. Now, given a finite-dimensional algebraic representation V of $\widetilde{\mathcal{P}}$, we can consider the associated homogeneous vector bundle $\mathcal{V}$ on X and the corresponding Euler-Poincaré characteristic (which is a virtual $\widetilde{\mathcal{G}}$-module)

$$\mathcal{X}(X,\mathcal{V}) := \sum_i (-1)^i H^i(X,\mathcal{V}).$$

Recall that $H^i(X,\mathcal{V})$ is determined in [9, Corollary 3.11] (and also in [12]).

We give a new definition of a fusion product $\otimes^F$ in the following.

Definition 3.2. For any positive integer ℓ, and $\lambda, \mu \in P_\ell^+$, define

$$[L(V(\lambda),\ell) \otimes^F L(V(\mu),\ell)]^* \cong \mathcal{X}(X,\mathcal{V}),$$

as virtual $\widetilde{\mathcal{G}}$-modules, where the $\widetilde{\mathcal{P}}$-module $V := (I_\ell(V(\lambda) \otimes V(\mu)))^*$ (cf. §1.1 for the notation I_ℓ).

Let $R_\ell(G)$ be the free Abelian group generated by $\{ L(V(\nu), \ell) : \nu \in P_\ell^+ \}$. Then $\otimes^F$ gives rise to a product in $R_\ell(G)$.

Let G be the (simple) simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$, and let $R(G)$ be its representation ring, i.e., $R(G)$ is the free Abelian group generated by the G-modules $\{ V(\lambda) : \lambda \in P^+ \}$, which is a ring under the usual tensor product of G-modules. Define the $\mathbb{Z}$-linear map

$$\beta \colon R(G) \longrightarrow R_\ell(G)$$

by $\beta(W)^* = \mathcal{X}(X, \mathcal{W}^*)$, where $\mathcal{W}$ is the homogeneous vector bundle on X associated to the $\widetilde{\mathcal{P}}$-module $I_\ell(W)$ and $\mathcal{W}^*$ is the dual vector bundle on X.

We have the following lemma.

LEMMA 3.3. *The kernel of β is an ideal of $R(G)$. Moreover, β is a homomorphism with respect to the product $\otimes^F$ in $R_\ell(G)$.*

In particular, $R_\ell(G)$ is an associative (and commutative) algebra under $\otimes^F$.

4. Comparison of the Two Fusion Products

We denote $R_\ell(G)$ equipped with the fusion product $\otimes^\bullet$ (resp. $\otimes^F$) by $(R_\ell(G), \otimes^\bullet)$ (resp. $(R_\ell(G), \otimes^F)$). Recall that the associativity of $(R_\ell(G), \otimes^\bullet)$ follows from the factorization rule (cf. [15]) for $\mathbb{P}^1$ with punctures.

Set (for $\lambda, \mu, \nu \in P_\ell^+$)

$$\overline{\chi}_\lambda(\nu, \mu) = \sum_{i \geqslant 1} \dim\Big(\operatorname{Hom}_{\mathfrak{g}}\Big(V(\lambda), H_i\big(\tilde{\mathfrak{u}}^-, L(V(\nu), \ell) \otimes V(\mu; 1)\big)\Big)\Big).$$

For any $1 \leqslant i \leqslant \operatorname{rank} \mathfrak{g}$, let $\omega_i \in P^+$ be the i-th fundamental weight corresponding to $\mathfrak{g}$. We have the following lemma.

LEMMA 4.1. *The products $\otimes^\bullet$ and $\otimes^F$ in $R_\ell(G)$ coincide if and only if for all $\lambda, \mu, \nu \in P_\ell^+$, $\overline{\chi}_\lambda(\nu, \mu) = 0$.*

In fact, the products $\otimes^\bullet$ and $\otimes^F$ in $R_\ell(G)$ coincide if and only if for all $\lambda, \omega_i, \nu \in P_\ell^+$, $\overline{\chi}_\lambda(\nu, \omega_i) = 0$.

As a consequence of the above lemma, together with some results of [1, Corollary 4.3], [3] and some partial determination of $H_i\big(\tilde{\mathfrak{u}}^-, L(V(\nu), \ell) \otimes V(\omega_i; 1)\big)$ for those ω_i such that $\langle \omega_i, \theta^\vee \rangle \leqslant 2$, we obtain the following result.

THEOREM 4.2. *For any simple (simply-connected) group G of type A_n, B_n, C_n, D_n or G_2, the products $\otimes^\bullet$ and $\otimes^F$ in $R_\ell(G)$ coincide. In particular, for these groups, the $\mathbb{Z}$-linear map $\beta \colon R(G) \to R_\ell(G)$ (cf. Definition 3.2) is an algebra homomorphism with respect to the product $\otimes^\bullet$ in $R_\ell(G)$.*

Remark 4.3. (a) Of course $\overline{\chi}_\lambda(\nu, \mu) = 0$, for all $\lambda, \mu, \nu \in P_\ell^+$, if Conjecture 2.3 is true. In particular, the validity of Conjecture 2.3 will imply that $\otimes^\bullet$ and $\otimes^F$ coincide in $R_\ell(G)$.

(b) The "in particular" statement of the above theorem is due to Faltings [5, Appendix], proved via case by case computation. In fact, this result of Faltings was the main motivation behind our work.

(c) It is likely that we can prove Theorem 4.2 for all G and all positive integer ℓ by combining our proof of the above Theorem and some results of Finkelberg [6]. (Some details in [6] are not clear to me as yet.)

Finally, we can prove the validity of Conjecture (2.3) for $G = SL(2)$.

PROPOSITION 4.4. *Conjecture 2.3 is true for the group $G = SL(2)$. In particular, Theorem 2.4 is true in this case. In fact, in this case, one has a rather precise description of $H_0\big(\tilde{\mathfrak{u}}^-, L(V(\nu), \ell) \otimes V(\mu; 1)\big)$ and hence of $H_i\big(\tilde{\mathfrak{u}}^-, L(V(\nu), \ell) \otimes V(\mu; 1)\big)$ for all $i \geqslant 0$.*

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