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Picard group of the moduli spaces of G-bundles

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Introduction

Let G be a simple simply-connected connected complex affine algebraic group and let C be a smooth irreducible projective curve of genus $\geqq 2$ over the field of complex numbers $\mathbb{C}$. Let $\mathfrak{M}$ be the moduli space of semistable principal G-bundles on C and let Pic $\mathfrak{M}$ be its Picard group, i.e., the group of isomorphism classes of algebraic line bundles on $\mathfrak{M}$. Following is our main result (which generalizes a result of Drezet-Narasimhan for $G = \mathrm{SL}(N)$ [DN] to any G).

(A) Theorem. *With the notation as above,* $\mathrm{Pic}\,(\mathfrak{M}) \approx \mathbb{Z}$.

A more precise result is obtained in Theorem (2.4) together with Theorem (4.9).

We use the above result and a result of Grauert-Riemenschneider to prove the following second main result of this paper.

(B) Theorem. *The dualizing sheaf ω of the moduli space $\mathfrak{M}$ is locally free. In particular, $\mathfrak{M}$ is a Gorenstein variety.*

Further, for any finite dimensional representation V of G, $H^i(\mathfrak{M}, \Theta(V)) = 0$, for all $i > 0$, where $\Theta(V)$ is the theta bundle on the moduli space $\mathfrak{M}$. In particular,

$$\mathscr{X}(\mathfrak{M}, \Theta(V)) = \dim H^0(\mathfrak{M}, \Theta(V)) ,$$

where $\mathscr{X}$ is the Euler-Poincaré characteristic.

In fact, we have a sharper result than the above (cf. Theorem 2.8).

We make essential use of the generalized flag variety X associated to the affine Kac-Moody group corresponding to G, which (i.e. X) parametrizes an algebraic family of G-bundles on C, and the fact that Pic $X \simeq \mathbb{Z}$. We also need to make use of the explicit construction of the moduli space $\mathfrak{M}$ via GIT.

1. Notation

Let G be a simple simply-connected connected complex affine algebraic group and let C be a smooth irreducible projective curve of genus $\geqq 2$ over the field of complex numbers $\mathbb{C}$. As in [KNR, Theorem 3.4], let $\mathfrak{M}$ be the moduli space of semistable principal G-bundles on C. Also, fix a point $p \in C$ and recall the definition of the generalized flag variety $X = \mathscr{G}/\mathscr{P}$ (associated to the affine Kac-Moody group $\mathscr{G}$ corresponding to the group G) from [KNR, Sect. 2.1], its open subset X^s and the morphism $\psi : X^s \to \mathfrak{M}$ from [loc. cit., Definition 6.1]. Also, recall the notation Γ from [loc. cit., Sect. 1.1] and the notation $\widetilde{W}, W, X_{\mathfrak{w}}$ from [loc. cit., Sect. 2.1].

For any ind-variety Y, by an *algebraic vector bundle of rank* r over Y, we mean an ind-variety E together with a morphism $\theta : E \to Y$ such that (for any n) $E_n \to Y_n$ is an algebraic vector bundle of rank r over the (finite dimensional) variety Y_n, where $\{Y_n\}$ is the filtration of Y giving the ind-variety structure and $E_n := \theta^{-1}(Y_n)$. If $r = 1$, we call E an *algebraic line bundle* over Y. For an introduction to ind-varieties, see [Ku2, Appendix B].

Let E and F be two algebraic vector bundles over Y. Then a morphism (of ind-varieties) $\varphi : E \to F$ is called a *bundle morphism* if the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

and moreover $\varphi_{|E_n} : E_n \to F_n$ is a bundle morphism for all n. In particular, we have the notion of isomorphism of vector bundles over Y.

We define Pic Y as the set of isomorphism classes of algebraic line bundles on Y. It is clearly an abelian group under the tensor product of line bundles.

For any set Y, I_Y denotes the identity map of Y.

2. Statement of the main theorems

We follow the notation from Sect. 1.

(2.1) Lemma. *The morphism $\psi : X^s \to \mathfrak{M}$ induces an injective map*

$$\psi^* : \mathrm{Pic}\,(\mathfrak{M}) \longrightarrow \mathrm{Pic}\,(X^s)\ .$$

Proof. Let $\mathfrak{L} \in \mathrm{Pic}\,(\mathfrak{M})$ be in the kernel of ψ^*, i.e., $\psi^*(\mathfrak{L})$ admits a nowhere-vanishing regular section σ on the whole of X^s. Fix $m \in \mathfrak{M}$ and a trivialization for $\mathfrak{L}_{|m}$. This canonically induces a trivialization for the bundle $\psi^*(\mathfrak{L})_{|\psi^{-1}(m)}$. In particular, the section $\sigma_{|\psi^{-1}(m)}$ can be viewed as a (regular) map $\sigma_m : \psi^{-1}(m) \to \mathbb{C}^*$. But $\psi^{-1}(m)$ is a certain union of Γ-orbits say $\psi^{-1}(m) = \underset{i\in I}{\cup} \Gamma x_i$, for $x_i \in X$ and moreover $\overline{\Gamma x_i} \cap \overline{\Gamma x_j} \neq \emptyset$, for any $i, j \in I$, where $\overline{\Gamma x_i}$ is the closure of Γx_i in X^s (cf. [KNR, Proof of Proposition 6.4]). Fixing $i \in I$, we get a regular map $\sigma_{m,i} : \Gamma \to \mathbb{C}^*$, defined as $\sigma_{m,i}(\gamma) = \sigma_m(\gamma x_i)$, for

$\gamma \in \Gamma$. Now by [Ku2, Proposition 2.4], $\sigma_{m,i}$ is a constant map for any $i \in I$, and hence $\sigma_m : \psi^{-1}(m) \to \mathbb{C}^*$ itself is a constant map. Thus the section σ descends to a set theoretic section $\hat{\sigma}$ of the line bundle $\mathfrak{L}$, which is regular by [KNR, Proposition 4.1 and Lemma 6.2]. Of course, the section $\hat{\sigma}$ does not vanish anywhere on $\mathfrak{M}$ (since σ was chosen to be nowhere-vanishing on X^s). This proves that $\mathfrak{L}$ is a trivial line bundle on $\mathfrak{M}$, thereby proving the lemma. □

It is clear that for any ind-variety Y, we have a natural map $\alpha : \text{Pic}\, Y \to \varprojlim_n \text{Pic}\,(Y_n)$.

(2.2) Lemma. $\text{Pic}\, X \approx \varprojlim_{\mathfrak{w} \in \widetilde{W}/W} \text{Pic}(X_{\mathfrak{w}}) \approx \mathbb{Z}$.

Proof. We will freely follow the notation from [KNR, Sect. 2.3]. Since the line bundles $\mathfrak{L}(d\chi_0)$ (for $d \in \mathbb{Z}$) (denoted in loc. cit. by $\mathscr{L}(d\chi_o)$) are, by construction, algebraic line bundles on X and moreover, for any $\mathfrak{w} \geqq \mathfrak{s}_o$, $\mathfrak{L}(\chi_0)_{|X_{\mathfrak{w}}}$ freely generates $\text{Pic}(X_{\mathfrak{w}})$, the surjectivity of the map α follows. Now we come to the injectivity of α:

Let $\mathfrak{L} \in \text{Ker}\, \alpha$. Fix a non-zero vector v_o in the fiber of $\mathfrak{L}$ over the base point $\mathfrak{e} \in X$. Then $\mathfrak{L}_{|X_{\mathfrak{w}}}$ being a trivial line bundle on each $X_{\mathfrak{w}}$, we can choose a nowhere-vanishing section $s_{\mathfrak{w}}$ of $\mathfrak{L}_{|X_{\mathfrak{w}}}$ such that $s_{\mathfrak{w}}(\mathfrak{e}) = v_o$. We next show that for any $\mathfrak{v} \geqq \mathfrak{w}$, $s_{\mathfrak{v}|X_{\mathfrak{w}}} = s_{\mathfrak{w}}$: Clearly $s_{\mathfrak{v}|X_{\mathfrak{w}}} = f s_{\mathfrak{w}}$, for some algebraic function $f : X_{\mathfrak{w}} \to \mathbb{C}^*$. But $X_{\mathfrak{w}}$ being projective and irreducible, f is constant and in fact $f \equiv 1$ since $s_{\mathfrak{v}}(\mathfrak{e}) = s_{\mathfrak{w}}(\mathfrak{e})$. This gives rise to a nowhere-vanishing regular section s of $\mathfrak{L}$ on the whole of X such that $s_{|X_{\mathfrak{w}}} = s_{\mathfrak{w}}$. From this it is easy to see that $\mathfrak{L}$ is isomorphic with the trivial line bundle on X. This proves that α is an isomorphism. Now the second isomorphism is proved in [KNR, Proposition 2.3]. □

We state the following very crucial 'lifting' result, the proof of which will be given in the next section.

(2.3) Proposition. *There exists a map* $\overline{\psi^*} : \text{Pic}\,(\mathfrak{M}) \to \text{Pic}\,(X)$, *making the following diagram commutative:*

$$\begin{array}{ccc} & \text{Pic}\,(\mathfrak{M}) & \\ \overline{\psi^*} \swarrow & & \searrow \psi^* \\ \text{Pic}\,(X) & \underset{i^*}{\longrightarrow} & \text{Pic}\,(X^s)\,, \end{array}$$

where i^* *is the canonical restriction map.*

As an easy consequence of the above proposition, Lemmas (2.1) and (2.2), we get the following main result of this paper.

(2.4) Theorem. *For any smooth projective irreducible curve* C *of genus* $\geqq 2$ *and simple simply-connected connected affine algebraic group* G, *the map* $\overline{\psi^*}$ *(as in the above proposition) is an injective group homomorphism.*

In particular, $\text{Pic}\,(\mathfrak{M}) \approx \mathbb{Z}$.

Proof. Injectivity of $\overline{\psi^*}$ follows from the injectivity of ψ^* (cf. Lemma 2.1) and the commutativity of the diagram in Proposition (2.3). By Proposition (2.3), Image $\psi^* \subset$ Image i^*. But since Pic $X \approx \mathbb{Z}$ (by Lemma 2.2), Image i^* is either finite or else Image $i^* \approx \mathbb{Z}$. Now since $\mathfrak{M}$ is a projective variety of dim > 0 (cf. [R1, Theorem 4.9]) and ψ^* is injective, Image i^* can not be finite, in particular, i^* is injective. Since ψ^* and i^* are group homomorphisms and i^* is injective, we get that $\overline{\psi^*}$ is a group homomorphism. This proves the theorem. □

(2.5) Definition. Let $n_{C,G} > 0$ be the least (positive) integer such that $\mathfrak{L}(n_{C,G}\chi_0) \in$ Image $\overline{\psi^*}$. Then of course

$$\mathit{Image}\, \overline{\psi^*} = \{\mathfrak{L}(dn_{C,G}\chi_0)\}_{d\in\mathbb{Z}}$$

We will be concerned with determining the number $n_{C,G}$ in Sect. 4.

(2.6) Remark. In the case when $G = SL(n, \mathbb{C})$, it is a result of Drezet–Narasimhan [DN] that Pic$(\mathfrak{M}) \approx \mathbb{Z}$.

We recall the following well known result. (We include a proof since we did not find it in the literature in this form.)

(2.7) Lemma. *Let Y be a Cohen–Macaulay projective variety and let $U \subset Y$ be an open subset such that* $\mathrm{codim}_Y(Y\backslash U) \geqq 2$. *Now let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two reflexive sheaves on Y such that ${\mathscr{S}_1}_{|_U} \approx {\mathscr{S}_2}_{|_U}$. Then the sheaf $\mathscr{S}_1$ is isomorphic with $\mathscr{S}_2$ on the whole of Y.*

Proof[1]. We recall the following two facts from Commutative Algebra.

Fact 1: If M, N are modules over a noetherian local ring with depth $M, N > 1$, and $0 \to M \to N \to K \to 0$ is an exact sequence, then depth $K > 0$.

Fact 2: If M is reflexive, then for any localisation $M_\mathfrak{p}$ of M at a prime ideal $\mathfrak{p}$, depth $M_\mathfrak{p} > 1$, unless the dimension of the local ring itself is less than 2 (i.e. M satisfies the ‘Serre condition’ S_2).

Let $i : U \hookrightarrow Y$ be the inclusion. Then from the above facts (and the assumptions of the lemma), one can check that $i_* i^* \mathscr{S}_j = \mathscr{S}_j$ (for $j = 1, 2$). Thus any homomorphism $i^*\mathscr{S}_1 \to i^*\mathscr{S}_2$ on U gives rise to a homomorphism $\mathscr{S}_1 \to \mathscr{S}_2$, i.e., Hom $(\mathscr{S}_1, \mathscr{S}_2) \to$ Hom $(i^*\mathscr{S}_1, i^*\mathscr{S}_2)$ is surjective. Injectivity is clear using reflexivity. This proves the lemma. □

We come to the following second main result of this paper.

(2.8) Theorem. *The dualizing sheaf ω of the moduli space $\mathfrak{M}$ is locally free. Moreover, $\overline{\psi^*}(\omega) = \mathfrak{L}(-2g\chi_0)$, where g is the dual Coxeter number of the Lie algebra $\mathfrak{g}$ of G (cf. [KNR, Remark 5.3]).*

In particular, $\mathfrak{M}$ is a Gorenstein variety. Further, for any line bundle $\mathfrak{L}$ on $\mathfrak{M}$ such that $\overline{\psi^*}(\mathfrak{L}) = \mathfrak{L}(d\chi_0)$ for some $d > -2g$, $H^i(\mathfrak{M}, \mathfrak{L}) = 0$, for all $i > 0$ So, for any finite dimensional representation V of G, $H^i(\mathfrak{M}, \Theta(V)) = 0$, for all $i > 0$, where $\Theta(V)$ is the theta bundle on the moduli space $\mathfrak{M}$.

[1] This proof is due to N. Mohan Kumar.

Proof. Let $\mathfrak{M}^o := \{E \in \mathfrak{M}; E$ is a stable G-bundle and $\operatorname{Aut} E =$ centre of $G\}$. Then $\mathfrak{M}^o$ is an open subset of the smooth locus of $\mathfrak{M}$ and, for any $E \in \mathfrak{M}^o$, the tangent space $T_E(\mathfrak{M}^o)$ can be identified with $H^1(C, \operatorname{Ad} E)$, where $\operatorname{Ad} E$ is the vector bundle on C associated to the principal G-bundle E via the adjoint representation Ad of G in its Lie algebra $\mathfrak{g}$. Also, on the set of stable bundles in the moduli space there are no identifications, i.e., if E_1 and E_2 are two stable G-bundles on C such that E_1 is S-equivalent to E_2, then E_1 is isomorphic with E_2 (as follows from the definition of S-equivalence, cf. [KNR, Sect. 3.3]). Moreover, for any $E \in \mathfrak{M}^o$, $H^0(C, \operatorname{Ad} E) = 0$. In particular, the fiber of the canonical bundle of $\mathfrak{M}^o$ at E can be identified with $\wedge^{\mathrm{top}}(H^1(C, \operatorname{Ad} E)^*)$, where $\wedge^{\mathrm{top}}$ is the top exterior power. This gives, from the definitions of the determinant bundle and the Θ-bundle (cf. [KNR, Sect. 3.8]), that

$$\operatorname{Det}(\operatorname{Ad})^*_{|\mathfrak{M}^o} = \Theta(\operatorname{Ad})^*_{|\mathfrak{M}^o} = \omega_{|\mathfrak{M}^o} .$$

But $\Theta(\operatorname{Ad})^*$ is a line bundle on the whole of $\mathfrak{M}$ (cf. [loc. cit., Sect. 3.8]). Since any line bundle is a reflexive sheaf (cf. [H, Exercise 5.1, p. 123]), $\Theta(\operatorname{Ad})^*$ is a reflexive sheaf on $\mathfrak{M}$. Since the dualizing sheaf ω of a normal variety is always reflexive; the moduli space $\mathfrak{M}$ is Cohen–Macaulay and normal (cf. [R1, Theorem 4.9]); and $\operatorname{codim}_{\mathfrak{M}}(\mathfrak{M} \backslash \mathfrak{M}^o) \geqq 2$ (unless the curve C is of genus 2 and $G = SL(2)$) (cf. [F, Theorem II.6]); we obtain from Lemma (2.7):

$$\omega \approx \Theta(\operatorname{Ad})^*, \text{ on the whole of } \mathfrak{M} . \tag{1}$$

(In the case of $G = SL(2)$ the validity of (1) is well known.) This of course gives that $\mathfrak{M}$ is a Gorenstein variety (by definition). Now the assertion that $\overline{\psi}^*(\omega) = \mathfrak{L}(-2g\chi_0)$ follows from [KNR, Theorem 5.4 and Lemma 5.2].

Finally we come to the proof of cohomology vanishing: By Serre duality [H, Corollary 7.7, Chap. III] (denoting $\dim \mathfrak{M} = n$),

$$\begin{aligned} H^i(\mathfrak{M}, \mathfrak{L})^* &\approx H^{n-i}(\mathfrak{M}, \mathfrak{L}^* \otimes \omega) \\ &= H^{n-i}(\mathfrak{M}, \mathfrak{L}^* \otimes \Theta(\operatorname{Ad})^*), \text{ by (1)} . \end{aligned} \tag{2}$$

But $\overline{\psi}^*(\mathfrak{L}^* \otimes \Theta(\operatorname{Ad})^*) = \mathfrak{L}((-d-2g)\chi_0)$. Now since $\operatorname{Pic}(\mathfrak{M}) \approx \mathbb{Z}$ (by Theorem 2.4), we get that the line bundle $\mathfrak{L} \otimes \Theta(\operatorname{Ad})$ is ample on $\mathfrak{M}$ (by assumption $d > -2g$).

The moduli space $\mathfrak{M}$ has rational singularities, as follows from [R1, Proof of Theorem 4.9] and a result of Boutot [Bo]. Now the vanishing of $H^i(\mathfrak{M}, \mathfrak{L})$ (for $i > 0$) follows from (2) and a result of Grauert-Riemenschneider [GR]. So the proof of the theorem is complete in view of [KNR, Theorem 5.4]. □

(2.9) Corollary. *For any finite dimensional representation V of G,*

$$\mathcal{X}(\mathfrak{M}, \Theta(V)) = \dim H^0(\mathfrak{M}, \Theta(V)) ,$$

where $\mathcal{X}$ is the Euler-Poincaré characteristic:

$$\mathcal{X}(\mathfrak{M}, \Theta(V)) = \sum_i (-1)^i \dim H^i(\mathfrak{M}, \Theta(V)) .$$

3. Extension of line bundles. Proof of Proposition (2.3)

(3.1). Recall the definition of the map $\varphi : \mathcal{G} \to \mathcal{X}_0$ from [KNR, Sect. 1] (where $\mathcal{X}_0$ denotes the set of isomorphism classes of principal G-bundles on C which are algebraically trivial restricted to $C^* := C \backslash p$). Fix an embedding $G \hookrightarrow SL(n)$, for some n. In particular, any principal G-bundle E on C gives rise to a vector bundle $\overline{E}$ of rank n on C (associated to the standard representation of $SL(n)$). For any integer $d \geqq 1$, define

$$X_d = \{g\mathcal{P} \in X : H^1(C, \overline{\varphi(g)} \otimes \mathcal{O}(-x + dp)) = 0 \text{ for all } x \in C\} ,$$

where $p \in C$ is the fixed base point. Then

$$X_1 \subset X_2 \subset \cdots .$$

(3.2) Lemma. *Each X_d is open in X. Moreover $X^s \subset X_{2h}$, where $X^s := \{g\mathcal{P} \in X : \varphi(g)$ is a semistable G-bundle$\}$, and h is the genus of the curve C.*

Proof. It suffices to prove that $X_d \cap X_{\mathfrak{w}}$ is open in $X_{\mathfrak{w}}$, for each $\mathfrak{w} \in \widetilde{W}/W$:

Recall the definition of the family of G-bundles $\mathcal{U} \to C \times X$ from [KNR, Proposition 2.8]. Consider the restriction $\mathcal{U}_{\mathfrak{w}}$ of the G-bundle $\mathcal{U} \to C \times X$ to $C \times X_{\mathfrak{w}}$ and let $\overline{\mathcal{U}_{\mathfrak{w}}}$ be the associated rank-n vector bundle (corresponding to the embedding $G \hookrightarrow \mathrm{SL}(n)$). Define a vector bundle $\widetilde{\mathcal{U}_{\mathfrak{w}}}$ on $C \times C \times X_{\mathfrak{w}}$ such that $\widetilde{\mathcal{U}_{\mathfrak{w}}}_{|x \times C \times X_{\mathfrak{w}}} = \mathcal{O}(-x + dp) \otimes \overline{\mathcal{U}_{\mathfrak{w}}}$ for each $x \in C$; and let $\pi : C \times C \times X_{\mathfrak{w}} \to C \times X_{\mathfrak{w}}$ be the projection on the two extreme factors. Applying the upper semi-continuity theorem [H, Chapter III, Sect. 12] to the morphism π and the locally free sheaf $\widetilde{\mathcal{U}_{\mathfrak{w}}}$ on $C \times C \times X_{\mathfrak{w}}$, we get that the set

$$S := \{(x, g\mathcal{P}) : H^1(C, \overline{\varphi(g)} \otimes \mathcal{O}(-x + dp)) \neq 0\}$$

is a closed subset of $C \times X_{\mathfrak{w}}$. In particular, $\pi_2(S)$ is a closed subset of $X_{\mathfrak{w}}$, where $\pi_2 : C \times X_{\mathfrak{w}} \to X_{\mathfrak{w}}$ is the projection on the second factor. It is easy to see that $X_d \cap X_{\mathfrak{w}} = X_{\mathfrak{w}} \setminus \pi_2(S)$. This proves that X_d is open in X.

For $g\mathcal{P} \in X^s$, $\overline{\varphi(g)}$ is a semistable vector bundle (cf. [RR, Theorem 3.18]), and hence the dual vector bundle $\overline{\varphi(g)}^*$ is also semistable. Now, by the Serre duality,

$$H^1(C, \overline{\varphi(g)} \otimes \mathcal{O}(-x + dp)) \approx H^0(C, \overline{\varphi(g)}^* \otimes \mathcal{O}(x - dp) \otimes K)^* .$$

Since $\overline{\varphi(g)}^*$ is semistable, $H^0(C, \overline{\varphi(g)}^* \otimes \mathcal{O}(x - dp) \otimes K) \neq 0$ implies that $d - 1 - \deg K \leqq 0$. In particular, if $d \geqq 2 + \deg K$, then $g\mathcal{P} \in X_d$. This proves the lemma since $\deg K = 2h - 2$. □

We have

$$\bigcup_{d \geqq 1} X_d = X ,$$

since each Schubert variety $X_{\mathfrak{w}}$ is contained in some large enough X_d (d of course depending upon $\mathfrak{w}$). This follows by the upper semi-continuity theorem (using an argument similar to the one used in the proof of the above lemma).

(3.3). Fix any $d \geqq 2h$. For all $m \geqq d$ and $g\mathscr{P} \in X_d$, we have

(1) $H^1(C, \overline{\varphi(g)} \otimes \mathcal{O}(mp)) = 0$, and
(2) $H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}(mp))$ generates the vector bundle $\overline{\varphi(g)} \otimes \mathcal{O}(mp)$ at every point of C.

Let $q_d := \dim H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}(dp))$. Then by Riemann-Roch theorem, $q_d = n(d+1-h)$. Denote by $\pi_d : \mathscr{F}_d \to X_d$ the $\mathrm{GL}(q_d)$-bundle such that for $g\mathscr{P} \in X_d$, $\pi_d^{-1}(g\mathscr{P})$ is the set of all the frames of the vector space $H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}(dp))$. We call $\mathscr{F}_d$ the *frame bundle associated to the family* $\mathscr{U}_{|X_d}$ (parametrized by X_d). Similarly, define the frame bundle $\pi_{d+1} : \mathscr{F}_{d+1} \to X_{d+1}$. Consider the parabolic subgroup $P = \{\theta \in \mathrm{GL}(q_{d+1}) : \theta\mathbb{C}^{q_d} = \mathbb{C}^{q_d}\}$ of $\mathrm{GL}(q_{d+1})$, where (for definiteness) $\mathbb{C}^{q_d} \hookrightarrow \mathbb{C}^{q_{d+1}}$ is sitting in the first q_d coordinates. We define the principal P-subbundle Q_d of $\mathscr{F}_{d+1|X_d}$ by

$$Q_d = \bigcup_{g\mathscr{P} \in X_d} \{s = (s_1, \ldots, s_{q_{d+1}}) \text{ a frame of } H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}((d+1)p)) \text{ such that } (s_1, \ldots, s_{q_d}) \text{ is a frame of } H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}(dp))\}\,.$$

(Observe that $H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}(dp))$ sits canonically inside $H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}((d+1)p))$ induced from the embedding $\overline{\varphi(g)} \otimes \mathcal{O}(dp) \hookrightarrow \overline{\varphi(g)} \otimes \mathcal{O}((d+1)p)$.) Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathscr{F}_d & \overset{\beta_d}{\twoheadleftarrow} & Q_d & \hookrightarrow & \mathscr{F}_{d+1} \\ \pi_d \downarrow & & & & \downarrow \pi_{d+1} \\ X_d & & \hookrightarrow & & X_{d+1}, \end{array}$$

where β_d takes any $s = (s_1, \ldots, s_{q_{d+1}}) \in Q_d$ to the frame $(s_1, \ldots, s_{q_d})$ of $H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}(dp))$. It is clear that β_d is a principal U-bundle, where $U := \{\theta \in \mathrm{GL}(q_{d+1}) : \theta_{|\mathbb{C}^{q_d}} = I\} \subset P$. Clearly U is a normal subgroup of P.

As in [KNR, Sect. 7.8], we have an irreducible smooth quasi-projective variety R_d with an action of $\mathrm{GL}(q_d)$, a family $\mathscr{W}_d$ of G-bundles on C parametrized by R_d and a lift of the $\mathrm{GL}(q_d)$-action to $\mathscr{W}_d$ (as bundle automorphisms), such that there exists a $\mathrm{GL}(q_d)$-equivariant morphism $\varphi_d : \mathscr{F}_d \to R_d$ with the property that the families $\pi_d^*(\mathscr{U}_{|X_d})$ and $\varphi_d^*(\mathscr{W}_d)$ are isomorphic. Moreover, let $R_d^s = \{x \in R_d : \mathscr{W}_d(x) := \mathscr{W}_{d|C\times x}$ is a semistable G-bundle$\}$ be the $\mathrm{GL}(q_d)$-invariant open subset of R_d. Then the canonical map $\theta_d : R_d^s \to \mathfrak{M}$ is surjective. Moreover, θ_d is $\mathrm{GL}(q_d)$-equivariant with respect to the trivial action of $\mathrm{GL}(q_d)$ on the moduli space $\mathfrak{M}$ (of semistable G-bundles on C). We recall the construction of R_d for its use in the sequel [R1, Sects. 3.8, 3.13.3]:

Let R_d^o be the set of locally free quotients E of $\mathbb{C}^{q_d} \otimes_{\mathbb{C}} \mathcal{O}_C$ of rank n and degree nd such that the canonical map $\mathbb{C}^{q_d} \approx H^0(\mathbb{C}^{q_d} \otimes_{\mathbb{C}} \mathcal{O}_C) \to H^0(E)$ is an isomorphism. Then R_d^o supports the tautological family $\widehat{\mathscr{W}_d}^o$ of rank-n vector bundles on C. Set $\mathscr{W}^o_d = \widehat{\mathscr{W}_d}^{\,o} \otimes_{\mathcal{O}_{C\times R^o_d}} \mathcal{O}_C(-dp)$. Now let

$$R_d = \{(x,\sigma) : x \in R^o_d \text{ and } \sigma \text{ is a reduction of the structure group of } \mathscr{W}^o_{d|C\times x} \text{ to } G\} .$$

Then clearly R_d supports a canonical family $\mathscr{W}_d$ of G-bundles on C and moreover $\mathrm{GL}(q_d)$ acts on $\mathscr{W}_d$ via its action on $\mathbb{C}^{q_d}$.

Using $H^1(C,E) = 0$, one proves that R_d is smooth and that the infinitesimal deformation map $T_t(R_d) \to H^1(C, \mathrm{Ad}\,(\mathscr{W}_{d|C\times t}))$ is surjective, where $T_t(R_d)$ is the tangent space at t to R_d.

(3.4) Proposition. *For any $d \geqq 2h$, the codimension of $R_d \backslash R^s_d$ in R_d is at least 2, where R_d is explicitly constructed as above.*

To prove the above proposition, we need the notion of the canonical reduction (or filtration) of a principal G-bundle on C. We choose a Borel subgroup B of G and a maximal torus $T \subset B$. By a *standard parabolic subgroup* we mean a parabolic subgroup P containing B. The following result is due to Ramanathan [R2, Proposition 1] (see also [Be]).

(3.5) Theorem. *Let E be a principal G-bundle on C. Then there exists a unique standard parabolic subgroup P of G and a unique reduction E_P of E to the subgroup P such that the following conditions hold:*

(1) *If U is the unipotent radical of P, then the P/U-bundle $E_{P/U}$, obtained from E_P by extension of the structure group via $P \to P/U$, is semistable. (Observe that P/U is reductive.)*
(2) *For any non-trivial character χ of P which is a non-negative linear combination of simple roots of B, the line bundle on C associated to E_P by χ has strictly positive degree.*

The unique reduction E_P of E as above is called the *canonical reduction*.

(3.6) Lemma. *Let E_P be the canonical reduction of a principal G-bundle E on C. Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of G and P respectively. Denote by $E_\mathfrak{s}$ the vector bundle associated to E_P by the natural representation of P on the vector space $\mathfrak{s} := \mathfrak{g}/\mathfrak{p}$. Then we have*

$$H^0(C, E_\mathfrak{s}) = 0 .$$

Proof. We may assume that $P \neq G$. Let $0 = V_0 \subset V_1 \subset \ldots \subset V_k = \mathfrak{s}$ be a filtration of $\mathfrak{s}$ by P-submodules V_i such that, for any $1 \leqq i \leqq k$, the P-module $W_i := V_i/V_{i-1}$ is irreducible. In particular, U acts trivially on W_i (cf. [Ku, Lemma 1]). If $\mathscr{V}_i$ is the vector bundle on C associated to E_P by the representation of P on V_i, then $E_\mathfrak{s}$ is filtered by the subbundles $\mathscr{V}_i$. We now show that $H^0(C, \mathscr{W}_i) = 0$ for all $1 \leqq i \leqq k$, where $\mathscr{W}_i := \mathscr{V}_i/\mathscr{V}_{i-1}$. This will of course prove the lemma.

Since the action of U on W_i is trivial, we obtain an (irreducible) representation of the reductive group P/U on W_i. Since $E_{P/U}$ is semistable, the vector bundles $\mathscr{W}_i$ are semistable (cf. [RR, Theorem 3.18]), and hence it is sufficient to show that $\deg(\mathscr{W}_i) < 0$: Now the weights of T on $\mathfrak{s}$ are of the form $\sum c_\alpha \alpha$ with $c_\alpha \leqq 0$ and $c_\alpha < 0$ for at least one $\alpha \notin I$, where I is the subset of the set of simple roots $\Pi = \{\alpha\}$ defining the parabolic subgroup P (i.e. I is the set of simple roots for P/U). It follows from this that the character of P defined by the determinant of the representation of P on W_i is non-trivial and is a non-positive linear combination of $\{\alpha\}_{\alpha\in\pi}$. By Condition (2) of Theorem (3.5), we see that $\deg(\mathscr{W}_i) < 0$. This completes the proof of the lemma. □

Let P be a standard parabolic subgroup of G and E_P be a reduction of the G-bundle E to P. For any character χ of P, denote by $E_{P,\chi}$ the line bundle on C associated to E_P by χ. Let $X(P)$ (resp. $X(T)$) denote the character group of P (resp. T). Then $X(T) = \oplus_{\alpha\in\Pi}\mathbb{Z}\omega_\alpha$, where ω_α is the fundamental weight defined by $\omega_\alpha(\beta^\vee) = \delta_{\alpha,\beta}$, for any simple coroot $\beta^\vee$. Moreover (since G is simply-connected) $X(P) = \oplus_{\alpha\notin I}\mathbb{Z}\omega_\alpha$. The map $\chi \mapsto \deg(E_{P,\chi})$ defines an element of $\mathrm{Hom}_{\mathbb{Z}}(X(P),\mathbb{Z})$, which in turn can be lifted to the element μ of $\mathrm{Hom}_{\mathbb{Z}}(X(T),\mathbb{Z})$ defined by $\mu(\omega_\alpha) = \deg(E_{P,\omega_\alpha})$ if $\alpha \notin I$ and $\mu(\omega_\alpha) = 0$ if $\alpha \in I$. We call μ the *type* of the reduction E_P.

Using the above lemma, one can prove the following proposition; the proof being similar to that of [PV, Theorem 4, p. 90].

(3.7) Proposition. *Let $\mathscr{W}$ be a family of G-bundles on C parametrized by a smooth variety S. Assume that at each point $t \in S$ the infinitesimal deformation map*

$$T_t(S) \to H^1(C, \mathrm{Ad}(\mathscr{W}_t))$$

is surjective, where $\mathscr{W}_t = \mathscr{W}_{|C\times t}$ and $T_t(S)$ is the tangent space at t to S. For $\mu \in \mathrm{Hom}(X(T),\mathbb{Z})$, let S_μ be the subset of S consisting of those points $t \in S$ such that the canonical reduction of $\mathscr{W}_t$ is of type μ. Then S_μ is non-empty only for finitely many μ. Moreover, S_μ is locally closed and smooth, and the normal space at $t \in S_\mu$ is given by $H^1(C, \mathscr{W}_{t,\mathfrak{s}})$, where $\mathscr{W}_{t,\mathfrak{s}}$ is the vector bundle associated to the canonical reduction $\mathscr{W}_{t,P}$ by the representation of P on $\mathfrak{s} := \mathfrak{g}/\mathfrak{p}$.

(3.8) Proof of Proposition (3.4). The family $\mathscr{W} = \mathscr{W}_d$ parametrized by R_d satisfies the hypothesis of the above proposition (3.7). So it suffices to prove that for $t \in R_d \backslash R_d^s$, we have $\dim H^1(C, \mathscr{W}_{t,\mathfrak{s}}) \geqq 2$:

By Lemma (3.6), $H^0(C, \mathscr{W}_{t,\mathfrak{s}}) = 0$ and hence by Riemann-Roch theorem,

(1) $$\dim H^1(C, \mathscr{W}_{t,\mathfrak{s}}) = -\deg \mathscr{W}_{t,\mathfrak{s}} + \dim(\mathfrak{s})(h-1)\,,$$

where recall that h is the genus of C. Further, since $t \in R_d \backslash R_d^s$, we have $\mathfrak{g} \neq \mathfrak{p}$. By the same argument, used in the proof of Lemma (3.6), $\deg \mathscr{W}_{t,\mathfrak{s}} < 0$. This gives (using 1 and the assumption that $h \geqslant 2$) that $\dim H^1(C, \mathscr{W}_{t,\mathfrak{s}}) \geqq 2$, proving Proposition (3.4). □

(3.9) Lemma. *Let H be an affine algebraic group acting algebraically on a smooth variety Y and let U be a H-stable open subset such that* $\mathrm{codim}_Y(Y\backslash U) \geqq 2$. *Then the canonical restriction map* $\mathrm{Pic}^H(Y) \to \mathrm{Pic}^H(U)$ *is an isomorphism, where* $\mathrm{Pic}^H(Y)$ *denotes the set of isomorphism classes of H-equivariant line bundles on Y.*

Proof. Let $\mathscr{L}$ be an H-equivariant line bundle on U. Since Y is smooth and $\mathrm{codim}_Y(Y\backslash U) \geqq 2$, $\mathscr{L}$ extends uniquely to a line bundle $\tilde{\mathscr{L}}$ on Y. We show that $\tilde{\mathscr{L}}$ is H-equivariant:

Fix $h \in H$ and an open subset $V \subset Y$ such that $\tilde{\mathscr{L}}_{|_V}$ is a trivial line bundle. In particular, the line bundle $\tilde{\mathscr{L}}_{|_{hV}}$ also is trivial (since by the H-equivariance of $\mathscr{L}$, $\tilde{\mathscr{L}}_{|_{h(U\cap V)}}$ is trivial and moreover $\mathrm{codim}_V(V\backslash U) \geqq 2$). Take a nowhere-vanishing section s_1 of $\tilde{\mathscr{L}}_{|_V}$ and s_2 of $\tilde{\mathscr{L}}_{|_{hV}}$. Now for any $x \in U \cap V$, $f_h(x)s_2(hx) = h(s_1(x))$, for some (unique) $f_h(x) \in \mathbb{C}^*$. Clearly the map $U \cap V \to \mathbb{C}^*$, taking $x \mapsto f_h(x)$ is a regular map, which extends to a regular map $\tilde{f}_h : V \to \mathbb{C}^*$ (since $\mathrm{codim}_V(V\backslash U) \geqq 2$). Define an action of h on $\tilde{\mathscr{L}}_{|_V}$ by

$$h(s_1(x)) = \tilde{f}_h(x)s_2(hx), \qquad \text{for all } x \in V\ .$$

By the uniqueness of extension, this action of h on $\tilde{\mathscr{L}}_{|_V}$ patches-up to give an action of h on the whole of $\tilde{\mathscr{L}}$. Further, as can be easily seen, this is a regular action of H on $\tilde{\mathscr{L}}$.

The injectivity of $\mathrm{Pic}^H(Y) \to \mathrm{Pic}^H(U)$ is easy to see: An H-equivariant section, which does not vanish anywhere on U, extends to a nowhere-vanishing section on Y (and by uniqueness of extension it is H-equivariant). □

(3.10) Lifting of line bundles from $\mathfrak{M}$ to X_d. Take any $d \geqq 2h$. Let $\mathfrak{L}$ be a line bundle on $\mathfrak{M}$. Pull back the line bundle $\mathfrak{L}$ via the $\mathrm{GL}(q_d)$-equivariant morphism $\theta_d : R_d^s \to \mathfrak{M}$ to get a $\mathrm{GL}(q_d)$-equivariant line bundle $\theta_d^*(\mathfrak{L})$ on R_d^s (cf. Sect. 3.3). By the above Lemma (3.9) and Proposition (3.4), $\theta_d^*(\mathfrak{L})$ extends to a $\mathrm{GL}(q_d)$-equivariant line bundle $\widehat{\theta_d^*(\mathfrak{L})}$ on R_d. Consider the diagram, where all the maps are $\mathrm{GL}(q_d)$-equivariant morphisms (the map i_d is the inclusion, φ_d and π_d are as in Sect. 3.3, and $\mathrm{GL}(q_d)$ acts trivially on X_d):

$$\begin{array}{ccc} \mathscr{F}_d & \xrightarrow{\varphi_d} R_d \xhookleftarrow{i_d} & R_d^s \\ {\scriptstyle \pi_d}\downarrow & & \downarrow{\scriptstyle \theta_d} \\ X_d & & \mathfrak{M} \end{array}$$

Now $\varphi_d^*(\widehat{\theta_d^*(\mathfrak{L})})$ being a $\mathrm{GL}(q_d)$-equivariant line bundle (and π_d being a principal $\mathrm{GL}(q_d)$-bundle) descends to give a line bundle (denoted) $\mathfrak{L}_d$ on X_d (cf. [Kr, Proposition 6.4]).

(3.11) Lemma. *For any line bundle $\mathfrak{L}$ on $\mathfrak{M}$ and $d \geqq 2h$*

$$\mathfrak{L}_{d+1\,|_{X_d}} \approx \mathfrak{L}_d\ , \ \textit{and} \quad \mathfrak{L}_{d\,|_{X^s}} \approx \psi^*(\mathfrak{L})\ ,$$

where $\psi : X^s \to \mathfrak{M}$ is the morphism as in Sect. 1 (cf. Lemma 3.2).

Proof. We will freely use the notation from Sect. 3.3. Let $X_{\mathfrak{w}}$ be a fixed Schubert variety, and denote the (reduced) variety $X_{\mathfrak{w}} \cap X_d$ by $Y = Y_{d,\mathfrak{w}}$. Then $Y^s := Y \cap X^s$ is an open non-empty (irreducible) subvariety of $X_{\mathfrak{w}}$. We denote by $\mathscr{F}_{d,Y}$, $\mathscr{F}_{d+1,Y}$ and $Q_{d,Y}$ the restrictions of $\mathscr{F}_d$, $\mathscr{F}_{d+1}$ and Q_d to Y, where Q_d is the P-subbundle of $\mathscr{F}_{d+1|X_d}$ as in Sect. 3.3. We show that $\mathfrak{L}_{d\,|Y} \approx \mathfrak{L}_{d+1\,|Y}$ and $\mathfrak{L}_{d\,|Y^s} \approx \psi^*(\mathfrak{L})_{|Y^s}$. This will of course prove the lemma.

We first show that

$$\mathfrak{L}_{d\,|Y^s} \approx \psi^*(\mathfrak{L})_{|Y^s} \ : \tag{1}$$

From the commutativity of the diagram (where $\mathscr{F}^s_{d,Y} := \pi_d^{-1}(Y^s)$, and π_d, φ_d, and ψ are the corresponding maps got by restriction, which we denote by the same symbols)

$$(D_1) \qquad \begin{array}{ccc} & \mathscr{F}^s_{d,Y} & \\ {\scriptstyle\pi_d}\swarrow & & \searrow{\scriptstyle\varphi_d} \\ Y^s & & R^s_d \\ {\scriptstyle\psi}\searrow & & \swarrow{\scriptstyle\theta_d} \\ & \mathfrak{M} & \end{array}$$

we see that the $\mathrm{GL}(q_d)$-linearizations on $\pi_d^*(\psi^*\mathfrak{L})$ and $\varphi_d^*(\theta_d^*\mathfrak{L})$ are the same. This shows that $\mathfrak{L}_{d\,|Y^s} \approx \psi^*(\mathfrak{L})_{|Y^s}$ (since π_d is a principal $\mathrm{GL}(q_d)$-bundle).

If H is an affine algebraic group and $\mathscr{H}$ an H-linearized line bundle on a principal H-bundle, we denote by $\mathscr{H}^H$ the line bundle on the base space (of the H-bundle) obtained by descending $\mathscr{H}$.

Let $\widetilde{\mathscr{W}}{}^o_d$ be the vector bundle on $C \times R_d$ which is the pull-back of $\widehat{\mathscr{W}}{}^o_d$ by the map $I_C \times \beta : C \times R_d \to C \times R^o_d$, where $\beta : R_d \to R^o_d$ is the canonical map. Let $\pi''_d : \mathscr{F}''_d \to R_d$ (resp. $\pi'_d : \mathscr{F}'_d \to R_d$) be the frame bundle of the vector bundle $(p_{R_d})_*(\widetilde{\mathscr{W}}{}^o_d \otimes \mathcal{O}(p))$ (resp. $(p_{R_d})_*(\widetilde{\mathscr{W}}{}^o_d)$), where $p_{R_d} : C \times R_d \to R_d$ is the projection on the second factor. Just as in Sect. 3.3, the inclusion

$$(p_{R_d})_*(\widetilde{\mathscr{W}}{}^o_d) \hookrightarrow (p_{R_d})_*(\widetilde{\mathscr{W}}{}^o_d \otimes \mathcal{O}(p))$$

defines a P-subbundle $Q'_d \subset \mathscr{F}''_d$ on R_d and a morphism $\beta'_d : Q'_d \to \mathscr{F}'_d$. Further, analogous to the map $\varphi_d : \mathscr{F}_d \to R_d$ there is a $\mathrm{GL}(q_{d+1})$-equivariant morphism $\varphi'_d : \mathscr{F}''_d \to R_{d+1}$. Thus we have the diagram:

$$(D_2) \qquad \begin{array}{ccc} & Q'_d & \\ {\scriptstyle\beta'_d}\swarrow & & \hookrightarrow \\ \mathscr{F}'_d & & \mathscr{F}''_d \\ {\scriptstyle\pi'_d}\downarrow & & \downarrow{\scriptstyle\varphi'_d} \\ R_d & & R_{d+1}\ . \end{array}$$

(Observe that β'_d is a principal U-bundle, π'_d is a principal $\mathrm{GL}(q_d)$-bundle and π''_d is a principal $\mathrm{GL}(q_{d+1})$-bundle.) Considering the commutative diagram (where $\mathscr{F}''^{s}_d := \pi''^{-1}_d(R^s_d)$)

$$(D_3)\qquad \begin{array}{ccccc} & & \mathscr{F}''^{s}_d & & \\ & {\scriptstyle \pi''_d}\swarrow & & \searrow{\scriptstyle \varphi'_d} & \\ R^s_d & & & & R^s_{d+1} \\ & {\scriptstyle \theta_d}\searrow & & \swarrow{\scriptstyle \theta_{d+1}} & \\ & & \mathfrak{M} & & \end{array}$$

we see, as above, that

$$(\varphi'^*_d \theta^*_{d+1}\mathfrak{L})^{\mathrm{GL}(q_{d+1})} \approx \theta^*_d(\mathfrak{L}) .$$

Since $\operatorname{codim}_{R_d}(R_d \backslash R^s_d) \geqq 2$ and R_d is smooth, we have

$$\widehat{\theta^*_d\mathfrak{L}} \approx (\varphi'^*_d(\widehat{\theta^*_{d+1}\mathfrak{L}}))^{\mathrm{GL}(q_{d+1})} .$$

Now

$$\begin{aligned} &(\varphi'^*_d(\widehat{\theta^*_{d+1}\mathfrak{L}}))^{\mathrm{GL}(q_{d+1})} \\ &\qquad \approx (\gamma^*_d(\widehat{\theta^*_{d+1}\mathfrak{L}}))^P \\ &\qquad \approx ((\gamma^*_d(\widehat{\theta^*_{d+1}\mathfrak{L}}))^U)^{\mathrm{GL}(q_d)} \\ &\qquad \approx \sigma^*((\gamma^*_d(\widehat{\theta^*_{d+1}\mathfrak{L}}))^U) , \end{aligned}$$

where $\gamma_d : Q'_d \to R_{d+1}$ is the restriction of φ'_d to Q'_d and $\sigma : R_d \to \mathscr{F}'_d$ is the canonical section, given by the isomorphism

$$\mathbb{C}^{q_d} = H^0(C, \mathbb{C}^{q_d} \otimes \mathcal{O}_C) \tilde{\to} H^0(C, \widetilde{\mathscr{W}}^o_d|_{C\times t})$$

for $t \in R_d$. Thus

$$(2)\qquad \widehat{\theta^*_d\mathfrak{L}} \approx \sigma^*((\gamma^*_d(\widehat{\theta^*_{d+1}\mathfrak{L}}))^U) .$$

Consider the following commutative diagram

$$(D_4)\qquad \begin{array}{ccccccc} Q_{d,Y} & \xrightarrow{\alpha} & Q'_d & \hookrightarrow & \mathscr{F}''_d \\ {\scriptstyle \beta_d}\downarrow & & \downarrow{\scriptstyle \beta'_d} & \searrow{\scriptstyle \gamma_d} & \downarrow{\scriptstyle \varphi'_d} \\ \mathscr{F}_{d,Y} & \xrightarrow{\delta} & \mathscr{F}'_d & & R_{d+1} \\ {\scriptstyle \pi_d}\downarrow & {\scriptstyle \varphi_d}\searrow & \downarrow{\scriptstyle \pi'_d} & & \\ Y & & R_d & & \end{array}$$

where $\delta := \sigma \circ \varphi_d$, and the map α is defined as follows: Let $g\mathscr{P} \in Y$ and let $s = (s_1, \ldots, s_{q_d}, \ldots, s_{q_{d+1}})$ be a frame of $H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}((d+1)p))$ such that $\bar{s} := (s_1, \ldots, s_{q_d})$ is a frame of $H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}(dp))$. We have a commutative diagram:

$$\begin{array}{ccccc} 0 \longrightarrow & H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}(dp)) & \longrightarrow & H^0(C, \overline{\varphi(g)} \otimes \mathcal{O}((d+1)p)) \\ & \downarrow & & \downarrow \\ 0 \longrightarrow & H^0(C, \widetilde{\mathscr{W}}^o_{d|C\times\varphi_d(\bar{s})}) & \longrightarrow & H^0(C, \widetilde{\mathscr{W}}^o_{d|C\times\varphi_d(\bar{s})} \otimes \mathcal{O}(p)) , \end{array}$$

where the vertical maps are isomorphisms. Observe that, under the first vertical isomorphism, the frame $\bar{s}$ is mapped to the frame $\delta(\bar{s})$. Now define $\alpha(s)$ to be the frame in $H^0(C, \widetilde{\mathscr{W}}^o_{d|C\times\varphi_d(\bar{s})} \otimes \mathcal{O}(p))$ which is the image of the frame s under the second vertical isomorphism. Then α is a U-equivariant morphism.

We claim that (as line bundles on $\mathscr{F}_{d,Y}$)

$$\varphi_d^*(\widehat{\theta_d^*\mathfrak{L}}) \approx (\alpha^*\gamma_d^*(\widehat{\theta_{d+1}^*\mathfrak{L}}))^U \quad : \tag{3}$$

This follows since

$$\begin{aligned}
&(\alpha^*\gamma_d^*(\widehat{\theta_{d+1}^*\mathfrak{L}}))^U \\
&\quad\approx \delta^*((\gamma_d^*(\widehat{\theta_{d+1}^*\mathfrak{L}}))^U) \\
&\quad\approx \varphi_d^*\sigma^*((\gamma_d^*(\widehat{\theta_{d+1}^*\mathfrak{L}}))^U) \\
&\quad\approx \varphi_d^*(\widehat{\theta_d^*\mathfrak{L}}) , \text{ using } (2) .
\end{aligned}$$

Now the bundle $\varphi_d^*(\widehat{\theta_d^*\mathfrak{L}})$ has a $\mathrm{GL}(q_d)$-linearization coming from the action of $GL(q_d)$ on $\widehat{\theta_d^*\mathfrak{L}}$ and (by definition of $\mathfrak{L}_d$) the bundle on Y obtained by descent is $\mathfrak{L}_{d|Y}$. On the other hand, the bundle $(\alpha^*\gamma_d^*(\widehat{\theta_{d+1}^*\mathfrak{L}}))^U$ has a $\mathrm{GL}(q_d)$-action given by the action of $P/U \approx \mathrm{GL}(q_d)$ arising from the action of P on $\alpha^*\gamma_d^*(\widehat{\theta_{d+1}^*\mathfrak{L}})$ (which in turn comes from the action of $\mathrm{GL}(q_{d+1})$, in particular, P on $\widehat{\theta_{d+1}^*\mathfrak{L}}$; observe that even though α is only U-equivariant, $\gamma_{d\circ\alpha}$ is P-equivariant) and the bundle on Y obtained by descent via π_d is $\mathfrak{L}_{d+1|Y}$. Now over $\mathscr{F}^s_{d,Y} := \pi_d^{-1}(Y^s)$, these two actions of $\mathrm{GL}(q_d)$ coincide (i.e. the isomorphism η of line bundles on $\mathscr{F}_{d,Y}$ as guaranteed by (3) is $\mathrm{GL}(q_d)$-equivariant on $\mathscr{F}^s_{d,Y}$), as is seen from the following commutative diagram (got from the diagrams D_1 and D_4) (where $Q^s_{d,Y} := \beta_d^{-1}(\mathscr{F}^s_{d,Y})$):

$$\begin{array}{ccccc}
 & & Q^s_{d,Y} & & \\
 & {\scriptstyle\beta_d}\swarrow & & \searrow{\scriptstyle\gamma_d\circ\alpha} & \\
\mathscr{F}^s_{d,Y} & \xrightarrow{\varphi_d} & R^s_d & & R^s_{d+1} \\
{\scriptstyle\pi_d}\downarrow & & {\scriptstyle\theta_d}\downarrow & \swarrow{\scriptstyle\theta_{d+1}} & \\
Y^s & \xrightarrow{\psi} & \mathfrak{M}. & &
\end{array}$$

Since Y^s is dense in Y, we have that $\mathscr{F}^s_{d,Y}$ is dense in $\mathscr{F}_{d,Y}$; in particular, the isomorphism η is $\mathrm{GL}(q_d)$-equivariant on the whole of $\mathscr{F}_{d,Y}$. Hence $\mathfrak{L}_{d|Y} \approx \mathfrak{L}_{d+1|Y}$. Denote this isomorphism by μ. Then the restriction of μ to Y^s is the identity map under the identification (1). From this it is easy to see that $\mathfrak{L}_d \approx \mathfrak{L}_{d+1|X_d}$. This completes the proof of the lemma. □

Finally we come to the

(3.12) Proof of Proposition (2.3). For any Schubert variety $X_\mathfrak{w}$, there exists a large enough $d(\mathfrak{w})$ such that $X_\mathfrak{w} \subset X_{d(\mathfrak{w})}$. Fix a line bundle $\mathfrak{L}$ on $\mathfrak{M}$. Let $\widehat{\mathfrak{L}_\mathfrak{w}}$ be the line bundle on $X_\mathfrak{w}$ defined by $\widehat{\mathfrak{L}_\mathfrak{w}} = \mathfrak{L}_{d(\mathfrak{w})|X_\mathfrak{w}}$. By Lemma (3.11), $\widehat{\mathfrak{L}_\mathfrak{w}}$ is

well defined and $\widehat{\mathfrak{L}_{\mathfrak{w}}}_{|X^s_{\mathfrak{w}}} \approx \psi^*(\mathfrak{L})_{|X^s_{\mathfrak{w}}}$, where $\psi : X^s \to \mathfrak{M}$ is the morphism as in Sect. 1. Moreover, for $\mathfrak{v} \leqq \mathfrak{w}$, $\widehat{\mathfrak{L}_{\mathfrak{w}}}_{|X_{\mathfrak{v}}} \approx \widehat{\mathfrak{L}_{\mathfrak{v}}}$. In particular, by Lemma (2.2), we get a line bundle $\widehat{\mathfrak{L}}$ on X with $\widehat{\mathfrak{L}}_{|X^s} \approx \psi^*(\mathfrak{L})$. This proves the proposition. □

4. Determination of Pic($\mathfrak{M}$)

(4.1) Definition [D, Sect. 2]. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two (finite dimensional) complex simple Lie algebras and $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. There exists a unique number $m_\varphi \in \mathbb{C}$, called the ***Dynkin index*** of the homomorphism φ, satisfying

$$\langle \varphi(x), \varphi(y) \rangle = m_\varphi \langle x, y \rangle \ , \ \text{ for all } x, y \in \mathfrak{g}_1 \ ,$$

where $\langle , \rangle$ is the Killing form on $\mathfrak{g}_1$ (and $\mathfrak{g}_2$) normalized so that $\langle \theta, \theta \rangle = 2$ for the highest root θ.

It is easy to see from [KNR, Lemma 5.2] that for a finite dimensional representation V of $\mathfrak{g}_1$ given by a Lie algebra homomorphism $\varphi : \mathfrak{g}1 \to sl(V)$, we have $m_\varphi = m_V$, where m_V is as in [KNR, Sect. 5.1] and $sl(V)$ is the Lie algebra of trace 0 endomorphisms of V.

By taking a representation V of G_2 such that $m_V \neq 0$, and using [KNR, Corollary 5.6], the following proposition follows easily.

(4.2) Proposition. *Let G_1, G_2 be two connected complex simple algebraic groups. Then for any algebraic group homomorphism $\phi : G_1 \to G_2$, the induced map at the third homotopy group level*

$$\phi_* : \pi_3(G_1) \approx \mathbb{Z} \longrightarrow \pi_3(G_2) \approx \mathbb{Z}$$

is given by the multiplication via the Dynkin index $m_{d\phi}$ of the induced Lie algebra homomorphism $d\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$, where $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) is the Lie algebra of G_1 (resp. G_2).

In particular, $m_{d\phi}$ is an integer.

(4.3) Remark. The integrality of m_ϕ is proved by Dynkin [D, Theorem 2.2], and so is the following lemma [D, Theorem 2.5], by a quite different (and long) argument.

(4.4) Lemma. *Let $\mathfrak{g}$ be a complex simple Lie algebra and let $V(\lambda)$ be an irreducible representation of $\mathfrak{g}$ with highest weight λ. Then the Dynkin index $m_{V(\lambda)}$ of the representation $V(\lambda)$ is given by*

$$m_{V(\lambda)} = (\|\lambda + \rho\|^2 - \|\rho\|^2) \frac{\dim_{\mathbb{C}} V(\lambda)}{\dim_{\mathbb{C}} \mathfrak{g}} \ ,$$

where ρ is the half sum of positive roots and the Killing form on $\mathfrak{g}$ is normalized (as earlier) so that $\|\theta\|^2 = 2$ for the highest root θ.

Proof. The representation $V = V(\lambda)$ of course gives rise to a Lie algebra homomorphism $\varphi = \varphi_V : \mathfrak{g} \to sl(V)$. Since $m_V = m_\varphi$ (cf. Sect. 4.1), for any $x, y \in \mathfrak{g}$

(1) $$m_V\langle x, y\rangle = \text{ trace } (\varphi(x) \circ \varphi(y)) .$$

Choose a basis $\{e_i\}$ of $\mathfrak{g}$ and let $\{e^i\}$ be the dual basis of $\mathfrak{g}$ with respect to the Killing form $\langle , \rangle$. Consider the Casimir element $\Omega := \sum_i e_i e^i \in U(\mathfrak{g})$. Then Ω acts on V via

(2) $$\Omega_V := \sum_i \varphi(e_i) \circ \varphi(e^i) .$$

But V being irreducible of highest weight λ,

(3) $$\Omega_V = (\|\lambda + \rho\|^2 - \|\rho\|^2) I_V ,$$

where I_V is the identity operator of V. In particular,

$$\begin{aligned} m_V &= \frac{1}{\dim \mathfrak{g}} \sum_i \text{ trace } (\varphi(e_i) \circ \varphi(e^i)) , && \text{by (1)} \\ &= \frac{1}{\dim \mathfrak{g}} \text{ trace } \Omega_V , && \text{by (2)} \\ &= \frac{1}{\dim \mathfrak{g}} (\|\lambda + \rho\|^2 - \|\rho\|^2) \dim V , && \text{by (3)} . \end{aligned}$$

This proves the lemma. □

We also need the following

(4.5) Lemma. *Let $\mathfrak{g}$ be a complex simple Lie algebra and let V and W be two finite dimensional representations of $\mathfrak{g}$. Then*

$$m_{V \otimes W} = m_V \dim W + m_W \dim V .$$

Proof. Write the characters

$$\begin{aligned} \text{ch } V &= \sum_\lambda n_\lambda e^\lambda , && \text{and} \\ \text{ch } W &= \sum_\mu m_\mu e^\mu , && \text{for some } n_\lambda,\ m_\mu \in \mathbb{Z}_+ . \end{aligned}$$

Then

$$\text{ch } (V \otimes W) = \sum_{\lambda, \mu} n_\lambda m_\mu e^{\lambda + \mu} .$$

Hence by [KNR, Lemma 5.2],

$$\begin{aligned} 2m_{V\otimes W} &= \sum_{\lambda,\mu} n_\lambda m_\mu \langle \lambda+\mu, \theta^\vee\rangle^2 \\ &= \sum n_\lambda m_\mu \langle \lambda, \theta^\vee\rangle^2 + \sum n_\lambda m_\mu \langle \mu, \theta^\vee\rangle^2 \\ &\quad +2\sum n_\lambda m_\mu \langle \lambda, \theta^\vee\rangle \langle \mu, \theta^\vee\rangle \\ &= 2\left(\sum_\mu m_\mu\right) m_V + 2\left(\sum_\lambda n_\lambda\right) m_W \\ &\quad +2\left(\sum_\lambda n_\lambda \langle \lambda, \theta^\vee\rangle\right)\left(\sum_\mu m_\mu \langle \mu, \theta^\vee\rangle\right) \\ &= 2(\dim W) m_V + 2(\dim V) m_W \qquad (1) \\ &\quad +2\left(\sum_\lambda n_\lambda \langle \lambda, \theta^\vee\rangle\right)\left(\sum_\mu m_\mu \langle \mu, \theta^\vee\rangle\right) . \end{aligned}$$

For any $h \in \mathfrak{h}$, define $\beta_V(h) = \sum_\lambda n_\lambda \langle \lambda, h\rangle$. Then the map $\beta_V : \mathfrak{h} \to \mathbb{C}$, $h \mapsto \beta_V(h)$ is W-equivariant (with the trivial action of W on $\mathbb{C}$). Hence, $\mathfrak{h}$ being an irreducible W-module,

(2) $$\beta_V \equiv 0 .$$

Combining (1) and (2), the lemma follows. □

(4.6) Definition. Let $\mathfrak{g}$ be a complex simple Lie algebra and let θ be the highest root (with respect to some choice of the set of positive roots). Express the associated coroot $\theta^\vee$ in terms of the simple coroots:

$$\theta^\vee = \sum_{i=1}^{\ell} m_i \alpha_i^\vee .$$

Now define $d = d(\mathfrak{g})$ to be the least common multiple of $\{m_i\}_{i=1,\dots,\ell}$. Then the number d is given as follows:

Type of $\mathfrak{g}$	d($\mathfrak{g}$)
A_ℓ ($\ell \geqq 1$), C_ℓ ($\ell \geqq 2$)	1
B_ℓ ($\ell \geqq 3$)	2
D_ℓ ($\ell \geqq 4$)	2
G_2	2
F_4	6
E_6	6
E_7	12
E_8	60

(4.7) Proposition. *For any finite dimensional representation V of $\mathfrak{g}$, the number $d(\mathfrak{g})$ divides m_V. Moreover, there exists an irreducible representation V_o of $\mathfrak{g}$ such that $d(\mathfrak{g}) = m_{V_o}$.*

Proof. Unfortunately, our proof is case by case. We follow the indexing convention as in [B, Planche I-IX]. We denote the i-th fundamental weight ($1 \leqq i \leqq \ell$) by ω_i.

Case 1. $A_\ell (\ell \geqq 1)$, $C_\ell (\ell \geqq 2)$: As in [KNR, Lemma 5.2], $m_{V_o} = 1$, for the standard $(\ell+1)$-dimensional representation V_o of A_ℓ. Similarly for the standard 2ℓ-dimensional representation V_o of C_ℓ (with highest weight ω_1), $m_{V_o} = 1$ (as can be seen from Lemma 4.4).

For a simply-connected group G, since the fundamental representations $\{V(\omega_i)\}_{1 \leqq i \leqq \ell}$ generate the representation ring $R(G)$ as an algebra (cf. [A, Theorem 6.41]), to prove that $d(\mathfrak{g})$ divides m_V for any $\mathfrak{g}$-module V, it suffices to show that $d(\mathfrak{g})$ divides $m_i := m_{V(\omega_i)}$ for all $1 \leqq i \leqq \ell$ (cf. Lemma 4.5). In the following calculations, we make use of Lemma (4.4) and [B, Planche I-IX] freely.

Case 2. B_ℓ ($\ell \geqq 3$): For $1 \leqq i \leqq \ell - 1$, $m_i = 2\binom{2\ell - 1}{i - 1}$, since $\dim V(\omega_i) = \binom{2\ell + 1}{i}$; and $m_\ell = 2^{\ell - 2}$.

In particular, $m_1 = 2$, so take $V_o = V(\omega_1)$.

Case 3. D_ℓ ($\ell \geqq 4$): For $1 \leqq i \leqq \ell - 2$, $m_i = 2\binom{2\ell - 2}{i - 1}$, since $\dim V(\omega_i) = \binom{2\ell}{i}$; and $m_{\ell-1} = m_\ell = 2^{\ell - 3}$.

In particular, $m_1 = 2$.

In the following calculations, $\dim V(\omega_i)$ is taken from [BMP].

Case 4. G_2: $m_1 = 2$, $m_2 = 8$.

(Observe that $V(\omega_2)$ is the adjoint representation of G_2 and hence m_2 can be calculated from [KNR, Lemma 5.2 and Remark 5.3].)

Case 5. F_4: $m_1, \ldots, m_4$ are respectively 18, 9×98, l26, and 6.

Case 6. E_6: $m_1, \ldots, m_6$ are respectively 6, 24, 150, 1800, 150, and 6.

Case 7. E_7: $m_1, \ldots, m_7$ are respectively 36, 360, 65×72, 2750×108, 104×165, 8×81, and 12.

Case 8. E_8: $m_1, \ldots, m_8$ are respectively 12×125, 4750×18, 49×108000, 75×111275472, 30×4720170, 45×39520, 15×980, and 60. □

(4.8) Remark. The values of m_i given above are also contained in [D], but some of his values are incorrect.

Combining Proposition (4.7) and Theorem (2.4) with the chart in Definition (4.6), we get the following strengthening of Theorem (2.4).

(4.9) Theorem. *With the notation and assumptions as in Theorem* (2.4), *consider the injective map* $\overline{\psi}^* : \mathrm{Pic}(\mathfrak{M}) \hookrightarrow \mathrm{Pic}(X) \approx \mathbb{Z}$. *Then*
(1) $\overline{\psi}^*$ *is surjective in the case where G is of type* A_ℓ ($\ell \geqq 1$), *and* $C_\ell (\ell \geqq 2)$.

(2) The order $\gamma = \gamma_G$ of the cokernel of $\overline{\psi^*}$ is bounded as follows:
(a) $G = B_\ell$ ($\ell \geqq 3$), $\gamma \leqq 2$
(b) $G = D_\ell$ ($\ell \geqq 4$), $\gamma \leqq 2$
(c) $G = G_2$, $\gamma \leqq 2$
(d) $G = F_4$, $\gamma \leqq 6$
(e) $G = E_6$, $\gamma \leqq 6$
(f) $G = E_7$, $\gamma \leqq 12$
(g) $G = E_8$, $\gamma \leqq 60$. □

(4.10) Remarks (added at the time of revision). (a) We have received a preprint by Y. Laszlo and C. Sorger " The line bundles on the stack of parabolic G-bundles over curves and their sections", which has some overlap with our paper. In particular, they calculate the Picard group of the moduli *stack* of parabolic G-bundles for the classical groups and G_2.
(b) We have shown that the order $\gamma = \gamma_G$ of the cokernel of $\overline{\psi^*}$ is precisely equal to 2 in the cases of $G = B_\ell$ ($\ell \geqq 3$) and $G = D_\ell$ ($\ell \geqq 4$), i.e., the theta bundle $\Theta(V_n)$ on the moduli space $\mathfrak{M} = \mathfrak{M}(n)$ of semistable Spin(n)-bundles ($n \geqq 7$) on the curve C does not admit a square root as a line bundle on the whole of $\mathfrak{M}$, where the standard $SO(n)$-representation V_n is to be considered as a representation of Spin(n) via $SO(n)$. We are not including the proof here as we have been informed that this has also been obtained recently by Beauville-Laszlo-Sorger (and prior to us). It is very likely that the bounds for γ given in Theorem (4.9) for all the other groups (G of type G_2, F_4, E_6, E_7, and E_8) are sharp as well.

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