Vol. 69, No. 1 DUKE MATHEMATICAL JOURNAL © January 1993

COHOMOLOGY OF QUANTUM GROUPS AT ROOTS OF UNITY

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0. Introduction. To any Cartan matrix (a_{ij}) of finite type of rank r, Drinfeld and Jimbo have associated a Hopf algebra U over the field $\mathbb{Q}(v)$ of rational functions, with generators E_i, F_i, K_i, K_i^{-1}; $i = 1, \ldots, r$, subject to the quantum analogues of Serre relations (see e.g. [L2, §1.1]). Lusztig [L1], [L2] introduced a $\mathbb{Z}[v, v^{-1}]$-form $U_{\mathbb{Z}}$, analogous to the Kostant $\mathbb{Z}$-form in the classical case, as the $\mathbb{Z}[v, v^{-1}]$-subalgebra of U generated by the elements

$$E_i^{(n)} := E_i^n/[n]!_{d_i}, \qquad F_i^{(n)} := F_i^n/[n]!_{d_i}, \qquad K_i^{\pm 1}; \qquad n \geqslant 0, i = 1, \ldots, r.$$

(The q-divided powers $E_i^{(n)}$, $F_i^{(n)}$ are defined, e.g., in [L2].) Then $U_{\mathbb{Z}}$ is a Hopf subalgebra of U (see [L2, §1.3]). Thus, for any $\mathbb{Z}[v, v^{-1}]$-algebra k, one obtains, by extension of scalars, a Hopf k-algebra $U_k := k \otimes_{\mathbb{Z}[v,v^{-1}]} U_{\mathbb{Z}}$.

In particular, let ξ be a primitive root of unity of odd order l and let $k := \mathbb{Q}(\xi)$ be the cyclotomic field, viewed as a $\mathbb{Z}[v, v^{-1}]$-algebra via the specialization $v \mapsto \xi$. The elements K_i^l $(i = 1, \ldots, r)$, are then central in U_k, and we set

$$U_\xi := U_k/\text{Ideal generated by } \{K_i^l - 1\}.$$

Lusztig further introduced the subalgebra $u_\xi \subset U_\xi$, generated by the elements E_i, F_i, $K_i^{\pm 1}$; $i = 1, \ldots, r$, called the *restricted enveloping algebra*. He showed that u_ξ is a Hopf subalgebra of U_ξ of dimension $l^{\dim \mathfrak{g}}$, where $\mathfrak{g}$ is the split semisimple Lie algebra over k associated to the Cartan matrix (a_{ij}).

Given an affine algebraic variety V over k, let $k[V]$ denote the algebra of regular functions on V. If V is a cone, then there is a natural grading on $k[V]$ by nonnegative integers. Let, in particular, $\mathcal{N} \subset \mathfrak{g}$ be the *nilpotent cone*, the subvariety of all the ad-nilpotent elements of $\mathfrak{g}$.

In this paper, we compute the cohomology $H^*(u_\xi, k)$ of the algebra u_ξ with trivial coefficients. (See the appendix for the definition of the cohomology of an algebra.) *The following restrictions on l (= order of the root of unity) are in force throughout the paper: l is odd and $l > h$, and moreover l is prime to* 3 *if* $\mathfrak{g}$ *has factors of the type G_2, where h is the Coxeter number of* $\mathfrak{g}$ (see, e.g., [J, Page 262]).

MAIN THEOREM. *$H^{\text{odd}}(u_\xi, k) = 0$, and there is a natural graded algebra isomorphism*

$$H^{2\bullet}(u_\xi, k) \simeq k^\bullet[\mathcal{N}].$$

Received 7 April 1992. Revision received 21 August 1992.

Remarks. (i) The above theorem is a "quantum version" of an earlier result of Friedlander-Parshall [FP], who computed the cohomology of the restricted enveloping algebras in finite characteristics.

(ii) After we proved the above result, we were informed by D. Kazhdan and M. Verbitsky that they obtained the same result in the special case $l =$ prime power (see [KV]), by reducing it to [FP]. Some partial results towards our theorem have also been obtained by Parshall-Wang in their recent preprint "Cohomology of Infinitesimal Quantum Groups I."

(iii) There is a natural $\mathfrak{g}$-module structure on each side of the isomorphism of the above theorem, and the isomorphism is compatible with that structure.

We now describe the contents of the paper in some detail:

In §1 various quantum algebras, with and without divided powers, are introduced. We make use of an increasing filtration on the Borel part b_ξ of the algebra u_ξ, given by DeConcini-Kac-Soibelman. The filtration is a quantized version of the standard increasing filtration on an enveloping algebra, and Gr b_ξ, the associated graded algebra, turns out to be a truncated algebra of skew polynomials [DK].

In §2 we compute the cohomology of b_ξ using a strategy similar to that of [FP]. We first find the cohomology of Gr b_ξ via Koszul duality considerations (see [BGS], [P]). The cohomology of b_ξ is then computed via a spectral sequence argument. We find that $H^*(b_\xi, k)$ is the symmetric algebra on $H^2(b_\xi, k)$ and that there is a canonical isomorphism $H^2(b_\xi, k) \simeq \mathfrak{n}^*$, where $\mathfrak{n}$ is the positive part of the Lie algebra $\mathfrak{g}$ or rather its "first Frobenius twist". The latter isomorphism is a certain transgression map from the cohomology of the "DeConcini-Kac center".

In §3 an induction technique is used to get the cohomology of u_ξ from the cohomology of b_ξ. The argument is based on a quantum version of Kempf vanishing, proved in [APW].

In §4 we define a functor F, assigning to any finite-dimensional U_ξ-module M a finite-dimensional $\mathfrak{a}$-module $F(M)$, where $\mathfrak{a} := Z_\mathfrak{g}(n)$ is the centralizer of the principal nilpotent n in $\mathfrak{g}$. We present a conjecture relating the functor F to the hypercohomology of perverse sheaves on an infinite-dimensional Grassmannian. The conjecture would imply in particular that, for any simple U_ξ-modules M_1, M_2 in the same linkage class, there is a natural isomorphism

$$\mathrm{Ext}^*_{U_\xi}(M_1, M_2) \simeq \mathrm{Hom}_\mathfrak{a}(F(M_1), F(M_2)).$$

In §5 (which is an appendix) we collect various known facts about the cohomology of algebras.

Acknowledgements. We thank E. Getzler, D. Kazhdan, and B. Tsygan for some helpful conversations, and S. D. Schack and J. D. Stasheff for some clarifications and references. We also thank the referee for a useful remark.

1. Preliminaries and notation. Throughout the paper, the ground field is the cyclotomic field k, and unless otherwise indicated, the tensor products are over k.

Let $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$ be the triangular decomposition of $\mathfrak{g}$. Let e_i (resp. f_i), $1 \leqslant i \leqslant r$, be the corresponding Chevalley generators of $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$), let $\alpha_1, \ldots, \alpha_r \in \mathfrak{h}^*$ be the set of simple roots of $\mathfrak{n}^+$, and let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. We denote the length of an element $w \in W$ by $l(w)$. Let $\mathfrak{h}^*_{\mathbb{Z}}$ denote the weight lattice in $\mathfrak{h}^*$ and let $\langle \cdot, \cdot \rangle$ be the unique W-invariant scalar product on $\mathfrak{h}^*$ normalized so that $d_i := \langle \alpha_i, \alpha_i \rangle/2$, $i = 1, \ldots, r$ are all positive integers with greatest common divisor $= 1$. Set $\mathfrak{h}^*_l := \mathfrak{h}^*_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/(l)$.

We fix a reduced expression of the longest element w_0 of W. This puts a total linear order $\prec$ on the set Δ_+ of positive roots (see e.g. [L2, Appendix]). Moreover, to each $\alpha \in \Delta_+$ and $n \geqslant 1$, Lusztig attached ([L1, §1.9], [L2, §4]) the elements $E_\alpha^{(n)}$, $F_\alpha^{(n)} \in U_{\mathbb{Z}}$. Their images in U_ξ are denoted by the same symbols.

Following [DK, §1.5], we introduce the $\mathbb{Q}[v, v^{-1}]$-subalgebra $\mathscr{U}_{\mathbb{Q}} \subset U$ generated by the elements $\{E_i, F_i, K_i, K_i^{-1}\}_{1 \leqslant i \leqslant r}$. Set $\mathscr{U}_k = k \otimes_{\mathbb{Q}[v, v^{-1}]} \mathscr{U}_{\mathbb{Q}}$ (where k is viewed as a $\mathbb{Q}[r, r^{-1}]$ algebra via the specialization $v \mapsto \xi$). It is shown in [DK, Corollary 3.1] that the elements $\{E_\alpha^l, F_\alpha^l\}_{\alpha \in \Delta^+}$ and $\{K_i^l\}_{1 \leqslant i \leqslant r}$ are central in $\mathscr{U}_k$. We set

$$\mathscr{U}_\xi := \mathscr{U}_k/\text{Ideal generated by } \{K_i^l - 1\}_{1 \leqslant i \leqslant r}$$

and let $\mathscr{Z} \subset \mathscr{U}_\xi$ be the subalgebra generated by $\{E_\alpha^l, F_\alpha^l\}_{\alpha \in \Delta^+}$. Finally, let $U(\mathfrak{g})$ be the classical enveloping algebra of the Lie algebra $\mathfrak{g}$.

The above-introduced algebras are related to each other by two isomorphisms which are due to Lusztig [L1], [L2] (together with [DK, §3.3]). He showed that the subalgebra u_ξ is normal in U_ξ. (See §5.2 for the definition of normal subalgebras.) Also $\mathscr{F}$ being central is clearly a normal subalgebra of $\mathscr{U}_\xi$. Furthermore, there are algebra isomorphisms:

$$\mathscr{U}_\xi // \mathscr{Z} \simeq u_\xi \quad \text{and} \quad U_\xi // u_\xi \simeq U(\mathfrak{g}). \tag{1}$$

(See §5.2 for the notation $//$.) The first isomorphism in (1) is induced by the natural map $\mathscr{U}_\xi \to u_\xi$ sending $E_i \mapsto E_i$, $F_i \mapsto F_i$ and $K_i^{\pm 1} \mapsto K_i^{\pm 1}$, $1 \leqslant i \leqslant r$. The second isomorphism in (1) is induced by the map $U_\xi \to U(\mathfrak{g})$, sending $E_i^{(l)} \mapsto e_i$, $F_i^{(l)} \mapsto f_i$, $1 \leqslant i \leqslant r$ and sending $(u_\xi)_+$ to zero, where the subscript $+$ denotes the augmentation ideal. The isomorphisms of (1) are "dual" to each other, in the sense that $\mathscr{Z}$ is a commutative Hopf algebra, dual in a certain sense to the cocommutative Hopf algebra $U(\mathfrak{g})$. (This is related to the "double construction" of Drinfeld.)

We denote by the superscripts $+$, $-$, and 0 the subalgebras of the algebras in question generated by the E's, the F's, and the K's respectively. (Actually, U_ξ^0 is generated by K_i, $\begin{pmatrix} K_i; c \\ t \end{pmatrix}$, $i = 1, \ldots, r$, $c \in \mathbb{Z}$ and $t \in \mathbb{Z}^+$; see [L2, §6].) There are corresponding triangular decompositions (see [L1], [L2], [DK]), e.g.,

$$U_\xi = U_\xi^+ \otimes U_\xi^0 \otimes U_\xi^-, \qquad \mathscr{U}_\xi = \mathscr{U}_\xi^+ \otimes \mathscr{U}_\xi^0 \otimes \mathscr{U}_\xi^-, \quad \text{and} \quad u_\xi = u_\xi^+ \otimes u_\xi^0 \otimes u_\xi^-. \tag{2}$$

We use the notation B_ξ (resp. $\mathscr{B}_\xi$, b_ξ) for the "Borel part", the subalgebra of U_ξ generated by $\{E_i^{(n)}, K_i\}_{1 \leqslant i \leqslant r, n \geqslant 0}$ (resp. the subalgebra of $\mathscr{U}_\xi$, u_ξ generated by $\{E_i, K_i\}_{1 \leqslant i \leqslant r}$). The isomorphisms (1) have their Borel and "$+$, $-$ part" analogues.

An essential role in our arguments is played by a certain increasing filtration on $\mathscr{B}_\xi$, a quantum analogue of the standard filtration on an enveloping algebra. The filtration was introduced in [DK] (and is based on a result of Soibelman). We shall not reproduce its construction and shall limit ourselves to the following proposition.

PROPOSITION 1.1 [DK, Prop. 1.7]. *The algebra $\mathscr{B}_\xi$ has a multiplicative filtration such that the associated graded algebra* Gr $\mathscr{B}_\xi$ *is generated by the homogeneous elements $\{E_\alpha, K_i\}_{\alpha\in\Delta_+, 1\leqslant i\leqslant r}$, subject to the defining relations*

$$K_iK_j = K_jK_i, \qquad K_i^l = 1, \qquad K_iE_\alpha = \xi^{\langle\alpha,\alpha_i\rangle}E_\alpha K_i \tag{3}$$

and

$$E_\alpha E_\beta = \xi^{\langle\alpha,\beta\rangle}E_\beta E_\alpha \qquad \text{whenever } \alpha \succ \beta. \tag{4}$$

The filtration on $\mathscr{B}_\xi$ gives rise to a filtration on the subalgebra $\mathscr{U}_\xi^+$, on the quotient algebra $\mathscr{B}_\xi//\mathscr{Z}^+ \simeq b_\xi$ (given by the Borel analogue of (1)), and its subalgebra u_ξ^+.
We obtain the following corollary.

COROLLARY 1.2. (i) Gr$(\mathscr{U}_\xi^+)$ *is generated by $\{E_\alpha\}_{\alpha\in\Delta^+}$ subject only to relation* (4).
(ii) Gr b_ξ *is generated by $\{E_\alpha, K_i\}_{\alpha\in\Delta^+, 1\leqslant i\leqslant r}$, subject to both relations* (3) *and* (4) *of Proposition* 1.1, *and also the relations*

$$E_\alpha^l = 0 \qquad \text{for all } \alpha \in \Delta^+. \tag{5}$$

(iii) Gr(u_ξ^+) *is generated by $\{E_\alpha\}$, subject to relations* (4) *and* (5).

Remark 1.3. One can use Proposition 1.1 to prove the first of the isomorphisms (1) as follows. The natural map $\mathscr{U}_\xi \to u_\xi$ is clearly surjective and factors through $\mathscr{U}_\xi//\mathscr{Z}$. It suffices to show that the resulting map is injective. This amounts to showing that $\dim(\mathscr{U}_\xi//\mathscr{Z}) \leqslant \dim u_\xi$: We first restrict our attention to the positive parts. We have $\dim \mathscr{U}_\xi^+//\mathscr{Z}^+ = \dim \mathrm{Gr}(\mathscr{U}_\xi^+//\mathscr{Z}^+)$, and the latter dimension is $\leqslant l^{\dim \mathfrak{n}^+}$; for the algebra $\mathrm{Gr}(\mathscr{U}_\xi^+//\mathscr{Z}^+)$ clearly satisfies the relations (4) and (5). On the other hand, Lusztig showed that $\dim u_\xi^+ = l^{\dim \mathfrak{n}^+}$. A similar argument gives the inequality for the "0" parts. Now the decompositions (2) give the inequality $\dim(\mathscr{U}_\xi//\mathscr{Z}) \leqslant \dim u_\xi$. ■

2. Cohomology of the algebra b_ξ.

2.1. *Cohomology of* Gr $\mathscr{U}_\xi^+$. Let Λ_ξ be the graded algebra with generators $\{\varepsilon_\alpha\}_{\alpha\in\Delta^+}$ (where we assign grade degree 1 to the generators ε_α) and the defining relations

$$\varepsilon_\alpha\cdot\varepsilon_\beta + \xi^{-\langle\alpha,\beta\rangle}\varepsilon_\beta\cdot\varepsilon_\alpha = 0 \qquad \text{whenever } \beta \prec \alpha$$

and

$$\varepsilon_\alpha^2 = 0 \qquad \text{for any } \alpha \in \Delta^+.$$

Thus, Λ_ξ is a q-analogue of the exterior algebra. We have the following proposition. (See Sections 1 and 5.1 for the notation.)

PROPOSITION. *There is a natural graded algebra isomorphism* $H^*(\mathrm{Gr}\,\mathscr{U}_\xi^+) \simeq \Lambda_\xi$.

Proof. View $\mathrm{Gr}\,\mathscr{U}_\xi^+$ as the graded algebra with a slightly different grading, where we assign degree 1 to all the generators $\{E_\alpha\}_{\alpha\in\Delta^+}$. The relations among E_α's are given by Corollary 1.2. A criterion [P, §5] (or [BGS, §2.10.3]) then shows that $\mathrm{Gr}\,\mathscr{U}_\xi^+$ is a Koszul algebra in the sense of [BGS, Def. 1.2.4]. The result now follows from [P, Theorem 2.5] (see also [BGS, Prop. 1.2.9]). ∎

2.2. *Cohomology of* $\mathscr{Z}^+$. Let $\mathfrak{N}$ be the k-vector space with basis $\{E_\alpha^l\}_{\alpha\in\Delta^+}$. Then the algebra $\mathscr{Z}^+$ (defined in §1) is canonically isomorphic to the symmetric algebra $S(\mathfrak{N})$. Hence $H^*(\mathscr{Z}^+) \simeq \Lambda(\mathfrak{N}^*)$, where $\Lambda(\mathfrak{N}^*)$ denotes the exterior algebra of the dual space.

Further $\mathscr{Z}^+$, being central, is a normal subalgebra of $\mathscr{U}_\xi^+$. Similarly, $\mathrm{Gr}\,\mathscr{Z}^+$ is a normal subalgebra of $\mathrm{Gr}\,\mathscr{U}_\xi^+$, where we take the induced filtration on $\mathscr{Z}^+$ from the filtration of $\mathscr{U}_\xi^+$. By §5.2, the "+" analogue of (1) of §1, and [DK, Corollary 3.3], there is a canonical u_ξ^+-action on $H^*(\mathscr{Z}^+)$ (resp. $\mathrm{Gr}\,u_\xi^+$-action on $\mathrm{Gr}\,\mathscr{Z}^+$). It is easy to see that $\mathrm{Gr}\,\mathscr{Z}^+$ is canonically isomorphic (as an algebra) to $\mathscr{Z}^+$.

LEMMA. *The* u_ξ^+*-action on* $H^*(\mathscr{Z}^+)$ *and the* $\mathrm{Gr}\,u_\xi^+$*-action on* $H^*(\mathrm{Gr}\,\mathscr{Z}^+)$ *are both trivial.*

Proof. The cohomology of $\mathscr{Z}^+$ may be computed via the standard Koszul complex:

$$\cdots \to \mathscr{Z}^+ \otimes_k \Lambda^1(\mathfrak{N}) \to \mathscr{Z}^+ \otimes_k \Lambda^0(\mathfrak{N}) \twoheadrightarrow k,$$

which is a $\mathscr{Z}^+$-free resolution of the $\mathscr{Z}^+$-module k. Since $\mathscr{U}_\xi^+$ is free as a $\mathscr{Z}^+$-module under multiplication, tensoring with $\mathscr{U}_\xi^+$ over $\mathscr{Z}^+$ yields an $\mathscr{U}_\xi^+$-free resolution of u_ξ^+ (using the "+" analogue of (1) of §1):

$$\cdots \to \mathscr{U}_\xi^+ \otimes_k \Lambda^1(\mathfrak{N}) \to \mathscr{U}_\xi^+ \otimes_k \Lambda^0(\mathfrak{N}) \twoheadrightarrow u_\xi^+ .$$

The u_ξ^+-action on $H^*(\mathscr{Z}^+)$ comes, by §5.2, from the $\mathscr{U}_\xi^+$-action on the following complex, where $\mathscr{U}_\xi^+$-action on $\mathrm{Hom}_{\mathscr{U}_\xi^+}(\mathscr{U}_\xi^+ \otimes \Lambda^i(\mathfrak{N}), k)$ is via the right multiplication on the $\mathscr{U}_\xi^+$ factor and the trivial action on $\Lambda^i(\mathfrak{N})$ and k.

$$\cdots \leftarrow \mathrm{Hom}_{\mathscr{U}_\xi^+}(\mathscr{U}_\xi^+ \otimes \Lambda^1(\mathfrak{N}), k) \leftarrow \mathrm{Hom}_{\mathscr{U}_\xi^+}(\mathscr{U}_\xi^+ \otimes \Lambda^0(\mathfrak{N}), k) \leftarrow 0.$$

The latter action clearly is the trivial action on the complex itself, proving the first claim of the lemma. The second claim is proved in the same way. ∎

The vector space $\mathfrak{N}$ can be identified naturally with the positive part $\mathfrak{n}^+$ of the Lie algebra $\mathfrak{g}$, under the map $E_\alpha^l \mapsto e_\alpha$, where e_α is a nonzero root vector in $\mathfrak{n}^+$ of root α. Thus the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ acts on $\mathfrak{N}$ (under the above identification) via the adjoint action, inducing an action on $S(\mathfrak{N}) \simeq \mathscr{Z}^+$ by derivations. This gives a $\mathfrak{b}$-action on $H^*(\mathscr{Z}^+) \simeq \Lambda(\mathfrak{N}^*)$. It coincides with the one induced by the coadjoint action on $\mathfrak{N}^*$.

2.3. *Cohomology of* Gr u_ξ^+. Corollary 5.3, combined with Lemma 2.2, yields a transgression map $\tau\colon \mathfrak{N}^* \simeq H^1(\mathrm{Gr}\,\mathscr{Z}^+) \to H^2(\mathrm{Gr}\,u_\xi^+)$. Since the image of τ is contained in the center of $H^*(\mathrm{Gr}\,u_\xi^+)$ (see Corollary 5.3), τ extends to an algebra homomorphism $S(\mathfrak{N}^*) \to H^{\mathrm{even}}(\mathrm{Gr}\,u_\xi^+)$. It can be easily seen that u_ξ^0 acts trivially on the image of τ (under the canonical action of u_ξ^0 on $H^*(\mathrm{Gr}\,u_\xi^+)$, induced from the isomorphism $\mathrm{Gr}\,b_\xi//\mathrm{Gr}\,u_\xi^+ \simeq u_\xi^0$ via Lemma 5.2.1).

PROPOSITION 2.3.1. *There exists a natural graded algebra isomorphism*

$$H^*(\mathrm{Gr}\,u_\xi^+) \simeq \Lambda_\xi \otimes S(\mathfrak{N}^*),$$

where Λ_ξ is as in §2.1 and the right-hand side is endowed with the tensor product graded algebra structure. (The elements of $\mathfrak{N}^*$ are assigned grade degree two.)

Remark. The right-hand side is also endowed with an u_ξ^0-action; u_ξ^0 acts trivially on $S(\mathfrak{N}^*)$ and acts on a generator $\varepsilon_\alpha \in \Lambda_\xi$ via the restricted weight $-\alpha$ (see e.g. [K, §4.4] for the definition of restricted weight). Then the isomorphism of the proposition commutes with the u_ξ^0-module structures.

Enumerate all the positive roots: $\beta_1, \beta_2, \ldots, \beta_N$ (where N is the total number of positive roots) according to the total linear order $\prec$ on Δ^+ (see §1). For each $j = 0, 1, 2, \ldots, N$, let $\mathfrak{N}_j$ be the subspace of $\mathfrak{N}$ spanned by the elements $E_{\beta_1}^l, \ldots, E_{\beta_j}^l$, and let $\mathscr{Z}_j \simeq S(\mathfrak{N}_j)$ be the subalgebra of $\mathscr{Z}^+$ generated by $\mathfrak{N}_j$. (We declare $\mathfrak{N}_0 = 0$.) Further, set $\mathrm{u}_j := \mathrm{Gr}(\mathscr{U}_\xi^+)//\mathscr{Z}_j$ where $\mathscr{Z}_j$ is identified as a subalgebra of $\mathrm{Gr}(\mathscr{Z}^+) \subset \mathrm{Gr}(\mathscr{U}_\xi^+)$ under the canonical identification $\mathscr{Z}^+ \simeq \mathrm{Gr}(\mathscr{Z}^+)$ (see §2.2).

Since $\mathscr{Z}_j$ (being central) is a normal subalgebra of $\mathrm{Gr}(\mathscr{U}_\xi^+)$, by Lemma 5.2.1, there is a canonical u_j-action on $H^*(\mathscr{Z}_j)$. One shows, repeating the proof of Lemma 2.2, that this action is actually trivial.

Proposition 2.3.1 is a special case of the following result.

PROPOSITION 2.3.2. *For each $j = 0, 1, 2, \ldots, N$, there is a natural graded algebra isomorphism*

$$H^*(\mathrm{u}_j) \simeq \Lambda_\xi \otimes_k S(\mathfrak{N}_j^*). \tag{1}$$

Further, this isomorphism commutes with the canonical u_ξ^0-actions.

2.4. *Proof of Proposition* 2.3.2. We proceed by induction on j. For $j = 0$, the statement reduces to Proposition 2.1. Now assume the validity of the proposition (by induction) for all $i \leqslant j$ and prove it for $j + 1$:

Let $A = A_{j+1}$ be the subalgebra of u_j generated by $E_{\beta_{j+1}}^l$. Then A is a polynomial algebra in one variable, which is normal (in fact central) in u_j, and we have, by Corollary 1.2(i), $\mathrm{u}_j//A \simeq \mathrm{u}_{j+1}$. Hence, we get a convergent spectral sequence (see §5.3) with

$$E_2^{p,q} = H^p(\mathrm{u}_{j+1}, H^q(A)) \Rightarrow H^{p+q}(\mathrm{u}_j). \tag{1}$$

Step 1. *The canonical algebra homomorphism $r^p\colon H^p(\mathrm{u}_{j+1}) \to H^p(\mathrm{u}_j)$ is surjective for any $p \geqslant 0$.*

Proof. By the induction hypothesis $H^*(\mathrm{u}_j) \simeq \Lambda_\xi \otimes S(\mathfrak{N}_j^*)$, and hence $H^*(\mathrm{u}_j)$ is an algebra generated by elements of degree $\leqslant 2$. So it suffices to prove the claim of Step 1 for $p = 1, 2$ only.

For any augmented algebra B, we have $H^1(B) \simeq (B_+/B_+ \cdot B_+)^*$, where B_+ is the augmentation of the ideal. From this we can easily see that the map r^1 is in fact an isomorphism.

Let R_j denote the space of relations for the algebra u_j (see §5.4). To prove the surjectivity of r^2, it suffices to show by §5.4 that the canonical map $R_j \to R_{j+1}$ is injective. But this is obvious from Corollary 1.2. ■

Step 2. *In the spectral sequence* (1), *we have $E_\infty^{p,q} = 0$, for all $q > 0$.*

Proof. There is a natural commutative diagram

$$\begin{tikzcd} E_2^{p-2,1} \arrow[r, "d_2"] & E_2^{p,0} \arrow[r, two heads] & E_\infty^{p,0} \arrow[r, hook] & H^p(\mathrm{u}_j) \\ & H^p(\mathrm{u}_{j+1}) \arrow[u, equal] \arrow[urr, "r^p"'] & & \end{tikzcd}$$

The surjectivity of r^p (guaranteed by Step 1) and the commutativity of the above diagram show that the inclusion $E_\infty^{p,0} \hookrightarrow H^p(\mathrm{u}_j)$ is an isomorphism. Hence, $E_\infty^{p,q} = 0$ for all $q > 0$; for the spectral sequence converges to $H^*(\mathrm{u}_j)$. ■

One shows, repeating the proof of Lemma 2.2, that the canonical u_{j+1}-action on $H^*(A)$ is trivial. Hence, there is a canonical transgression map $H^1(A) \to H^2(\mathrm{u}_{j+1})$ (see Corollary 5.3). Observe that $\dim H^1(A) = 1$. Choose a vector $v \in H^2(\mathrm{u}_{j+1})$ as the image of a nonzero element in $H^1(A)$. It belongs to the center of $H^*(\mathrm{u}_{j+1})$ by Corollary 5.3.

Step 3. *The kernel of the homomorphism $r: H^*(\mathrm{u}_{j+1}) \to H^*(\mathrm{u}_j)$ is equal to the* (*two-sided*) *ideal generated by v.*

Proof. First, observe that $E_2^{p,0}/d_2(E_2^{p-2,1}) \simeq E_3^{p,0} \simeq E_\infty^{p,0}$. (The last isomorphism follows since $H^q(A) = 0$, for all $q > 1$.) But $E_\infty^{p,0} \simeq H^p(\mathrm{u}_j)$, by the proof of Step 2. Now use the fact that d_2 is a derivation (see the proof of Corollary 5.3). This completes the proof of Step 3. ■

Step 4. *The algebra homomorphism $r: H^*(\mathrm{u}_{j+1}) \to H^*(\mathrm{u}_j)$ admits a graded algebra splitting that commutes with the canonical action of u_ξ^0.*

Proof. We have, by induction, $H^*(\mathrm{u}_j) \simeq \Lambda_\xi \otimes S(\mathfrak{N}_j^*)$. We have only to lift the generators of the algebras Λ_ξ and $S(\mathfrak{N}_j^*)$ to $H^*(\mathrm{u}_{j+1})$ and to check that the relations among them are preserved. To lift the generators of $S(\mathfrak{N}_j^*)$, observe that there is a commutative diagram (see §2.3)

$$\begin{tikzcd} \mathfrak{N}_{j+1}^* \arrow[r] \arrow[d] & H^2(\mathrm{u}_{j+1}) \arrow[d, "r"] \\ \mathfrak{N}_j^* \arrow[r] & H^2(\mathrm{u}_j) \end{tikzcd}$$

where the horizontal arrows are the appropriate transgression maps and the projection $\mathfrak{N}_{j+1}^* \twoheadrightarrow \mathfrak{N}_j^*$ is induced by the canonical imbedding $\mathfrak{N}_j \hookrightarrow \mathfrak{N}_{j+1}$. Hence, any splitting of the projection $\mathfrak{N}_{j+1}^* \twoheadrightarrow \mathfrak{N}_j^*$ provides a lifting of the generators of $S(\mathfrak{N}_j^*)$ to $H^2(\mathrm{u}_{j+1})$. Moreover, Corollary 5.3 shows that the image of the transgression map is central in $H^*(\mathrm{u}_{j+1})$; in particular, the lifted generators are central in $H^*(\mathrm{u}_{j+1})$.

Next, view the generators $\{\varepsilon_\alpha\}_{\alpha \in \Delta^+}$ of Λ_ξ as elements of $H^1(\mathrm{u}_j)$ (by the induction hypothesis). We have already observed in the proof of Step 1 that the map $r^1\colon H^1(\mathrm{u}_{j+1}) \to H^1(\mathrm{u}_j)$ is an (u_ξ^0-equivariant) isomorphism. We use the inverse isomorphism to transfer the elements $\varepsilon_\alpha \in H^1(\mathrm{u}_j)$ to the elements $\tilde{\varepsilon}_\alpha \in H^1(\mathrm{u}_{j+1})$. Proving the claim of Step 4 amounts to showing that the elements $\tilde{\varepsilon}_\alpha$ satisfy the relations of Proposition 2.1.

Let us first prove that $\tilde{\varepsilon}_\alpha \cdot \tilde{\varepsilon}_\beta + \xi^{-\langle\alpha,\beta\rangle}\tilde{\varepsilon}_\beta \cdot \tilde{\varepsilon}_\alpha = 0$ whenever $\alpha \succ \beta$: We know that

$$r(\tilde{\varepsilon}_\alpha \cdot \tilde{\varepsilon}_\beta + \xi^{-\langle\alpha,\beta\rangle}\tilde{\varepsilon}_\beta \cdot \tilde{\varepsilon}_\alpha) = \varepsilon_\alpha \cdot \varepsilon_\beta + \xi^{-\langle\alpha,\beta\rangle}\varepsilon_\beta \cdot \varepsilon_\alpha = 0.$$

So by the claim of Step 3, we get

$$\tilde{\varepsilon}_\alpha \cdot \tilde{\varepsilon}_\beta + \xi^{-\langle\alpha,\beta\rangle}\tilde{\varepsilon}_\beta \cdot \tilde{\varepsilon}_\alpha = c \cdot v \qquad \text{for some } c \in k. \tag{2}$$

To show that $c = 0$, observe that the left-hand side of (2) has restricted weight $-(\alpha + \beta)$ (with respect to the u_ξ^0-action) whereas the right-hand side has restricted weight 0. But for any positive roots $\alpha \neq \beta$, either $-1 \leqslant \langle\alpha, \beta^\vee\rangle \leqslant 1$ or $-1 \leqslant \langle\beta, \alpha^\vee\rangle \leqslant 1$ (see [Bo, p. 278]). In the first case, we have $1 \leqslant \langle\alpha + \beta, \beta^\vee\rangle \leqslant 3$, hence $\alpha + \beta \notin l \cdot \mathfrak{h}_{\mathbb{Z}}^*$, for $l > 3$. (By our assumption $l > h$; and further $h \geqslant 3$ for every simple Lie algebra $\mathfrak{g}$, except when $\mathfrak{g} = sl(2)$. Hence for any $\mathfrak{g}$ other than $sl(2)$, we have $l > 3$. For $\mathfrak{g} = sl(2)$, since there is only one positive root, the relation is vacuously satisfied.) This forces the constant c in (2) to vanish.

To prove the other relation, i.e., $\tilde{\varepsilon}_\alpha^2 = 0$, we similarly need to observe that (since l is odd) $2\alpha \notin l \cdot \mathfrak{h}_{\mathbb{Z}}^*$, for any positive root α. ■

Step 5. The element v introduced in Step 3 is not a zero-divisor in $H^*(\mathrm{u}_{j+1})$.

Proof. By Step 2, we have $E_3^{p,1} \cong E_\infty^{p,1} = 0$; hence the differential d_2 restricted to $E_2^{p,1}$, in the spectral sequence (1), is injective. But for $x \otimes y \in H^p(\mathrm{u}_{j+1}) \otimes H^1(A) = E_2^{p,1}$ we have $d_2(x \otimes y) = (-1)^p x \cdot d_2 y = (-1)^p x \cdot v$ (see the proof of Corollary 5.3). ■

The results of Steps 3–5 complete the proof of Proposition 2.3.2 and hence, in particular, that of Proposition 2.3.1. ■

2.5. *Cohomology of b_ξ.* For any $\lambda \in \mathfrak{h}_l^*$ (see §1), let $k(\lambda)$ denote the 1-dimensional u_ξ^0-module corresponding to the restricted weight λ. We denote by the same symbol $k(\lambda)$ the 1-dimensional representation of b_ξ (and Gr b_ξ) obtained via the canonical projection to u_ξ^0.

Following is the main result of this section, where as usual ρ is the half sum of positive roots, to be considered canonically as an element of $\mathfrak{h}_l^*$.

THEOREM. (i) $H^{\mathrm{odd}}(b_\xi) = 0$, *and there is a natural graded algebra isomorphism*

$$H^{2\cdot}(b_\xi) \simeq S^{\cdot}(\mathfrak{N}^*).$$

(ii) *For any* $w \in W$, $H^*(b_\xi, k(\rho - w\rho))$ *is a free module over* $H^*(b_\xi)$ *of rank* 1 *generated by an element of degree* $l(w)$, *the length of* w.

(iii) $H^*(b_\xi; k(\lambda)) = 0$, *if* $\lambda \in \mathfrak{h}_l^*$ *is not of the form* $\rho - w\rho$, *for any* $w \in W$.

COROLLARY. $H^p(b_\xi, k(\rho - w\rho)) = 0$, *unless* $p - l(w) \in 2 \cdot \mathbb{Z}_+$.

Proof of the theorem. The subalgebra u_ξ^+ is normal in b_ξ, and moreover $b_\xi//u_\xi^+ \simeq u_\xi^0$. Hence, for any b_ξ-module M, we have a natural isomorphism $H^0(b_\xi, M) \simeq H^0(u_\xi^+, M)^{u_\xi^0}$, where the superscript u_ξ^0 stands for u_ξ^0-invariants. Furthermore, the assignment $N \mapsto N^{u_\xi^0}$ is an exact functor on the category of u_ξ^0-modules, for u_ξ^0 is a finite-dimensional semisimple algebra. Hence, the natural isomorphism as above can be extended to the corresponding derived functors, giving a natural isomorphism

$$H^i(b_\xi, M) \simeq H^i(u_\xi^+, M)^{u_\xi^0} \qquad \text{for any } i \geqslant 0.$$

In particular, for any $\lambda \in \mathfrak{h}_l^*$ we get

$$H^i(b_\xi, k(\lambda)) \simeq [H^i(u_\xi^+) \otimes k(\lambda)]^{u_\xi^0}. \tag{1}$$

Using now the filtration of u_ξ^+ (see §1) and Proposition 5.5, we get a spectral sequence with

$$E_1^{p,q} = H_{(p)}^{p+q}(\mathrm{Gr}\, u_\xi^+) \Rightarrow H^{p+q}(u_\xi^+).$$

This spectral sequence is compatible with the u_ξ^0-actions. Hence, the isomorphism (1), combined with Proposition 2.3.1 and the remark following it, yields a spectral sequence with E_1-term

$$[H^*(\mathrm{Gr}\, u_\xi^+) \otimes k(\lambda)]^{u_\xi^0} \simeq [\Lambda_\xi \otimes k(\lambda)]^{u_\xi^0} \otimes S(\mathfrak{N}^*) \Rightarrow H^*(b_\xi, k(\lambda)). \tag{2}$$

Observe next that any restricted weight, occuring in the pth graded component Λ_ξ^p of Λ_ξ, is of the form $-(\beta_1 + \cdots + \beta_p)$, for some *distinct* positive roots $\beta_1, \ldots, \beta_p$. We claim that for $\lambda = \rho - w\rho$, one has

$$\dim[\Lambda_\xi^p \otimes k(\lambda)]^{u_\xi^0} = \begin{cases} 1 & \text{if } p = l(w) \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

The claim amounts to the following statement: Let $\{\gamma_1, \ldots, \gamma_p\}$ and $\{\gamma'_1, \ldots, \gamma'_q\}$ be two sets of distinct positive roots (γ_i is allowed to be equal to γ'_j). If $\gamma_1 + \cdots + \gamma_p = \gamma'_1 + \cdots + \gamma'_q \bmod l \cdot \mathfrak{h}_{\mathbb{Z}}^*$, then the equality actually holds in $\mathfrak{h}_{\mathbb{Z}}^*$. (Here we are using our assumption $l > h$ and l is odd.) The proof of this statement is a slight variation of the argument contained in [J, Part II, Lemma 12.10]. (Since l is not assumed to be a prime, the argument loc. cit. can not be taken verbatim.) We briefly outline the argument: As in [J], we may assume that $\gamma := \gamma_1 + \cdots + \gamma_p - (\gamma'_1 + \cdots + \gamma'_q)$ is of the form $\gamma = l\beta$, for some (fundamental) miniscule weight β, and moreover $\gamma \leqslant 2\rho$ (as elements of $\mathfrak{h}^*_{\mathbb{Z}}$) (and of course γ is in the root lattice). From the list of miniscule weights (see e.g. [Bo, Exercise 15, §4, Chap. 6]) and the description of the fundamental weights as in [Bo, Planche 1–9], we see that $\gamma = l\beta$ is not in the root lattice for any odd l and any miniscule β in the Lie algebras of type B, C, D, and E_7. For the miniscule weights β in the Lie algebras of type A and E_6, one can easily see that the condition $\gamma = l\beta \leqslant 2\rho$ is violated, for any $l > h$. The remaining simple Lie algebras (of type E_8, G_2, and F_4) have no miniscule weights at all.

It follows from (3) that (for $\lambda = \rho - w\rho$) all the nonvanishing E_1-terms in the spectral sequence (2) are of the same parity. Hence, the spectral sequence collapses at the E_1-term itself, giving rise to a graded space isomorphism

$$E_\infty \simeq \mathrm{Gr}(H^*(b_\xi, k(\rho - w\rho))) \simeq E_1 \simeq S^{(*-l(w))/2}(\mathfrak{N}^*). \tag{4}$$

Further, there is (as at the beginning of §2.3) a natural transgression map $\tau\colon \mathfrak{N}^* \simeq H^1(\mathscr{Z}^+) \to H^2(u_\xi^+)$. It is easy to see that u_ξ^0 acts trivially on $H^*(\mathscr{Z}^+)$. In particular, τ being a u_ξ^0-module map, Image $\tau \subset [H^2(u_\xi^+)]^{u_\xi^0} \simeq H^2(b_\xi)$. Furthermore, the subalgebra A generated by the image of $\mathfrak{N}^*$ in $H^*(u_\xi^+)$ is commutative (see Lemma 2.2 and Corollary 5.3) and hence is a quotient of $S(\mathfrak{N}^*)$. But (taking $w = e$) the corresponding graded subalgebra Gr $A \subset E_\infty \simeq E_1$ can be seen to be isomorphic, via (4), with the right-hand side of (4), and hence Gr A coincides with E_∞. This proves parts (i) and (ii) of the theorem.

We come to part (iii):

$$\begin{aligned} H^p(b_\xi, k(\lambda)) &= \mathrm{Ext}^p_{b_\xi}(k, k(\lambda)) \\ &\simeq \mathrm{Ext}^p_{b_\xi}(k(-\lambda), k) && \text{(see [}B\text{, Propn. 3.1.8])} \\ &\simeq \mathrm{Ext}^p_{u_\xi}(u_\xi \otimes_{b_\xi} k(-\lambda), k) && \text{(see the proof of Lemma 5.2.1).} \end{aligned}$$

Now part (iii) follows from the linkage principle [APW2, §2.9], for $-\lambda$ is not linked to 0 unless $\lambda = \rho - w\rho$. ∎

Remark. For the Lie algebra of type A_r, considering the positive roots $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_r\}$, we see that their sum is equal to $(r + 1)\omega_1$, where ω_1 is the first fundamental weight. Since $h = r + 1$ (for A_r), we see that the restriction $l > h$ is necessary in the argument given above to prove (3).

2.6. *An adjoint action on $H^*(b_\xi)$.* The algebra b_ξ is a normal subalgebra of the algebra B_ξ, and we have (see §1) $B_\xi // b_\xi \simeq U(\mathfrak{b})$, the classical enveloping algebra of the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. This gives, by §5.2 and [L2, Lemma 8.8], a canonical $U(\mathfrak{b})$-action on $H^*(b_\xi)$. On the other hand, there is a canonical $U(\mathfrak{b})$-action on the algebra $S(\mathfrak{n}^*)$ induced by the adjoint $\mathfrak{b}$-action on $\mathfrak{n} := \mathfrak{n}^+$.

LEMMA. *The isomorphism $H^{2\cdot}(b_\xi) \simeq S^\cdot(\mathfrak{n}^*)$ of Theorem 2.5 (identifying $\mathfrak{N}$ with $\mathfrak{n}$; see §2.2) commutes with the $U(\mathfrak{b})$-actions.*

Proof. Fix the divided power $E_i^{(l)} \in U_{\mathbb{Z}}$. Let D_i be the derivation of the algebra $B_{\mathbb{Z}}$, the Borel part of $U_{\mathbb{Z}}$, given by $D_i\colon u \mapsto E_i^{(l)} \cdot u - u \cdot E_i^{(l)}$. It is a remarkable fact [DK, §3.4] that the derivation D_i descends to a derivation of the algebra $\mathscr{B}_\xi$ (no divided powers!) to be denoted by D_i again. Moreover, it is shown in [DK] that $D_i(\mathscr{U}_\xi^+) \subset \mathscr{U}_\xi^+$ and $D_i(\mathscr{Z}^+) \subset \mathscr{Z}^+$, so that D_i induces a derivation on $u_\xi^+ \simeq \mathscr{U}_\xi^+//\mathscr{Z}^+$. A derivation on the algebra gives rise to a derivation on its cohomology. Thus, for any $p \geqslant 0$, there is a natural D_i-action on $H^p(\mathscr{Z}^+)$, $H^p(u_\xi^+)$, and on $H^p(b_\xi)$. Now, recall the isomorphism $\mathscr{Z}^+ \simeq S(\mathfrak{N})$ from §2.2. It is shown in [DK, §3] that the D_i-action on $\mathfrak{N} \subset \mathscr{Z}^+$ (modulo $\bigoplus_{p>1} S^p(\mathfrak{N})$) corresponds to the adjoint action of the element $e_i \in \mathfrak{n}$ on $\mathfrak{N} \simeq \mathfrak{n}$. Hence, the canonical isomorphism $H^1(\mathscr{Z}^+) \simeq \mathfrak{n}^*$ is compatible with the D_i-action on the left-hand side and the coadjoint action of e_i on the right-hand side. Furthermore, the transgression map $H^1(\mathscr{Z}^+) \to H^2(u_\xi^+)$ commutes with the D_i-action. The last part of the proof of Theorem 2.5 shows now that the D_i-action on $H^*(b_\xi)$ corresponds to the coadjoint action of e_i on $S(\mathfrak{n}^*)$ via the isomorphism of the theorem.

Finally, the derivation D_i on $B_{\mathbb{Z}}$ induces (an inner) derivation on B_ξ such that $D_i(b_\xi) \subset b_\xi$. It follows from Lemma 5.2.2 that the canonical action on $H^*(b_\xi)$ of the image of $E_i^{(l)}$ in $B_\xi//b_\xi$ coincides with the one induced by D_i. This shows that the isomorphism $H^*(b_\xi) \simeq S(\mathfrak{n}^*)$ of Theorem 2.5 commutes with the action of e_i. Hence it commutes with the $U(\mathfrak{n})$-action. Compatibility of the isomorphism with the $\mathfrak{h}$-action is easy and is left to the reader. ■

3. Cohomology of u_ξ. We compute here the cohomology of the algebra u_ξ, along the lines of [J, Part II, Chapter 12]. The computation relies heavily on some results of Andersen-Polo-Wen.

Recall first that the algebra u_ξ is normal in U_ξ, and we have (see item (1) of §1) $U_\xi//u_\xi \simeq U(\mathfrak{g})$. Hence, there is a natural $U(\mathfrak{g})$-module structure on $H^*(u_\xi)$ (see §5.2).

Here is the main result of the paper.

THEOREM. (i) *$H^{\mathrm{odd}}(u_\xi) = 0$, and there is a natural graded algebra isomorphism*

$$H^{2\cdot}(u_\xi) \simeq k^\cdot[\mathscr{N}].$$

where $N \subset \mathfrak{g}$ is the nilpotent cone.

(ii) *The natural $\mathfrak{g}$-action on the left-hand side of the above isomorphism corresponds to the standard coadjoint action of $\mathfrak{g}$ on $k[\mathscr{N}]$.*

We first recall the construction of the induction functor [APW].

For any (left) U_ξ-module M, set

$$M^f = \{x \in M \mid F_i^{(p)}x = E_i^{(p)}x = 0, \text{for all } 1 \leqslant i \leqslant r \text{ and all } p \geqslant p(x) \gg 0\}.$$

By the commutation relations [L2, §§5, 6], M^f is an U_ξ-submodule of M. We similarly define M^f for any $U(\mathfrak{g})$-module M. Let $\mathscr{C}_{U_\xi}$ (resp. $\mathscr{C}_{U(\mathfrak{g})}$) be the category of all the U_ξ-modules (resp. $U(\mathfrak{g})$-modules) M such that $M = M^f$. Similarly, for a B_ξ-module N, set

$$N^f = \{x \in N \mid E_i^{(p)}x = 0, \text{ for all } 1 \leqslant i \leqslant r \text{ and all } p \geqslant p(x) \gg 0\}.$$

(An analogous definition is given for N^f, for any $U(\mathfrak{b})$-module N.) Let $\mathscr{C}_{B_\xi}$ (resp. $\mathscr{C}_{U(\mathfrak{b})}$) be the category of all the B_ξ-modules (resp. $U(\mathfrak{b})$-modules) N such that $N = N^f$.

Define the induction functor $\operatorname{Ind} = \operatorname{Ind}_{B_\xi}^{U_\xi}: \mathscr{C}_{B_\xi} \to \mathscr{C}_{U_\xi}$ by $\operatorname{Ind} M = [\operatorname{Hom}_{B_\xi}(U_\xi, M)]^f$, where $\operatorname{Hom}_{B_\xi}(U_\xi, M)$ is made into a U_ξ-module under

$$(xf)(y) = f(yx) \qquad \text{for } f \in \operatorname{Hom}_{B_\xi}(U_\xi, M), \qquad x, y \in U_\xi.$$

Define the functor $\overline{\operatorname{Ind}} = \operatorname{Ind}_{U(\mathfrak{b})}^{U(\mathfrak{g})}: \mathscr{C}_{U(\mathfrak{b})} \to \mathscr{C}_{U(\mathfrak{g})}$ in a similar way.

Proof of the theorem. Given an U_ξ-module M, set $I_{u_\xi}(M) := H^0(u_\xi, M) = M^{u_\xi}$, as the subspace of u_ξ-invariants. There is a natural $U(\mathfrak{g})$-module structure on $I_{u_\xi}(M)$ arising as in §5.2. It is easy to see that the assignment $M \mapsto I_{u_\xi}(M)$ yields a functor $I_{u_\xi}: \mathscr{C}_{U_\xi} \to \mathscr{C}_{U(\mathfrak{g})}$. Similarly, one defines a functor $I_{b_\xi}: \mathscr{C}_{B_\xi} \to \mathscr{C}_{U(\mathfrak{b})}$. It is easy to check that the diagram

(1)
$$\begin{tikzcd} & \mathscr{C}_{U_\xi} \arrow[dr, "I_{u_\xi}"] & \\ \mathscr{C}_{B_\xi} \arrow[ur, "\operatorname{Ind}"] \arrow[dr, "I_{b_\xi}"'] & & \mathscr{C}_{U(\mathfrak{g})} \\ & \mathscr{C}_{U(\mathfrak{b})} \arrow[ur, "\overline{\operatorname{Ind}}"'] & \end{tikzcd}$$

commutes. Observe next that all the functors in the above diagram are left-exact and all the categories involved are abelian categories with enough injectives (see [APW2, §0.8]). Furthermore, it is shown in [APW2, §0.8] that the functor Ind takes injectives into injectives and also, by an argument similar to [J, Part I, proof of Lemma 6.4], I_{b_ξ} takes injectives into injectives. Thus there are two spectral sequences (given by Grothendieck, for the composition of two functors; see e.g. [J, Part I, §4.1]) converging to the same limit, with E_2-terms

$$'E_2^{p,q} = (R^p I_{u_\xi})(R^q \operatorname{Ind}(k))$$

and

$$''E_2^{p,q} = (R^p\, \overline{\operatorname{Ind}})(R^q I_{b_\xi}(k)),$$

where k is treated as the trivial B_ξ-module.

Observe that $R^p I_{u_\xi}(\cdot) = H^p(u_\xi, \cdot)$. Further, $R^q \operatorname{Ind}(k) = 0$, for all $q > 0$ by [AW, Theorem 5.3] and $R^0 \operatorname{Ind}(k) = k$. Hence, the first spectral sequence collapses at the E_2-term, and we have $'E_\infty \simeq H^*(u_\xi, k)$. In the second spectral sequence, we have $R^\cdot I_{b_\xi}(k) = H^\cdot(b_\xi, k) \simeq S(\mathfrak{n}^*)$ by Theorem 2.5. Furthermore, the canonical $U(\mathfrak{b})$-module structure on $R^\cdot I_{b_\xi}(k)$ corresponds to the coadjoint action on $S(\mathfrak{n}^*)$ by Lemma 2.6. Hence $''E_2^{p,q} = 0$ for all $p > 0$ by [J, Part II, Lemma 12.12(a)]. Thus, the second spectral sequence also degenerates at the E_2-term, and we have (see [J, Part II, §12.14])

$$''E_\infty \simeq \overline{\operatorname{Ind}}(S(\mathfrak{n}^*)) \simeq H^0(G(k)/B(k), S(\mathfrak{n}^*)) \simeq k[\mathcal{N}], \tag{2}$$

where $G(k)$ stands for the split (semisimple) simply-connected Lie group over k with Lie algebra $\mathfrak{g}$ and $B(k)$ stands for the Borel subgroup of $G(k)$ corresponding to the subalgebra $\mathfrak{b} \subset \mathfrak{g}$.

Comparing the two spectral sequences yields a graded space isomorphism

$$H^*(u_\xi) \simeq {}'E_\infty \simeq {}''E_\infty \simeq k[\mathcal{N}]. \tag{3}$$

Next, we construct a natural $\mathfrak{g}$-equivariant algebra homomorphism $\overline{\varphi}$: $H^*(u_\xi) \to k[\mathcal{N}]$ as follows. The canonical restriction map $H^*(u_\xi) \to H^*(b_\xi)$ is a $U(\mathfrak{b})$-equivariant algebra homomorphism and hence induces a $U(\mathfrak{g})$-equivariant algebra homomorphism φ: $\overline{\operatorname{ind}}(H^*(u_\xi)) \to \overline{\operatorname{Ind}}(H^*(b_\xi))$. As we saw in the preceding discussion (see (2)), the target algebra is canonically isomorphic to $k[\mathcal{N}]$. On the other hand, $H^*(u_\xi)$ being a $U(\mathfrak{g})$-module, we have $\overline{\operatorname{Ind}}(H^*(u_\xi)) \simeq H^*(u_\xi)$. Now one checks easily that the resulting map $\overline{\varphi}$: $H^*(u_\xi) \to k[\mathcal{N}]$ from φ (under these identifications) is an isomorphism, because of the isomorphisms (3). But since $\overline{\varphi}$ is a $\mathfrak{g}$-equivariant algebra homomorphism, the theorem follows. ■

4. Relation to perverse sheaves on a Grassmannian. The results of the previous sections hold for any field F containing the cyclotomic field k. In this section we choose an imbedding $k \hookrightarrow \mathbb{C}$ and take $\mathbb{C}$ to be the ground field throughout. Let G be the simply-connected (semisimple) complex Lie group corresponding to the Lie algebra $\mathfrak{g}$.

4.1. *A functor.* We have defined, in the course of the proof of the main theorem in §3, a functor (derived functor of I_{u_ξ}) assigning to any U_ξ-module $M \in \mathscr{C}_{U_\xi}$, the graded $U(\mathfrak{g})$-module $H^*(u_\xi, M) \in \mathscr{C}_{U(\mathfrak{g})}$. By definition, the $U(\mathfrak{g})$-action on any object of $\mathscr{C}_{U(\mathfrak{g})}$ is locally finite. Hence, it can be exponentiated to an algebraic G-action, so that $H^*(u_\xi, M)$ becomes a G-module. In addition, the cohomology $H^*(u_\xi, M)$ has a natural $H^*(u_\xi)$-module structure (see §5.1). The G-module and the $H^*(u_\xi)$-module structures are compatible; i.e.,

$$g(a \cdot x) = g(a) \cdot g(x) \qquad \text{for any } g \in G,\ a \in H^*(u_\xi),\ x \in H^*(u_\xi, M). \tag{1}$$

One can show that $H^*(u_\xi, M)$ is finitely generated over $H^*(u_\xi)$, provided M is finite-dimensional. The isomorphism $H^{2\cdot}(u_\xi) \simeq \mathbb{C}^\cdot[\mathcal{N}]$ of the main theorem (§3) makes $H^*(u_\xi, M)$ into a finitely generated $\mathbb{C}^\cdot[\mathcal{N}]$-module. The $\mathbb{C}[\mathcal{N}]$-module $H^*(u_\xi, M)$ gives rise to a G-equivariant coherent sheaf on $\mathcal{N}$ (because of (1)).

There is a canonical $\mathbb{C}^*$-action on $\mathcal{N}$ by multiplication, commuting with the adjoint G-action ($\mathbb{C}^* := \mathbb{C}\backslash\{0\}$). The gradation on the cohomology puts a $\mathbb{C}^*$-equivariant structure on the corresponding sheaf on $\mathcal{N}$. Thus, $H^*(u_\xi, M)$ may be viewed as a $\mathbb{C}^* \times G$-equivariant coherent sheaf on $\mathcal{N}$.

By G-equivariance, the sheaf $H^*(u_\xi, M)$ is locally trivial on the (open) conjugacy class O consisting of the regular nilpotents in $\mathcal{N}$. Let $n \in \mathcal{N}$ be a fixed regular nilpotent and let $\mathfrak{a}$ denote the centralizer of n in $\mathfrak{g}$. Then $\mathfrak{a}$ is an abelian Lie algebra of dimension $= rk\ G$. Set

$$F(M) = H^*(u_\xi, M)_{|n},$$

the geometric fibre of the locally free sheaf $H^*(u_\xi, M)_{|O}$ at n. There is a natural action of the isotropy group $A^\sim \subset \mathbb{C}^* \times G$ of the point n on the space $F(M)$, arising from the $\mathbb{C}^* \times G$-equivariant structure on the sheaf. The centralizer of n in G is canonically contained in $A^\sim$, and hence $F(M)$ is an $\mathfrak{a}$-module. There is also a copy of the group $\mathbb{C}^*$ contained in $A^\sim$, constructed as follows: Let $SL_2(\mathbb{C}) \to G$ be a group homomorphism associated to n via the Jacobson-Morozov theorem and let $\mathbb{C}^* \hookrightarrow SL_2(\mathbb{C})$, $z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$, be the diagonal imbedding. The composition of these two homomorphisms gives a map $\varphi\colon \mathbb{C}^* \to G$ such that $\varphi(z)\cdot n \cdot \varphi(z)^{-1} = z^2 \cdot n$, $z \in \mathbb{C}^*$. Define a homomorphism $\mathbb{C}^* \to \mathbb{C}^* \times G$ by $z \mapsto (z^{-2}, \varphi(z))$. The image of this homomorphism has n as a fixed point. The weight-space decomposition of $F(M)$, with respect to the action of this $\mathbb{C}^*$, puts a $\mathbb{Z}$-gradation on $F(M)$. Similarly, the subalgebra $\mathfrak{a}$ is stable with respect to the adjoint action of the image of φ. This action puts a gradation on $\mathfrak{a}$. The gradation on $\mathfrak{a}$ is compatible with that on $F(M)$, making $F(M)$ a graded $\mathfrak{a}$-module.

4.2. *A category of perverse sheaves.* Let $T \subset B \subset G$ be the maximal torus and the Borel subgroup of G, corresponding to the fixed triangular decomposition of the Lie algebra $\mathfrak{g}$ (see §1). Let $Q := \mathrm{Hom}(\mathbb{C}^*, T)$ be the group of algebraic homomorphisms $\lambda\colon \mathbb{C}^* \to T$. Then Q can canonically be thought of as the coroot lattice (for the simply-connected group G). The Weyl group W acts naturally on Q. The *affine Weyl group* W_{aff} is defined as the semidirect product $Q \rtimes W$. The map $\lambda \cdot w \mapsto \lambda$ gives a canonical bijection $W_{\mathrm{aff}}/W \simeq Q$.

Let LG be the group of all the polynomial loops $f\colon \mathbb{C}^* \to G$, L^+G the subgroup of those loops that have no pole at $0 \in \mathbb{C}$, and $I \subset L^+G$ the (Iwahori) subgroup consisting of all those loops $f \in L^+G$ such that $f(0) \in B$. Set $\mathrm{Gr} = LG/L^+G$, the generalized Grassmannian. The space Gr is the union of a sequence $\mathrm{Gr}_1 \subset \mathrm{Gr}_2 \subset \cdots$ of subsets, all of which have a compatible structure of finite-dimensional projective varieties over $\mathbb{C}$.

There is an obvious I-action on Gr on the left. Each I-orbit is contained in a certain subset Gr_i and is biregular isomorphic to a finite-dimensional affine space. Any homomorphism $\lambda \in Q$ can of course be viewed as an element of LG. Let $\lambda \cdot L^+G$ be the corresponding point in Gr and let $C_\lambda := I \cdot \lambda \cdot L^+G/L^+G$ be the I-orbit of that point. The assignment $\lambda \mapsto C_\lambda$ sets up a bijection of the set Q with the set of all the I-orbits in Gr. We can also parametrize the orbits by the cosets W_{aff}/W via the canonical bijection $W_{\mathrm{aff}}/W \simeq Q$. The orbits form a stratification of Gr.

The varieties Gr_i $(i = 1, 2, \ldots)$ being finite-dimensional, it makes sense to consider a perverse sheaf on Gr whose support is contained in a certain Gr_i. Let $P(\mathrm{Gr})$ denote the abelian category of all those perverse sheaves on Gr, which are constant along the I-orbits and supported on a finite union of orbits. For $\bar{\lambda} \in W_{\mathrm{aff}}/W$, let $IC_{\bar{\lambda}}$ denote the intersection cohomology complex on the closure of the orbit corresponding to $\bar{\lambda}$. The complexes $IC_{\bar{\lambda}}$, $\bar{\lambda} \in W_{\mathrm{aff}}/W$, are the simple objects of the category $\mathfrak{P}(\mathrm{Gr})$. Further, let $H^*(\mathrm{Gr}) = H^*(\mathrm{Gr}, \mathbb{C})$ be the singular cohomology of Gr. There is a natural graded algebra isomorphism $H^*(\mathrm{Gr}) \simeq S(\mathfrak{a})$ constructed in [Gi, §1.7]. (The gradation on $\mathfrak{a}$ is the one defined in §4.1.) Given $\mathscr{F} \in \mathfrak{P}(\mathrm{Gr})$, let $H^*(\mathscr{F})$ denote the hypercohomology of $\mathscr{F}$. Then $H^*(\mathscr{F})$ is a finite-dimensional graded vector space. The space $H^*(\mathscr{F})$ has a natural graded $H^*(\mathrm{Gr})$-module structure and hence a graded $\mathfrak{a}$-module structure. Thus, the assignment $\mathscr{F} \mapsto H^*(\mathscr{F})$ yields a functor: $\mathfrak{P}(\mathrm{Gr}) \rightsquigarrow$ graded $\mathfrak{a}$-modules.

4.3. *A conjecture.* Assume from now on, in this section, that $\mathfrak{g}$ is simply laced and identify Q with the root lattice in $\mathfrak{h}^*$. Define a W_{aff}-action on $\mathfrak{h}^*$ by the formula

$$w_a(v) = l\lambda + wv \qquad \text{for } w_a = \lambda \cdot w \in Q \rtimes W \text{ and } v \in \mathfrak{h}^*.$$

A finite-dimensional simple U_ξ-module is said to be *linked* to the trivial representation (see [APW]) if its highest weight is of the form $w_a(-\rho) - \rho$, for some $w_a \in W_{\mathrm{aff}}$. Let $\mathscr{C}$ be the abelian category formed by all those finite-dimensional U_ξ-modules, all of whose simple subquotients are linked to the trivial representation. The simple objects of $\mathscr{C}$ are precisely those irreducible U_ξ-modules with *dominant* highest weights of the form $w_a(-\rho) - \rho$. This forces w_a to be the maximal element in its coset $\in W_{\mathrm{aff}}/W$.

Combining the recent results of Kazhdan-Lusztig [KL] with those of Casian [Ca], one obtains the following result.

THEOREM. *There is an equivalence of categories $\mathscr{P}\colon \mathscr{C} \rightsquigarrow \mathfrak{P}(\mathrm{Gr})$. Under this equivalence, the irreducible U_ξ-module with (dominant) highest weight $w_a(-\rho) - \rho$, $w_a \in W_{\mathrm{aff}}$, goes to the complex $IC_{\overline{w_a}}$, where $\overline{w_a} := w_a \bmod W \in W_{\mathrm{aff}}/W$.*

We can now make the following conjecture.

CONJECTURE. *There is a natural isomorphism of graded $\mathfrak{a}$-modules*

$$H^*\mathscr{P}(M) \simeq F(M) \qquad \textit{for any } M \in \mathscr{C}.$$

Let $M, N \in \mathscr{C}$. We have $H^*(u_\xi, M) \simeq \mathrm{Ext}^*_{U_\xi}(U_\xi \otimes_{u_\xi} \mathbb{C}, M)$ (see the proof of Lemma 5.2.1). Hence, the Yoneda product

$$\mathrm{Ext}^*_{U_\xi}(U_\xi \otimes_{u_\xi} \mathbb{C}, M) \times \mathrm{Ext}^*_{U_\xi}(M, N) \to \mathrm{Ext}^*_{U_\xi}(U_\xi \otimes_{u_\xi} \mathbb{C}, N)$$

yields a natural morphism

(1) $$\mathrm{Ext}^i_{U_\xi}(M, N) \to \mathrm{Hom}^i_{H^*(u_\xi)}(H^*(u_\xi, M), H^*(u_\xi, N)),$$

where Hom^i stands for the space of morphisms of graded modules shifting the grading by i. Taking the fibres at $n \in \mathscr{N}$ of the corresponding coherent sheaves (see §4.1), we get a morphism

(2) $$\mathrm{Ext}^*_{U_\xi}(M, N) \to \mathrm{Hom}^*_{\mathfrak{a}}(F(M), F(N)).$$

The above conjecture, combined with a theorem of [Gi2], would imply the following result (see also [BGS, Prop. 3.3.9(ii)]).

COROLLARY. *For any simple U_ξ-modules M, $N \in \mathscr{C}$, the morphism* (2) *is an isomorphism, provided the above conjecture holds.*

APPENDIX

5. Cohomology of associative algebras. Fix a field k. By an algebra we always mean an associative k-algebra with unit.

5.1. Let A be an augmented algebra. Given a left A-module M, define the *cohomology of M* (see [CE, ch. X]) by

$$H^i(A, M) := \mathrm{Ext}^i_A(k, M) \qquad i \geqslant 0,$$

where k is viewed as an A-module via the augmentation. Let M itself be a k-algebra such that A acts trivially on M. Then there is a cup product on $H^*(A, M)$ (see [G, §7]) making it into a graded k-algebra. We have, in particular, the graded algebra $H^*(A, k)$, to be abbreviated as $H^*(A)$.

For any A-module M, there is the Yoneda product [B, §2.6] $H^*(A) \otimes H^*(A, M) \to H^*(A, M)$ making $H^*(A, M)$ a graded $H^*(A)$-module. The Yoneda product on $H^*(A)$ coincides with the cup product [GS, §13.7].

5.2. Given an augmented algebra B, let B_+ denote the augmentation ideal. Let A be a subalgebra of B and $A_+ := A \cap B_+$ the augmentation ideal in A. The subalgebra A is said to be *normal* if $B \cdot A_+ = A_+ \cdot B$. In this case $B \cdot A_+$ is of course a two-sided ideal in B, and we set $B//A := B/B \cdot A_+$, which is an augmented algebra again.

LEMMA 5.2.1. *Assume that B is a flat right A-module* (*under the right multiplication*). *Then, for any B-module N and any $i \geqslant 0$, there is a natural $\bar{B}$-module structure on $H^i(A, N)$, where $\bar{B} := B//A$.*

Proof. We have $\mathrm{Hom}_A(k, \cdot) = \mathrm{Hom}_B(B \otimes_A k, \cdot)$. This yields, since B is flat over A, that $\mathrm{Ext}^i_A(k, N) \simeq \mathrm{Ext}^i_B(B \otimes_A k, N)$. But $B \otimes_A k \simeq \bar{B}$, and the (left) $\bar{B}$-action on $\mathrm{Ext}^i_B(\bar{B}, N)$ is induced by the right multiplication of $\bar{B}$ on itself. ■

We now describe the action of Lemma 5.2.1 in a special case: Let d be an element of B such that $d \cdot a - a \cdot d \in A$, for any $a \in A$. Then the assignment $a \mapsto D(a) := d \cdot a - a \cdot d$ is a derivation of A. The derivation D can be extended canonically to a derivation on the Bar-complex of A, hence induces a derivation $\hat{D}$ of the algebra $H^*(A)$. On the other hand, let $\bar{d} \in \bar{B}$ be the image of d. We have the following lemma.

LEMMA 5.2.2. *The action of $\bar{d}$ on $H^*(A)$, arising from Lemma 5.2.1, coincides with the derivation $\hat{D}$.*

Proof. Proof is left to the reader (use the bar-resolution to compute $H^*(A)$). ■

5.3. Retain the notation and assumptions of §5.2. (In particular, we assume that B is a flat right A-module.)

PROPOSITION [CE, Chap. 16, Theorem 6.1]. *For any $\bar{B}$-module M and a B-module N, there is a convergent spectral sequence with*

$$E_2^{p,q} = \mathrm{Ext}^p_{\bar{B}}(M, H^q(A, N)) \Rightarrow \mathrm{Ext}^{p+q}_B(M, N). \quad ■$$

Taking $M = N = k$, the differential $d_2\colon E_2^{0,1} \to E_2^{2,0}$ in the spectral sequence yields the following result.

COROLLARY. *Assume that the canonical $\bar{B}$-action on $H^*(A)$ is trivial. Then there is a natural transgression map*

$$H^1(A) \to H^2(\bar{B}),$$

and its image is contained in the center of $H^*(\bar{B})$.

Proof. Only the second statement requires proof. Observe first that the cup product (§5.1) on $H^*(\bar{B}, H^*(A))$, arising from the algebra structure on $H^*(A)$, makes the above spectral sequence with

$$E_2^{p,q} = H^p(\bar{B}, H^q(A)) \Rightarrow H^{p+q}(B)$$

a multiplicative spectral sequence. Furthermore, the $\bar{B}$-action on $H^*(A)$ being trivial (by assumption), we get a canonical graded algebra isomorphism

$$E_2^{*,*} = H^*(\bar{B}, H^*(A)) \simeq H^*(\bar{B}) \otimes H^*(A), \tag{1}$$

where the right-hand side is endowed with the tensor-product graded algebra structure.

For any $x \in H^p(\bar{B})$ and $y \in H^1(A)$, we have

$$(x \otimes 1) \cdot (1 \otimes y) = (-1)^p (1 \otimes y) \cdot (x \otimes 1).$$

Since the differential d_2 in the spectral sequence (1) is a derivation [G, §7] and since $d_2(E_2^{p,0}) = 0$, we get

$$\begin{array}{ccc} d_2((x \otimes 1) \cdot (1 \otimes y)) & = d_2 x \cdot y + (-1)^p x \cdot d_2 y = (-1)^p x \cdot d_2 y \\ \Vert & \\ (-1)^p d_2((1 \otimes y) \cdot (x \otimes 1)) & = (-1)^p (d_2 y \cdot x - y \cdot d_2 x) = (-1)^p d_2 y \cdot x. \end{array}$$

This completes the proof. ■

5.4. Let I be a homogeneous two-sided ideal in the tensor algebra $T(V)$ of a finite-dimensional k-vector space V (under the natural grading on $T(V)$). Set $A = T(V)/I$ and $R = I/(T^+(V) \cdot I + I \cdot T^+(V))$, where $T^+(V)$ is the augmentation ideal of $T(V)$. The following lemma is well known and can easily be proved.

LEMMA. *If I has no degree 1 elements, then* $H^2(A) \simeq \mathrm{Hom}_k(R, k)$.

5.5. Let A be an augmented algebra with a multiplicative filtration $k = A_0 \subset A_1 \subset \cdots$, such that $\bigcup A_i = A$, and let $\mathrm{Gr}\, A$ be the associated graded algebra. Pick a graded resolution of k by free $\mathrm{Gr}\, A$-modules. Using such a resolution for computing cohomology, one gets an extra grading: $H^i(\mathrm{Gr}\, A) = \Pi_{p \geqslant 0} H^i_{(p)}(\mathrm{Gr}\, A)$ on each cohomology group.

The following result seems to be known; its proof is similar to that of [J, Part I, Prop. 9.13].

PROPOSITION. *There is a natural convergent spectral sequence with*

$$E_1^{p,q} = H_{(p)}^{p+q}(\mathrm{Gr}\, A) \Rightarrow H^{p+q}(A).$$

5.6. Let A be a Hopf algebra with comultiplication $\Delta\colon A \to A \otimes A$ and antipode $\mathscr{S}\colon A \to A$. The counit $\varepsilon\colon A \to k$ gives an augmentation on A. Given an A-bimodule M, define the *adjoint A-action* on M by the formula

$$\mathrm{ad}\, a\colon x \mapsto \sum_i a_i^1 \cdot x \cdot \mathscr{S}(a_i^2) \qquad \text{for } a \in A,\, x \in M,$$

where $\Delta(a) = \sum a_i^1 \otimes a_i^2$. Let M^{ad} denote the (left) A-module structure on M thus obtained.

Viewing A itself as an A-bimodule, we get an A-module A^{ad}. The multiplication map on A gives a morphism of A-modules $m\colon A^{ad} \otimes A^{ad} \to A^{ad}$. Define a graded algebra structure on $H^*(A, A^{ad})$ via the composition

$$H^*(A, A^{ad}) \otimes H^*(A, A^{ad}) \simeq H^*(A \otimes A, A^{ad} \otimes A^{ad})$$
$$\xrightarrow{\Delta^*} H^*(A, A^{ad} \otimes A^{ad}) \xrightarrow{m^*} H^*(A, A^{ad}).$$

The following result is an extension to Hopf algebras of a well-known fact in group cohomology.

PROPOSITION. *For any A-bimodule M, there is a natural graded space isomorphism*

$$\mathrm{Ext}^*_{A\text{-bimod}}(A, M) \simeq H^*(A, M^{ad}).$$

Proof. We have

$$H^0(A, M^{ad}) = \{m \in M \mid \mathrm{ad}\, a(m) = 0, \forall a \in A_+\}, \tag{1}$$

$$\mathrm{Ext}^0_{A\text{-bimod}}(A, M) = \{m \in M \mid a \cdot m = m \cdot a, \forall a \in A\}. \tag{2}$$

The right-hand side of (2) is clearly contained in the right-hand side of (1). For the opposite inclusion, write $\Delta(a) = \sum a_i^1 \otimes a_i^2$ and $(Id \otimes \Delta)(\Delta(a)) = \sum a_i' \otimes a_i'' \otimes a_i'''$. Then we have $m \cdot a = m \cdot \sum \varepsilon(a_i^1) a_i^2 = \sum \varepsilon(a_i^1) \cdot m \cdot a_i^2$, for any $m \in M$. If m belongs to the right-hand side of (1), then the latter expression equals

$$\sum a_i' \cdot m \cdot (\mathscr{S}(a_i'') \cdot a_i''') = \sum a_i^1 \cdot m \cdot \varepsilon(a_i^2) = \sum a_i^1 \varepsilon(a_i^2) \cdot m = a \cdot m.$$

Hence (1) = (2), and the functors $\mathrm{Ext}^0_{A\text{-bimod}}(A, \cdot^{ad})$ and $H^0(A, \cdot)$ coincide on the category of A-bimodules. Hence the derived functors coincide, and the proposition follows. ■

We obtain the following corollary, which seems to be known (but we could not find a reference).

COROLLARY. *For any Hopf algebra A with antipode, $H^*(A)$ is a commutative algebra.*

Proof. By a result of Gerstenhaber [G, §7], the algebra $\mathrm{Ext}^*_{A\text{-bimod}}(A, A)$, the Hochschild cohomology of A, is a commutative algebra for any A. It follows from the proposition above that $H^*(A, A^{ad})$ is commutative, for any Hopf algebra A. But the augmentation map split off the imbedding $k \hookrightarrow A^{ad}$ as an A-module homomorphism (observe that k is A-stable under the adjoint representation). Hence, the natural morphism $H^*(A, k) \to H^*(A, A^{ad})$ is injective, and the result follows. ■

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