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A refinement of the PRV conjecture

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1. Introduction

This short note is a continuation of [K]; the notation of which (cf. [K; § 0]) we adopt here often without explanation. In particular, recall that G is any complex semi-simple simply-connected algebraic group with a fixed Borel subgroup B and complex maximal torus $T \subset B$. Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be the Lie algebras of $G \supset B \supset T$ respectively. We have the associated Weyl group W, and $D \subset \mathfrak{h}^*$ denotes the set of dominant integral weights. For $\lambda \in D$, $V(\lambda)$ denotes the (finite dimensional) irreducible $\mathfrak{g}$-module over $\mathbb{C}$ with highest weight λ. Further, for any integral weight λ, $\bar{\lambda}$ denotes the unique element in D in the W-orbit of λ.

We will also assume familiarity with the contents of [K], but let us recall the following main theorem of [K]; which was conjectured by Parthasarathy-Ranga Rao-Varadarajan (in short PRV): (Actually we proved a strengthened version of this conjecture, due to Kostant; cf. [K; Theorem 2.10].)

1.1 **Theorem** [K; Theorem 2.10]. *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. Fix $\lambda, \mu \in D$ and any $w \in W$. Then the irreducible $\mathfrak{g}$-module $V(\overline{\lambda + w\mu})$ occurs with multiplicity at least one in $V(\lambda) \otimes V(\mu)$.* □

Now the aim of this note is to prove the following refinement of the above theorem; which was conjectured recently by D.N. Verma (cf. [K; Remark 2.12]).

For any $\lambda \in \mathfrak{h}^*$, let $W_\lambda := \{w \in W \colon w\lambda = \lambda\}$ be the stabilizer of λ.

1.2 **Theorem** (A refinement of the PRV conjecture). *Fix $\lambda, \mu \in D$ and consider the map $\eta \colon W_\lambda \backslash W / W_\mu \to D$, defined by $\eta(W_\lambda v W_\mu) = \overline{\lambda + v\mu}$, for any $v \in W$. Then, for any $w \in W$, the irreducible $\mathfrak{g}$-module $V(\overline{\lambda + w\mu})$ occurs in $V(\lambda) \otimes V(\mu)$ with multiplicity at least equal to $\# \eta^{-1}(\eta(W_\lambda w W_\mu))$, where $\#$ denotes the order.* □

As an immediate consequence of the above theorem, we obtain the following

1.3 **Corollary.** *With the notation as in the above theorem, the number of irreducible components $m_{\lambda,\mu}$ of $V(\lambda) \otimes V(\mu)$ is at least as much as the order of the double coset space $W_\lambda \backslash W / W_\mu$. (Observe that $W_\lambda = W_\mu = \{e\}$, if we assume λ and μ to be both regular.)* □

1.4 *Remark.* As shown by a table in [K; Remark 2.12], the multiplicity of $V(\overline{\lambda+w\mu})$ in $V(\lambda)\otimes V(\mu)$ could be strictly greater than $\#\eta^{-1}(\eta(W_\lambda\, w W_\mu))$ in some cases. □

The following proposition provides an interesting class of tensor product of two irreducible representations, where the inequality in Corollary (1.3) is actually an equality.

1.5 **Proposition.** *If we assume (say) $V(\mu)$ to be a miniscule representation, then for any irreducible representation $V(\lambda)$,*

$$m_{\lambda,\mu}=\# W_\lambda\backslash W/W_\mu.$$

(Recall that an irreducible representation is called miniscule if all its weights form a single W-orbit.)

Proof. For any $\lambda, \mu\in D$, clearly

$$m_{\lambda,\mu}=\dim \operatorname{Hom}_{\mathfrak{n}}(V(\lambda)^*, V(\mu)), \qquad \text{where } \mathfrak{n} \text{ is the nil-radical of } \mathfrak{b}.$$

By Joseph's theorem [K; Theorem 2.5], applied to the lowest weight vector of $V(\lambda)^*$ (actually this particular case of Joseph's result is due to Harish-Chandra), we have for any $V(\mu)$ (not necessarily miniscule):

$$\dim \operatorname{Hom}_{\mathfrak{n}}(V(\lambda)^*, V(\mu))=\dim\{a\in V(\mu): X_\alpha^{1+\langle\alpha^\vee,\lambda\rangle}a=0, \text{ for all positive roots } \alpha\},$$

where X_α is some non-zero root vector corresponding to the root α.

Now assuming $V(\mu)$ to be miniscule and applying Joseph's result again; this time to the (extremal) weight vectors of $V(\mu)$, we get:

$$m_{\lambda,\mu}=\# S,$$

where

$$S=S_{\lambda,\mu}:=\{\bar{w}=w \bmod W_\mu\in W/W_\mu: \langle\alpha_i^\vee, \lambda+w\mu\rangle\geqq 0 \text{ for all the simple co-roots } \alpha_i^\vee\}.$$

We next observe that the canonical orbit map γ: $W/W_\mu\to W_\lambda\backslash W/W_\mu$, restricted to S; is injective:

Let $v\in W_\lambda$ and $w\in W$ be such that both of $\bar{w}$ and $\overline{vw}\in S$. Let r_j be a simple reflection in W_λ such that $v_1:=r_j v<v$, then, by the definition of S,

$$\begin{aligned}0\leqq\langle\alpha_j^\vee, \lambda+vw\mu\rangle&=\langle v^{-1}\alpha_j^\vee, \lambda+w\mu\rangle\\&=-\langle v_1^{-1}\alpha_j^\vee, \lambda+w\mu\rangle\leqq 0\end{aligned}$$

(The last inequality is due to the fact that $v_1^{-1}\alpha_j^\vee$ is a positive co-root and $\bar{w}\in S$.)

Hence we get (since $\lambda(\alpha_j^\vee)=0$):

$$vw\mu=v_1 w\mu \tag{*}$$

In particular $\overline{v_1 w}\in S$. Now repeating the above process with v replaced by v_1 and so on (cf. Lemma 2.1), we get (by (*)) $\overline{vw}=\bar{w}$. This proves the assertion that the map $\gamma|_S$ is injective. In particular, we obtain $m_{\lambda,\mu} \leqq \# W_\lambda \backslash W/W_\mu$. So the proposition follows from Corollary (1.3). Actually the inequality $m_{\lambda,\mu} \geqq \# W_\lambda \backslash W/W_\mu$, in this case (i.e., $V(\mu)$ is miniscule), can be directly obtained by observing that $\bar{w} \in S$, for any element w of minimal length in its double coset $W_\lambda w W_\mu$ (use the fact that $V(\mu)$ being miniscule $\langle \alpha^v, \mu \rangle \geqq -1$, for any root α; cf. [B; Exercise 24, p. 226]). □

2. Proof of Theorem (1.2)

In this section we fix once and for all $\lambda, \mu \in D$. Let P_λ (resp. P_μ) denote the parabolic subgroup $BW_\lambda B$ (resp. $BW_\mu B$) of G. (In view of the following lemma it is indeed a subgroup.)

2.1 **Lemma.** *For* $\lambda \in D$, *the subgroup* $W_\lambda \subset W$ *(defined in* § 1) *is generated by the simple reflections it contains.*

This lemma is well known; see, e.g., [B; Chap. V, § 3.3]. □

2.2 **Lemma.** *The double coset space* $W_\lambda \backslash W/W_\mu$ *parametrizes the G-orbits in* $G/P_\lambda \times G/P_\mu$ *bijectively, where G acts diagonally.*

The correspondence is given by $W_\lambda w W_\mu \mapsto G.$ $(1 \bmod P_\lambda,\ w \bmod P_\mu)$; *where (by abuse of notation) w also denotes any lift of w in the normalizer* $N(T)$ *of the torus T.*

Proof. As is easy to see, the map: $G/P_\lambda \times G/P_\mu \to P_\lambda \backslash G/P_\mu$, defined by $(g_1 \bmod P_\lambda,\ g_2 \bmod P_\mu) \mapsto P_\lambda (g_1^{-1} g_2) P_\mu$, induces a bijection from the set of the G-orbits in $G/P_\lambda \times G/P_\mu$ with the double coset space $P_\lambda \backslash G/P_\mu$. Now the lemma follows from Borel-Tits [BT; Corollaire 5.20]. □

For any $w \in W$ and $\lambda, \mu \in D$, $\tilde{X}_w$ (resp. $\tilde{X}_w^P$, where P stands for the pair (P_λ, P_μ)) denotes the closure of the G-orbit $G.$ $(1 \bmod B,\ w \bmod B)$ in $G/B \times G/B$ (resp. the closure of the G-orbit $G.$ $(1 \bmod P_\lambda,\ w \bmod P_\mu)$ in $G/P_\lambda \times G/P_\mu$); $\mathscr{L}_w^P(\lambda \boxtimes \mu)$ is the restriction of the external tensor product $\mathscr{L}^P(\lambda \boxtimes \mu)$, of the line bundles $\mathscr{L}^{P_\lambda}(\lambda)$ on G/P_λ with $\mathscr{L}^{P_\lambda}(\mu)$ on G/P_μ (where $\mathscr{L}^{P_\lambda}(\lambda)$ is associated to the one dimensional representation $\mathbb{C}_{-\lambda}$ of P_λ, cf. [K; § 1.1]), to the subvariety $\tilde{X}_w^P$; and $\mathscr{L}_w(\lambda \boxtimes \mu)$ is the pullback of $\mathscr{L}_w^P(\lambda \boxtimes \mu)$ via the projection: $\tilde{X}_w \to \tilde{X}_w^P$ (got by the restriction of the canonical projection π: $G/B \times G/B \to G/P_\lambda \times G/P_\mu$). With this notation we have the following:

2.3 **Lemma.** *For any* $w \in W$, *the canonical pullback map*

$$\phi\colon H^0(\tilde{X}_w^P, \mathscr{L}_w^P(\lambda \boxtimes \mu)) \to H^0(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu)),$$

got from the map $\pi|_{\tilde{X}_w}$, *is an isomorphism.*

Proof. Since $\pi^{-1}(\tilde{X}_w^P)$ is a G-stable closed irreducible subvariety of $G/B \times G/B$, by Lemma (2.2), there exists a $\hat{w} \in W$ such that $\pi^{-1}(\tilde{X}_w^P) = \tilde{X}_{\hat{w}}$. Since $\pi|_{\tilde{X}_{\hat{w}}}$: $\tilde{X}_{\hat{w}}$

$\to \tilde{X}^P_w$ is a (smooth) proper morphism with connected fibres, we obtain that the canonical map:

$$H^0(\tilde{X}^P_w, \mathscr{L}^P_w(\lambda \boxtimes \mu)) \to H^0(\tilde{X}_{\hat{w}}, \mathscr{L}_{\hat{w}}(\lambda \boxtimes \mu))$$

is an isomorphism.

Clearly $\tilde{X}_w$ is a subvariety of $\tilde{X}_{\hat{w}}$(i.e., $w \leqq \hat{w}$) and hence, by [K; Theorem 1.5(c)], the canonical restriction map:

$$H^0(\tilde{X}_{\hat{w}}, \mathscr{L}_{\hat{w}}(\lambda \boxtimes \mu)) \to H^0(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu))$$

is surjective. But the map $\pi|_{\tilde{X}_w}$: $\tilde{X}_w \to \tilde{X}^P_w$ being surjective, the induced map ϕ is injective.

So we obtain the following commutative triangle:

$$\begin{array}{ccc} H^0(\tilde{X}_{\hat{w}}, \mathscr{L}_{\hat{w}}(\lambda \boxtimes \mu)) & \twoheadrightarrow & H^0(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu)) \\ & \nwarrow \wr \qquad \phi \nearrow & \\ & H^0(\tilde{X}^P_w, \mathscr{L}^P_w(\lambda \boxtimes \mu)) & \end{array}$$

The surjectivity of ϕ follows from the surjectivity of the other two maps in the above triangle. This proves the lemma. □

2.4 **Lemma.** *Fix $\hat{w} \in W$ and let w be an element of the least possible length in the double coset $W_\lambda \hat{w} W_\mu$* (even though we do not need, such a w is unique). *Then the* $\mathfrak{g}$*-module $V(\overline{\lambda + \hat{w}\mu})$ does not occur in $H^0(\tilde{X}_v, \mathscr{L}_v(\lambda \boxtimes \mu))^*$, for any $v < w$.*

Proof. The proof of this lemma is analogous to the proof of [K; Proposition 2.13]; whose notation we adopt freely. By [K; Theorem 1.5(c)], we can assume, without loss of generality, that $v \xrightarrow{\beta} w$. With the notation as in the proof of [K; Proposition 2.13], it suffices to prove:

(a) $\langle v\mu, \beta^v \rangle > 0$:

It is easy to see that $\langle v\mu, \beta^v \rangle \geqq 0$. Now, if possible, assume that $\langle v\mu, \beta^v \rangle = 0$: Then $e_{w\mu} = e_{v\mu}$, i.e., $v^{-1} w \mu = \mu$ and hence $v^{-1} w \in W_\mu$. In particular $W_\lambda w W_\mu = W_\lambda v W_\mu$; which is a contradiction to the assumption that w is of minimal length in its double coset. This proves (a).

(b) $\langle v\mu, \beta^v \rangle \geqq 1 - \langle \lambda + w\mu, \beta^v \rangle$; provided $\langle \lambda + w\mu, \beta^v \rangle < 0$:

Since $\langle v\mu, \beta^v \rangle = -\langle w\mu, \beta^v \rangle$, it suffices to prove that $\langle \lambda, \beta^v \rangle \geqq 1$. If not; let $\langle \lambda, \beta^v \rangle = 0$. Then the reflection corresponding to β, $v_\beta \in W_\lambda$ and hence $W_\lambda w W_\mu = W_\lambda v_\beta w W_\mu = W_\lambda v W_\mu$; which again is a contradiction to the choice of w. This proves (b), and thus finishes the proof of the lemma. □

Fix a $w \in W$ and let $\{W_\lambda w_1 W_\mu, \ldots, W_\lambda w_n W_\mu\}$ be distinct double cosets such that $\eta(W_\lambda w_i W_\mu) = \overline{\lambda + w\mu}$, for all $1 \leqq i \leqq n$; where η is as defined in Theorem (1.2). With this notation, we have the following crucial:

2.5 **Proposition.** *The irreducible* $\mathfrak{g}$*-module* $V(\overline{\lambda + w\mu})$ *occurs in*

$$H^0\left(\bigcup_{i=1}^{n} \tilde{X}^P_{w_i}, \mathscr{L}^P(\lambda \boxtimes \mu)|_{\bigcup_{i=1}^{n} \tilde{X}^P_{w_i}}\right)^*$$

with multiplicity exactly equal to n; *where* $\tilde{X}^P_{w_i} \subset G/P_\lambda \times G/P_\mu$ *is the subvariety and* $L^P(\lambda \boxtimes \mu)$ *is the line bundle on* $G/P_\lambda \times G/P_\mu$ *defined earlier, and the union* $\bigcup_{i=1}^{n} \tilde{X}^P_{w_i} \subset G/P_\lambda \times G/P_\mu$ *is taken with the reduced subscheme structure.*

Proof. We assume, by induction on k, that $V(\overline{\lambda + w\mu})$ occurs in $H^0(Y_k, \mathscr{L}^P(\lambda \boxtimes \mu)|_{Y_k})^*$ with multiplicity exactly equal to k, where $Y_k := \bigcup_{i=1}^{k} \tilde{X}^P_{w_i}$. The case $k=1$ is the content of [K; Theorem 2.10 and Proposition 2.9], in view of Lemma (2.3).

Consider the exact sheaf sequence:

$$0 \to \mathscr{I}_{Y_k}(Y_{k+1}) \to \mathcal{O}_{Y_{k+1}} \to \mathcal{O}_{Y_k} \to 0, \qquad (\mathscr{S}_1)$$

where $\mathscr{I}_{Y_k}(Y_{k+1})$ is the ideal sheaf of the closed subvariety Y_k in Y_{k+1}.

On tensoring the sequence $(\mathscr{S}_1)$ with the locally free sheaf $\mathscr{L}^P(\lambda \boxtimes \mu)|_{Y_{k+1}}$ and taking cohomology, we obtain the following long exact sequence:

$$\begin{aligned} 0 &\to H^0(Y_{k+1}, \mathscr{I}_{Y_k}(Y_{k+1}) \otimes \mathscr{L}^P(\lambda \boxtimes \mu)|_{Y_{k+1}}) \\ &\to H^0(Y_{k+1}, \mathscr{L}^P(\lambda \boxtimes \mu)|_{Y_{k+1}}) \to H^0(Y_k, \mathscr{L}^P(\lambda \boxtimes \mu)|_{Y_k}) \\ &\to H^1(Y_{k+1}, \mathscr{I}_{Y_k}(Y_{k+1}) \otimes \mathscr{L}^P(\lambda \boxtimes \mu)|_{Y_{k+1}}) \to \dots \end{aligned} \qquad (\mathscr{S}_2)$$

But, as is fairly easy to see,

$$\mathscr{I}_{Y_k}(Y_{k+1}) = \mathscr{I}_{Y_k}(Y_k \cup \tilde{X}^P_{w_{k+1}}) \approx \mathscr{I}_{Y_k \cap \tilde{X}^P_{w_{k+1}}}(\tilde{X}^P_{w_{k+1}}),$$

where the intersection $Y_k \cap \tilde{X}^P_{w_{k+1}}$ is the scheme theoretic intersection, and the sheaf $\mathscr{I}_{Y_k \cap \tilde{X}^P_{w_{k+1}}}(\tilde{X}^P_{w_{k+1}})$, which is defined on $\tilde{X}^P_{w_{k+1}}$, is extended to the whole of Y_{k+1} by defining it to be zero on the open set $Y_{k+1} \backslash \tilde{X}^P_{w_{k+1}}$. In particular, we get by [H, Chap. III, Lemma 2.10]:

$$\begin{aligned} &H^p(Y_{k+1}, \mathscr{I}_{Y_k}(Y_{k+1}) \otimes \mathscr{L}^P(\lambda \boxtimes \mu)|_{Y_{k+1}}) \\ &\approx H^p(\tilde{X}^P_{w_{k+1}}, \mathscr{I}_{Y_k \cap \tilde{X}^P_{w_{k+1}}}(\tilde{X}^P_{w_{k+1}}) \otimes \mathscr{L}^P_{w_{k+1}}(\lambda \boxtimes \mu)), \quad \text{for all } p \geqq 0. \end{aligned} \qquad (I)$$

Similarly, consider the long exact sequence corresponding to the sheaf sequence $0 \to \mathscr{I}_{Y_k \cap \tilde{X}^P_{w_{k+1}}}(\tilde{X}^P_{w_{k+1}}) \to \mathcal{O}_{\tilde{X}^P_{w_{k+1}}} \to \mathcal{O}_{Y_k \cap \tilde{X}^P_{w_{k+1}}} \to 0$:

$$\begin{aligned} 0 &\to H^0(\tilde{X}^P_{w_{k+1}}, \mathscr{I}_{Y_k \cap \tilde{X}^P_{w_{k+1}}}(\tilde{X}^P_{w_{k+1}}) \otimes \mathscr{L}^P_{w_{k+1}}(\lambda \boxtimes \mu)) \\ &\to H^0(\tilde{X}^P_{w_{k+1}}, \mathscr{L}^P_{w_{k+1}}(\lambda \boxtimes \mu)) \to H^0(Y_k \cap \tilde{X}^P_{w_{k+1}}, \mathscr{L}^P(\lambda \boxtimes \mu)|_{Y_k \cap \tilde{X}^P_{w_{k+1}}}) \\ &\to H^1(\tilde{X}^P_{w_{k+1}}, \mathscr{I}_{Y_k \cap \tilde{X}^P_{w_{k+1}}}(\tilde{X}^P_{w_{k+1}}) \otimes \mathscr{L}^P_{w_{k+1}}(\lambda \boxtimes \mu)) \to 0. \end{aligned} \qquad (\mathscr{S}_3)$$

(The last zero is due to the fact that $H^1(\tilde{X}^P_{w_{k+1}}, \mathscr{L}^P_{w_{k+1}}(\lambda \boxtimes \mu))=0$; by [K; Theorem 1.5(b) and remark 1.6(a)].)

Now by an argument identical to the proof of [R_1; Theorem 3] (see also [R_2; corollary 1.11]), together with [MR_2; Theorem 1] and [MR_1; Proposition 4], the intersection $Y_k \cap \tilde{X}^P_{w_{k+1}}$ is reduced. Since $Y_k \cap \tilde{X}^P_{w_{k+1}}$ is a G-stable closed subvariety of $G/P_\lambda \times G/P_\mu$, we can write it as the union $\bigcup_{j=1}^{m} \tilde{X}^P_{v_j}$, for some $v_j \in W$. In fact since $\tilde{X}_{w_{k+1}}$ is mapped onto $\tilde{X}^P_{w_{k+1}}$, under the canonical projection $\pi: G/B \times G/B \to G/P_\lambda \times G/P_\mu$, we can choose the v_j's so that for any j, $v_j \leqq w_{k+1}$. Of course we can assume that w_{k+1} is of minimal length in its double coset. Further, by Lemma (2.2), the $\tilde{X}^P_{w_i}$'s (for $1 \leqq i \leqq n$) are all distinct and moreover by Lemmas (2.3), (2.4), and [K; Theorem 2.10 and Proposition 2.9] no $\tilde{X}^P_{w_i}$ can contain any $\tilde{X}^P_{w_{i'}}$ for $1 \leqq i \neq i' \leqq n$. Hence $Y_k \cap \tilde{X}^P_{w_{k+1}}$ is properly contained in $\tilde{X}^P_{w_{k+1}}$ and consequently $v_j < w_{k+1}$, for any $1 \leqq j \leqq m$. But then, by Lemmas (2.3) and (2.4), the isotypical component in $H^0(Y_k \cap \tilde{X}^P_{w_{k+1}}, \mathscr{L}^P(\lambda \boxtimes \mu)|_{Y_k \cap \tilde{X}^P_{w_{k+1}}})$ corresponding to the irreducible module $V(\overline{\lambda + w\mu})^*$ is 0. In particular, from the exact sequence $(\mathscr{S}_3)$, the isotypical component in $H^1(\tilde{X}^P_{w_{k+1}}, \mathscr{I}_{Y_k \cap \tilde{X}^P_{w_{k+1}}}(\tilde{X}^P_{w_{k+1}}) \otimes \mathscr{L}^P_{w_{k+1}}(\lambda \boxtimes \mu))$ corresponding to $V(\overline{\lambda + w\mu})^*$ is 0 and $V(\overline{\lambda + w\mu})^*$ occurs with multiplicity one in $H^0(\tilde{X}^P_{w_{k+1}}, \mathscr{I}_{Y_k \cap \tilde{X}^P_{w_{k+1}}}(\tilde{X}^P_{w_{k+1}}) \otimes \mathscr{L}^P_{w_{k+1}}(\lambda \boxtimes \mu))$. Now the induction gets completed from the exact sequence $(\mathscr{S}_2)$ and the isomorphism (I). This proves the proposition. □

(2.6) *Remark.* The above proposition provides a refinement of the strengthened PRV conjecture, due to Kostant (cf. [K; Theorem 2.10 and Proposition 2.9]).

(2.7) *Proof of Theorem* (1.2). Write $\eta^{-1}(\eta(W_\lambda w W_\mu))$ as the set of distinct double cosets $\{W_\lambda w_1 W_\mu, \ldots, W_\lambda w_n W_\mu\}$. By Proposition (2.5), it suffices to prove that the canonical restriction map

$$\psi: H^0(G/P_\lambda \times G/P_\mu, \mathscr{L}^P(\lambda \boxtimes \mu)) \to H^0\left(\bigcup_{i=1}^{n} \tilde{X}^P_{w_i}, \mathscr{L}^P(\lambda \boxtimes \mu)|_{\bigcup_{i=1}^{n} \tilde{X}^P_{w_i}}\right)$$

is surjective.

To prove this, we use char. p methods: we denote by the same symbols G, P_λ, $\tilde{X}^P_{w_i}$, $\mathscr{L}^P(\lambda \boxtimes \mu)$ the corresponding objects defined over an algebraically closed field of char. $p>0$. We first observe that the line bundle $\mathscr{L}^P(\lambda \boxtimes \mu)$ on $G/P_\lambda \times G/P_\mu$ is ample. Further all the subvarieties $\tilde{X}^P_{w_i} \subset G/P_\lambda \times G/P_\mu$ are "simultaneously compatibly Frobenius split" (cf. [MR_2; Theorem 1] together with [MR_1; Proposition 4]). In particular, the union $\bigcup_{i=1}^{n} \tilde{X}^P_{w_i} \subset G/P_\lambda \times G/P_\mu$ is compatibly Frobenius split. Now the assertion, that the map ψ is surjective in char. p, follows from [MR_1; Proposition 3]. But then, by semicontinuity, ψ is surjective in char. 0.

This completes the proof of Theorem (1.2). □

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Note added in proof

I learnt from G. Lusztig that he has recently obtained a different proof of the PRV conjecture (i.e., Theorem 1.1); by using his results on the intersection homology of generalized Schubert varieties associated to affine Kac-Moody groups