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Proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture

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Introduction

The aim of this paper is to settle in affirmative the following conjecture (C_1), generally attributed to K.R. Parthasarathy-R. Ranga Rao-V.S. Varadarajan (henceforth called the PRV conjecture) formulated in the sixties as a fallout of [PRV], or more precisely its strengthened form (C_2), due to Kostant, stated below.

(C_1) *The PRV conjecture*. Let $\mathfrak{g}$ be a finite dimensional complex semi-simple Lie algebra with associated Weyl group W and let $V(\lambda)$ and $V(\mu)$ be two finite dimensional irreducible $\mathfrak{g}$-modules with highest weights λ and μ respectively. Then, for any $w \in W$, the irreducible $\mathfrak{g}$-module $V(\overline{\lambda + w\mu})$ (cf. § 0), with extremal weight $\lambda + w\mu$, occurs with multiplicity at least one in $V(\lambda) \otimes V(\mu)$.

(C_2) *The strengthened PRV conjecture* (*due to Kostant*). With the notations as in (C_1), $V(\overline{\lambda + w\mu})$ occurs with multiplicity exactly one in the submodule $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu}) \subset V(\lambda) \otimes V(\mu)$, defined in Lemma (2.8).

Now we describe the contents of the paper in more detail:

Section (0) is devoted to setting up the notations to be followed throughout the paper.

Section (1): Let G be a (finite dimensional) complex semi-simple group with a (fixed) Borel subgroup B, maximal torus $T \subset B$, and Weyl group W. For any B-variety X, one can define a G-variety $\tilde{X} = G \times_B X$. In particular, for any sequence of simple reflections $\mathfrak{w} = (r_{i_1}, \ldots, r_{i_n})$ (resp. for any $w \in W$) one has the G-variety $\tilde{Z}_{\mathfrak{w}}$ (resp. $\tilde{X}_w$), associated to the Bott-Samelson-Demazure-Hansen B-variety $Z_{\mathfrak{w}}$ (resp. the Schubert variety $X_w := \overline{BwB/B} \subset G/B$). One can also identify the G-variety $\tilde{X}_w$ with the closure of a G-orbit in $G/B \times G/B$ (G acting diagonally). Now given two integral weights λ, μ; we can define a line bundle $\mathscr{L}(\lambda \boxtimes \mu)$ on $G/B \times G/B$ (and its restriction $\mathscr{L}_w(\lambda \boxtimes \mu)$ to $\tilde{X}_w$), which is the external tensor product of the line bundles $\mathscr{L}(\lambda)$ and $\mathscr{L}(\mu)$ resp. (on G/B (cf. § 1.1)). The main result of this section (Theorem (1.5)) asserts (among other results) that for λ, μ dominant:

(a) $H^p(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu))=0$, for all $p>0$ and any $w \in W$, and

(b) for any $v \leqq w$; the canonical restriction map: $H^0(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu)) \to H^0(\tilde{X}_v, \mathscr{L}_v(\lambda \boxtimes \mu))$ is surjective.

This result is obtained as an easy consequence of our Proposition (1.2); on the cohomology vanishing of $\tilde{Z}_{\mathfrak{w}}$ with respect to some line bundles. This proposition is analogous to our crucial proposition [Ku; Proposition 2.3] for $Z_{\mathfrak{w}}$ and its (Proposition (1.2)) proof also is quite analogous; as given in [Ku; § 4].

Section (2) is devoted to the proof of the strengthened PRV conjecture stated above. The brief idea of the proof is as follows:

Fix a $w \in W$ and dominant integral weights λ, μ. By a result of Bott (actually only its special case; which is an easy consequence of Peter-Weyl theorem), and the Leray spectral sequence for the map: $\tilde{X}_w \to G/B$; we identify (Theorem (2.2))

$$(*) \qquad H^0(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu)) \text{ with } \sum_{\theta \in D} V(\theta)^* \otimes \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w(\mu), V(\theta))$$

(cf. § 0 for various notations). Now we use a result of Joseph (Theorem (2.5)), which gives a precise knowledge of the annihilator (in $U(\mathfrak{n})$) of any extremal weight vector in any finite dimensional irreducible $\mathfrak{g}$-module; to conclude that $\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w(\mu), V(\overline{\lambda+w\mu}))$ is exactly one dimensional. So $V(\overline{\lambda+w\mu})$ occurs with multiplicity one in $H^0(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu))^*$ by $(*)$. But now, we use the result (b) stated above to show (cf. Proposition (2.9)) that $H^0(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu))^*$ is a sub-module of $V(\lambda) \otimes V(\mu)$; in fact it is precisely the $\mathfrak{g}$-module $U(\mathfrak{g})(e_\lambda \otimes e_{w\mu}) \subset V(\lambda) \otimes V(\mu)$, defined in Lemma (2.8). Hence $V(\overline{\lambda+w\mu})$ occurs with multiplicity exactly one in $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$; thus proving the strengthened PRV conjecture (C_2), in particular the PRV conjecture (C_1).

We further show (Proposition (2.13)) that if we assume that λ and μ are both dominant regular, then the irreducible module $V(\overline{\lambda+w\mu})$ does not occur in $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{v\mu})$, for any $v<w$. Finally in this section, we compute the character (with respect to the action of the torus T) of the $\mathfrak{g}$-module $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$ in terms of the Demazure operators (cf. Theorem (2.14)). This gives a generalization of (an old) result of Brauer; see [K_1]. (This paper of Kostant also surveys various results known at that time on the decomposition of tensor product of two finite dim. irreducible representations.)

Section (3) is not directly connected with the PRV conjecture (at least it is not used in the proof of the conjecture). We include it here because the proof of the main (and the only) theorem of this section (Theorem (3.1)) uses only the ideas developed earlier to prove the PRV conjecture. Theorem (3.1) asserts that for any $w \in W$, any integral weight λ, and any dominant integral μ; the $(-\lambda)$-th weight space in the Lie algebra homology $H_p(\mathfrak{n}, V_w(\mu))$ is isomorphic with the

G-invariants in $H^p(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))$. Study of the Lie algebra homology $H_*(\mathfrak{n}, V_w(\mu))$ has recently been initiated by Joseph [J_2]. We hope that this identification will shed some light on this.

(Added at the time of Revision) We have also proved, by a very similar method, the strengthened PRV conjecture (C_2) for any symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, provided we assume that λ is regular. The details will appear elsewhere. □

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Contents

0. Notations

We will adhere (often without explaining again) to the following notational conventions throughout the paper:

$\mathfrak{g}$	= a complex finite dimensional semi-simple Lie algebra
$\mathfrak{b}$	= a fixed Borel subalgebra of $\mathfrak{g}$
$\mathfrak{n}$	= the nil-radical $[\mathfrak{b}, \mathfrak{b}]$ of $\mathfrak{b}$
$\mathfrak{h}$	= a fixed Cartan subalgebra $\subset \mathfrak{b}$
$U(\mathfrak{a})$ (for any Lie algebra $\mathfrak{a}$)	= the universal enveloping algebra of $\mathfrak{a}$
G	= Simply-connected complex (semi-simple) group with Lie algebra $\mathfrak{g}$
B	= the Borel subgroup of G associated to $\mathfrak{b}$
N	= the unipotent radical $[B, B]$ of B
T	= the (complex) maximal torus of G associated to $\mathfrak{h}$
Δ	= the set of roots for the pair $(\mathfrak{g}, \mathfrak{h})$
Δ_+	= the set of (positive) roots for the pair $(\mathfrak{b}, \mathfrak{h})$
W	= the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$
$\{\alpha_1, \ldots, \alpha_l\}$	denotes the set of simple roots $\subset \Delta_+$
$\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$	denotes the corresponding (simple) co-roots; $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$
$\{r_1, \ldots, r_l\}$	denotes the (simple) reflections $\in W$ corresponding to the simple roots $\{\alpha_1, \ldots, \alpha_k\}$ respectively

$\mathfrak{h}^*_{\mathbb{Z}}$	= the set of integral weights, i.e., $\{\lambda \in \mathfrak{h}^*: \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$, for all $i\}$
D	= the set of dominant integral weights, i.e., $\{\lambda \in \mathfrak{h}^*_{\mathbb{Z}}: \langle \lambda, \alpha_i^\vee \rangle \geqq 0$, for all $i\}$
$\mathbb{C}_\lambda$ (for $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$)	denotes the one dimensional B-module, such that the torus T acts by the character e^λ and (of course) the unipotent-radical N acts trivially
$V(\lambda)$ (for $\lambda \in D$)	= the (finite dimensional) irreducible $\mathfrak{g}$-module with highest weight λ
$V(\bar{\lambda})$ (for any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$)	= the (finite dimensional) irreducible $\mathfrak{g}$-module with highest weight $\bar{\lambda}$, where $\bar{\lambda}$ is the unique element in D in the W-orbit of λ
$V_w(\lambda)$ (for $\lambda \in D$ and $w \in W$)	= the $U(\mathfrak{b})$-span of any non-zero extremal weight vector $e_{w\lambda}$ (of weight $w\lambda$) in $V(\lambda)$
ρ	= the (unique) element in D, such that $\langle \rho, \alpha_i^\vee \rangle = 1$, for all simple co-roots $\alpha_i^\vee$.

Unless otherwise stated, vector spaces and tensor products are over $\mathbb{C}$. For a vector space V, V^* denotes its dual $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Given any sequence $\mathfrak{w} = (r_{i_1}, \ldots, r_{i_n})$ of simple reflections, by $m(\mathfrak{w})$ we mean $r_{i_1} \ldots r_{i_n} \in W$. The sequence $\mathfrak{w}$ is said to be reduced, if $m(\mathfrak{w}) = r_{i_1} \ldots r_{i_n}$ is a reduced decomposition of $m(\mathfrak{w})$.

1. Basic 'cohomology vanishing' results for line bundles on the closures of G-orbits in $G/B \times G/B$

(1.1) *Basic constructions.* We adopt the notations from [Ku; § 2.1 and § 2.2].

For any B-variety X, we denote by $\tilde{X}$ the G-variety $G \times_B X$, i.e., it is the total space of the fiber bundle with fiber X, associated to the principal B-bundle: $G \to G/B$. For any B-varieties X, Y and a B-morphism $\phi: X \to Y$, there is a canonical G-morphism $\tilde{\phi}: \tilde{X} \to \tilde{Y}$.

In particular, for any sequence (not necessarily reduced) $\mathfrak{w} = (r_{i_1}, \ldots, r_{i_n})$ of simple reflections and any $1 \leqq j \leqq n$, we have the G-varieties $\tilde{Z}_{\mathfrak{w}}$ and $\tilde{Z}_{\mathfrak{w}(j)}$ and a canonical inclusion $\tilde{Z}_{\mathfrak{w}(j)} \hookrightarrow \tilde{Z}_{\mathfrak{w}}$, where the (Bott-Samelson-Demazure-Hansen) B-varieties $Z_{\mathfrak{w}}$ and $Z_{\mathfrak{w}(j)}$ and a B-morphism: $Z_{\mathfrak{w}(j)} \hookrightarrow Z_{\mathfrak{w}}$ are defined in [Ku; § 2.1]. Of course $\tilde{Z}_{\mathfrak{w}}$ (and $\tilde{Z}_{\mathfrak{w}(j)}$) is smooth. For any $w \in W$, we also have the G-variety $\tilde{X}_w$, where X_w is the closure of the Bruhat cell $BwB/B \subset G/B$. Moreover, for any $v \leqq w$, we have a canonical inclusion $\tilde{X}_v \hookrightarrow \tilde{X}_w$, induced from the inclusion $X_v \hookrightarrow X_w$. Further, there are G-morphisms (G acting on $G/B \times G/B$ diagonally):

$$\tilde{\theta}_{\mathfrak{w}}: \tilde{Z}_{\mathfrak{w}} \to G/B \times G/B$$

and

$$\tilde{d}_w: \tilde{X}_w \to G/B \times G/B,$$

defined by

$$\tilde{\theta}_{\mathfrak{w}}(g, z) = (g \bmod B, g\,\theta_{\mathfrak{w}}(z)), \quad \text{for } g \in G \text{ and } z \in Z_{\mathfrak{w}}$$

and

$$\tilde{d}_w(g, x) = (g \bmod B, g \cdot x), \quad \text{for } g \in G \text{ and } x \in X_w \subset G/B;$$

where the map $\theta_{\mathfrak{w}}: Z_{\mathfrak{w}} \to G/B$ is defined in [Ku; § 2.1]. Clearly the map $\tilde{\theta}_{\mathfrak{w}}$ (resp. $\tilde{d}_w$) is well defined, i.e., it descends to $\tilde{Z}_{\mathfrak{w}}$ (resp. $\tilde{X}_w$). It can be easily seen that the map $\tilde{d}_w$ is a closed immersion and its image is the closure of the G-orbit of the point $(e, w \bmod B)$ in $G/B \times G/B$.

For any integral weight $\lambda \in \mathfrak{h}^*$, we denote (as in [Ku; § 2.2]) by $\mathscr{L}(\lambda)$, the line bundle on G/B associated to the principal B-bundle: $G \to G/B$ by the (one dimensional) representation $\mathbb{C}_{-\lambda}$ of B (cf. § 0). Now given two $\lambda, \mu \in \mathfrak{h}^*_{\mathbb{Z}}$, we denote by $\mathscr{L}(\lambda \boxtimes \mu)$, the line bundle on $G/B \times G/B$ which is the external tensor product of the line bundles $\mathscr{L}(\lambda)$ and $\mathscr{L}(\mu)$ respectively. We further denote by $\mathscr{L}_{\mathfrak{w}}(\lambda \boxtimes \mu)$ (resp. $\mathscr{L}_w(\lambda \boxtimes \mu)$) the pull back of $\mathscr{L}(\lambda \boxtimes \mu)$ by the map $\tilde{\theta}_{\mathfrak{w}}$ (resp. $\tilde{d}_w$).

Now we can state the following analogue of our crucial proposition (Proposition (2.3)) in [Ku]:

(1.2) **Proposition.** *Let $\mathfrak{w} = (r_{i_1}, \ldots, r_{i_n})$ be any sequence of simple reflections and let $1 \leqq j \leqq k \leqq n$ be such that the subsequence $(r_{i_j}, \ldots, r_{i_k})$ is reduced. Choose any $\lambda, \mu \in D$ (cf. § 0). Then, with the notations as above, we have:*

$$H^p\left(\tilde{Z}_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda \boxtimes \mu) \otimes \mathcal{O}_{\tilde{Z}_{\mathfrak{w}}}\left[-\bigcup_{q=j}^{k} \tilde{Z}_{\mathfrak{w}(q)}\right]\right) = 0, \quad \textit{for all } p > 0.$$

We also have:

$$H^p(\tilde{Z}_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda \boxtimes \mu)) = 0, \quad \textit{for all } p > 0. \quad \square$$

The proof of the above proposition is identical to the proof of the analogous proposition given in [Ku; Sect. 4], if we observe the following:

(1.3) **Lemma.** *For any sequence $\mathfrak{w} = (r_{i_1}, \ldots, r_{i_n})$ (not necessarily reduced), the canonical bundle $K_{\tilde{Z}_{\mathfrak{w}}}$ of $\tilde{Z}_{\mathfrak{w}}$ is isomorphic with*

$$\mathscr{L}_{\mathfrak{w}}(-\rho \boxtimes -\rho) \otimes \mathcal{O}_{\tilde{Z}_{\mathfrak{w}}}[-\partial \tilde{Z}_{\mathfrak{w}}], \quad \text{where } \partial \tilde{Z}_{\mathfrak{w}} = \bigcup_{q=1}^{n} \tilde{Z}_{\mathfrak{w}(q)}.$$

Proof. The lemma follows fairly easily by making successive use of [R_1; Lemma 3 in § 1] together with the fact that $K_{G/B} = \mathscr{L}(-2\rho)$. $\square$

(1.4) **Corollary** (of Proposition 1.2). *Let $\mathfrak{w} = (r_{i_1}, \ldots, r_{i_n})$ be any sequence. Then for any $1 \leqq j \leqq n$ and any $\lambda, \mu \in D$, the canonical map: $H^0(\tilde{Z}_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda \boxtimes \mu)) \to H^0(\tilde{Z}_{\mathfrak{w}(j)}, \mathscr{L}_{\mathfrak{w}(j)}(\lambda \boxtimes \mu))$ is surjective.*

Proof. Apply Proposition (1.2) to the cohomology exact sequence, corresponding to the sheaf sequence:

$0 \to \mathcal{O}_{\tilde{Z}_{\mathfrak{w}}}[-\tilde{Z}_{\mathfrak{w}(j)}] \to \mathcal{O}_{\tilde{Z}_{\mathfrak{w}}} \to \mathcal{O}_{\tilde{Z}_{\mathfrak{w}(j)}} \to 0$ tensored with the locally free sheaf $\mathscr{L}_{\mathfrak{w}}(\lambda \boxtimes \mu)$. $\square$

In the case when $\mathfrak{w}$ is a reduced sequence, the image of the map $\tilde{\theta}_{\mathfrak{w}}: \tilde{Z}_{\mathfrak{w}} \to G/B \times G/B$ is precisely equal to $\tilde{d}_w(\tilde{X}_w)$, where $w = m(\mathfrak{w})$. By slight abuse of notation, we denote the map $\tilde{\theta}_{\mathfrak{w}}$ considered as a map: $\tilde{Z}_{\mathfrak{w}} \to \tilde{X}_w$, again, by $\tilde{\theta}_{\mathfrak{w}}$.

(Since $\tilde{d}_w$ is a closed immersion, no confusion is likely.) Then $\tilde{\theta}_{\mathfrak{w}}$ is a birational surjective morphism.

We have the following main theorem of this section:

(1.5) **Theorem.** *Fix $v \leqq w \in W$ and $\lambda, \mu \in D$. Then with the notations as above:*

(a) *For any reduced sequence $\mathfrak{w}$ with $m(\mathfrak{w}) = w$, the map $\tilde{\theta}_{\mathfrak{w}}: \tilde{Z}_{\mathfrak{w}} \to \tilde{X}_w$ is a rational resolution.*

In particular, for any locally free sheaf $\mathscr{S}$ on $\tilde{X}_w$, one has:

$$H^p(\tilde{X}_w, \mathscr{S}) \simeq H^p(\tilde{Z}_{\mathfrak{w}}, \tilde{\theta}^*_{\mathfrak{w}}(\mathscr{S})), \quad \text{for all } p \geqq 0.$$

(b) $H^p(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu)) = 0$, *for all $p > 0$.*

(c) *The canonical restriction map:*

$$H^0(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu)) \to H^0(\tilde{X}_v, \mathscr{L}_v(\lambda \boxtimes \mu)) \text{ is surjective.}$$

(d) *The variety $\tilde{X}_w$ is projectively normal and also arithmetically Cohen-Macaulay in the projective embedding given by any $\mathscr{L}(\lambda \boxtimes \mu)$, for both λ and μ dominant regular.*

Of course $\tilde{X}_w$ is normal and Cohen-Macaulay since X_w is.

Proof. By a lemma of Kempf [D; §5, Proposition 2] and the Leray spectral sequence corresponding to the map $\tilde{\theta}_{\mathfrak{w}}$, (a) follows from Proposition (1.2). (Observe that, because of a result of Grauert-Riemenschneider in Char. 0 the higher direct images of the canonical bundle $K_{\tilde{Z}_{\mathfrak{w}}}$ of $\tilde{Z}_{\mathfrak{w}}$ under $\tilde{\theta}_{\mathfrak{w}}$ are all zero.)

(b) is a consequence of (a) and Proposition (1.2).

(c) follows from (a) and Corollary (1.4).

One can prove (d), e.g., by an argument identical to the one given in [Ku; §2] to prove the corresponding facts for X_w's. □

(1.6) *Remarks.* (a) An analogue of the above theorem for $\tilde{X}_w$ replaced by $\widetilde{X^P_w}$ (where $P \supset B$ is any parabolic subgroup and X^P_w is the B-variety $\overline{BwP/P} \subset G/P$) can be easily formulated and deduced from the case of the Borel subgroup B given above.

(b) A. Ramanathan has told me that the varieties $\tilde{X}_w \subset G/B \times G/B$ are 'compatibly Frobenius split'. A particular case of this; when $w = e$, i.e. $\tilde{X}_w$ is the diagonal in $G/B \times G/B$, is proved in [R_2]. In particular, the above theorem holds good in arbitrary characteristic. See [MR]; [RR]; and [R_1].

2. Proof of the strengthened PRV conjecture

With the notations as in §0, we recall the following theorem of Bott [B; Theorem I], a simple proof of which is given by Kostant in [K_2; §6].

(2.1) **Theorem.** *For any finite dimensional algebraic B-module M and any* $p \geqq 0$, *there is a G-module isomorphism:*

$$H^p(G/B, \mathcal{M}) \approx \sum_{\theta \in D} V(\theta)^* \otimes [H^p(\mathfrak{n}, V(\theta) \otimes M)]^{\mathfrak{h}},$$

where we put the trivial G-module structure on the $\mathfrak{h}$*-invariants* $[H^p(\mathfrak{n}, V(\theta)\otimes M)]^{\mathfrak{h}}$ *of the Lie algebra cohomology of* $\mathfrak{n}$ *with coefficients in the* $\mathfrak{b}$ *(and hence* $\mathfrak{n}$*)-module* $V(\theta)\otimes M$ *and* $\mathcal{M}$ *denotes the locally free sheaf on G/B associated to the B-module M.* □

As a consequence of the above theorem of Bott, we derive the following very useful result:

(2.2) **Theorem.** *With the notations as in* §1; *for any* $w \in W$, $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$ *and* $\mu \in D$: $H^0(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))$ *is canonically G-module isomorphic with*

$$\sum_{\theta \in D} V(\theta)^* \otimes \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w(\mu), V(\theta)),$$

where we put the trivial G-module structure on $\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w(\mu), V(\theta))$ *and* $V_w(\mu)$ *and* $\mathbb{C}_\lambda$ *are as defined in* §0.

Proof. By the Leray spectral sequence (rather the definition of the direct image sheaf; since we are dealing only with H^0 here), corresponding to the canonical fibration $\pi = \pi_w : \tilde{X}_w \to G/B$, we get that $H^0(\tilde{X}_w, L_w(\lambda \boxtimes u)) \approx H^0(G/B, \pi_* \mathcal{L}_w(\lambda \boxtimes \mu))$, where π_* is the direct image sheaf on G/B.

Since the line bundle $\mathcal{L}_w(\lambda \boxtimes \mu)$ on the G-space $\tilde{X}_w$ is a G-equivariant line bundle and the map π is G-equivariant, for any $\bar{g} = g \bmod B \in G/B$, the vector spaces $H^q(\pi^{-1}(\bar{g}), \mathcal{L}_w(\lambda \boxtimes \mu)|_{\pi^{-1}(\bar{g})})$ (for any q, in particular for $q=0$) are mutually isomorphic. In particular they have the same dimension. So, by the semi-continuity theorem [H; Chap. III, Corollary 12.9], $\mathcal{S} = \pi_* \mathcal{L}_w(\lambda \boxtimes \mu)$ is a locally free sheaf on G/B with stalk, over any point $\bar{g} \in G/B$,

$$(*) \qquad \mathcal{S}_{\bar{g}} = H^0(\pi^{-1}(\bar{g}), \mathcal{L}_w(\lambda \boxtimes \mu)|_{\pi^{-1}(\bar{g})}).$$

Moreover, from the description $(*)$ of $\mathcal{S}_{\bar{g}}$, it is not difficult to see that the locally free sheaf (vector bundle) $\mathcal{S}$ on G/B is associated to the representation of B on $H^0(\pi^{-1}(\bar{e}), \mathcal{L}_w(\lambda \boxtimes \mu)|_{\pi^{-1}(\bar{e})})$, where e is the identity of G (cf. [A_1; Lemma 3.3]). But since $\tilde{d}_w(\pi^{-1}(\bar{e})) = e \times X_w \subset G/B \times G/B$ (cf. §1.1), $H^0(\pi^{-1}(\bar{e}), \mathcal{L}_w(\lambda \boxtimes \mu)|_{\pi^{-1}(\bar{e})})$ can be canonically identified (as B-modules) with $M_w := \mathbb{C}_{-\lambda} \otimes H^0(X_w, \mathcal{L}_w(\mu))$, where the line bundle $\mathcal{L}_w(\mu)$ is the restriction of $\mathcal{L}(\mu)$ to X_w. Summing up; we have obtained that the direct image sheaf $\pi_*(\mathcal{L}_w(\lambda \boxtimes \mu))$ is a locally free sheaf on G/B associated to the B-module M_w. (Exactly similar argument also proves that the higher direct image sheaf $R^q \pi_*(\mathcal{L}_w(\lambda \boxtimes \mu))$ on G/B is locally free and is associated to the B-module $\mathbb{C}_{-\lambda} \otimes H^q(X_w, \mathcal{L}_w(\mu))$.)

This gives the following:

$$(\mathrm{I}_1) \qquad H^0(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)) \approx H^0(G/B, \mathcal{M}_w),$$

where $\mathscr{M}_w$ is the locally free sheaf on G/B associated to the B-module M_w. Now by Theorem (2.1) (in fact only its special case for $p=0$; which is a trivial consequence of Peter-Weyl theorem), we get by (I_1):

$$H^0(\tilde{X}_w, \mathscr{L}_w(\lambda\boxtimes\mu)) \approx \sum_{\theta\in D} V(\theta)^*\otimes[H^0(\mathfrak{n}, V(\theta)\otimes M_w)]^{\mathfrak{h}}$$
$$\approx \sum_{\theta\in D} V(\theta)^*\otimes \mathrm{Hom}_{\mathfrak{b}}(M_w^*, V(\theta)).$$

Now the theorem follows from the fact that

$$(\mathrm{I}_2) \qquad H^0(X_w, \mathscr{L}_w(\mu))^* \approx V_w(\mu),$$

which of course is a consequence of Demazure character formula. (Observe that the assumption, that μ is dominant, has been used here.) □

As a consequence of the above Theorem (2.2) and Theorem (1.5)(c), we get the following interesting corollary:

(2.3) **Corollary.** *For any $v\leqq w$ and any λ, μ, $\theta\in D$; the canonical restriction map:* $\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda\otimes V_w(\mu), V(\theta))\to \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda\otimes V_v(\mu), V(\theta))$ *is surjective.* □

(2.4) **Proposition.** *For any $w\in W$ and λ, $\mu\in D$;* $\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda\otimes V_w(\mu), V(\overline{\lambda+w\mu}))$ *(cf. § 0) is one dimensional (over $\mathbb{C}$).* □

Before we can prove the above proposition, we recall the following result due to Joseph:

(2.5) **Theorem** $[\mathrm{J}_1; §3.5]$. *For any $w\in W$ and $\mu\in D$; the ($U(\mathfrak{n})$-module) map: $U(\mathfrak{n})\to V_w(\mu)$, defined by $X\mapsto X\cdot e_{w\mu}$, has kernel precisely equal to the left $U(\mathfrak{n})$-ideal*

$$\sum_{\alpha\in\Delta_+} U(\mathfrak{n})\cdot X_\alpha^{k_\alpha+1},$$

where X_α is any non-zero root vector in $\mathfrak{g}$ corresponding to the root α and k_α is defined as follows:

$$k_\alpha = k_\alpha^\mu(w) = 0, \qquad \text{if } \langle\alpha^v, w\mu\rangle \geqq 0$$
$$= -\langle\alpha^v, w\mu\rangle, \qquad \text{otherwise.} \quad □$$

(2.6) *Remark.* Actually Joseph proves the above theorem for $V_w(\mu)$ replaced by $\mathscr{D}_w(\mathbb{C}_\mu)$ (the notation $\mathscr{D}_w(\mathbb{C}_\mu)$ is as in $[\mathrm{J}_1]$); but by the normality of X_w, $\mathscr{D}_w(\mathbb{C}_\mu)\approx V_w(\mu)$.

(2.7) *Proof of Proposition* (2.4). Since $V_w(\mu)$ is a $U(\mathfrak{n})$-cyclic module generated by the element $e_{w\mu}$ (of weight $w\mu$); $\mathbb{C}_\lambda$ is of weight λ; and the $\lambda+w\mu$ weight space in $V(\overline{\lambda+w\mu})$ is one dimensional, we clearly have $\dim \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda\otimes V_w(\mu), V(\overline{\lambda+w\mu})\leqq 1$.

Now define a $U(\mathfrak{n})$-map

$$\sigma: U(\mathfrak{n})\to V(\overline{\lambda+w\mu}) \quad \text{by} \quad X\mapsto X\cdot e_{\lambda+w\mu},$$

where $e_{\lambda+w\mu}$ is some fixed non-zero vector of weight $\lambda+w\mu$ in $V(\overline{\lambda+w\mu})$. We also have a surjective map $\tau: U(\mathfrak{n}) \to \mathbb{C}_\lambda \otimes V_w(\mu)$, defined by

$$X \mapsto X\cdot(1_\lambda \otimes e_{w\mu}), \quad \text{where } 1_\lambda \text{ is some fixed non-zero element in } \mathbb{C}_\lambda.$$

To prove the proposition; it suffices to show that the map σ factors through τ to give the map (call) σ'. (Observe that the map σ' will automatically be an $\mathfrak{h}$-module map, since the weight of $1_\lambda \otimes e_{w\mu} = \lambda + w\mu =$ weight of $e_{\lambda+w\mu}$.)

Define, for any $\alpha \in \Delta_+$,

$$\begin{aligned} k'_\alpha &= 0, && \text{if } \langle \alpha^v, \lambda+w\mu\rangle \geqq 0 \\ &= -\langle \alpha^v, \lambda+w\mu\rangle, && \text{otherwise.} \end{aligned}$$

To obtain the map σ', it suffices to show (by Theorem (2.5)) that $k_\alpha \geqq k'_\alpha$, for all $\alpha \in \Delta_+$. But it follows trivially since $\langle \alpha^v, \lambda\rangle \geqq 0$ (as λ is dominant, by assumption). $\square$

Fix any $\lambda, \mu \in D$. By the Borel-Weil theorem, there is a G-module (in fact a $G\times G$-module) isomorphism ξ

$$\begin{array}{ccc} [V(\lambda)\otimes V(\mu)]^* & \xrightarrow{\xi} & H^0(G/B\times G/B, \mathscr{L}(\lambda\boxtimes\mu)) \\ & \searrow^{\sim} & \wr\wr \\ & & H^0(G/B, \mathscr{L}(\lambda))\otimes H^0(G/B, \mathscr{L}(\mu)), \end{array}$$

got by tensoring the Borel-Weil isomorphisms: $V(\lambda)^* \to H^0(G/B, \mathscr{L}(\lambda))$ and $V(\mu)^* \to H^0(G/B, \mathscr{L}(\mu))$. On composition with the canonical restriction map: $H^0(G/B\times G/B, \mathscr{L}(\lambda\boxtimes\mu)) \to H^0(\tilde{X}_w, \mathscr{L}_w(\lambda\boxtimes\mu))$, we get a G-module map

$$\xi_w: [V(\lambda)\otimes V(\mu)]^* \to H^0(\tilde{X}_w, \mathscr{L}_w(\lambda\boxtimes\mu)).$$

Since $\widetilde{BwB/B} := G\times_B (BwB/B)$ sits as a (Zariski) open subset of $\tilde{X}_w$, the following lemma is trivial to prove:

(2.8) **Lemma.** *Ker* $\xi_w = \{f \in [V(\lambda)\otimes V(\mu)]^*: f|_{U(\mathfrak{g})\cdot(e_\lambda\otimes e_{w\mu})} \equiv 0\}$, *where* $U(\mathfrak{g})\cdot(e_\lambda \otimes e_{w\mu})$ *denotes the* $U(\mathfrak{g})$*-span of the vector* $e_\lambda\otimes e_{w\mu}$ *in* $V(\lambda)\otimes V(\mu)$.

Recall that e_λ (resp. $e_{w\mu}$) is some non-zero extremal weight vector in $V(\lambda)$ of weight λ (resp. in $V(\mu)$ of weight $w\mu$). $\square$

By Theorem (1.5)(c) (taking $w=w_0$ and $v=w$), the map ξ_w is surjective and hence dualizing the above lemma, we get the following crucial:

(2.9) **Proposition.** *For any* $w\in W$ *and* $\lambda, \mu \in D$;

$$H^0(\tilde{X}_w, \mathscr{L}_w(\lambda\boxtimes\mu))^* \approx U(\mathfrak{g})\cdot(e_\lambda\otimes e_{w\mu}) \hookrightarrow V(\lambda)\otimes V(\mu). \quad \square$$

Now combining Theorem (2.2) with Propositions (2.4) and (2.9), we get the following one of the main theorems of this paper; which was the strengthened PRV conjecture (C_2) due to Kostant:

(2.10) **Theorem.** *For any finite dimensional semi-simple Lie algebra* $\mathfrak{g}$, *any* $\lambda, \mu \in D$, *and* $w \in W$; *the irreducible* $\mathfrak{g}$-*module* $V(\overline{\lambda + w\mu})$ *(with extremal weight* $\lambda + w\mu$) (cf. § 0) *occurs with multiplicity exactly one inside the* $\mathfrak{g}$-*submodule* $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$ *of* $V(\lambda) \otimes V(\mu)$, *where* $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$ *is as defined in Lemma* (2.8).

In particular, the $\mathfrak{g}$-*module* $V(\overline{\lambda + w\mu})$ *occurs with multiplicity at least one in* $V(\lambda) \otimes V(\mu)$. □

As a particular case of the above theorem (taking w to be the longest element w_0 of W), of course, one gets the following result due to PRV:

(2.11) **Theorem** [PRV; Corollary 1 to Theorem 2.1]. $V(\lambda + w_0 \mu)$ *occurs with multiplicity exactly one in* $V(\lambda) \otimes V(\mu)$. □

(2.12) *Remark.* The stronger question, whether $V(\overline{\lambda + w\mu})$ occurs with multiplicity exactly one inside $V(\lambda) \otimes V(\mu)$ for any $w \in W$, has a negative answer, as D.N. Verma has observed. Such counter-examples already exist for $\lambda = \mu = \rho$ and for any rank-2 group (see below for $G = SL(3)$ and G_2). In fact verma has made the following conjecture:

For any $\lambda \in \mathfrak{h}^*$, *let* $W_\lambda = \{w \in W : w\lambda = \lambda\}$ *be the stabilizer. Fix* $\lambda, \mu \in D$; *and consider the map* $\eta: W_\lambda \backslash W / W_\mu \to D$, *defined by* $\eta(w) = \overline{\lambda + w\mu}$, *for any* $w \in W$. *With these notations, he conjectures that, for any* $w \in W$,

$$(*) \qquad m_{\eta(w)} \geqq \# \eta^{-1}(\eta(w)),$$

where $m_{\eta(w)}$ *denotes the multiplicity of* $V(\eta(w))$ *in the decomposition of* $V(\lambda) \otimes V(\mu)$.

We now reproduce the complete decomposition of $V(\rho) \otimes V(\rho) = \sum_{v \in D} m_v V(v)$, in the cases of $G = SL(3)$ and G_2, in the following two tables; where χ_i denotes the i-th fundamental weight: $\chi_i(\alpha_j^v) = \delta_{i,j} (1 \leqq i, j \leqq l = 2)$.

(a) $G = SL(3)$

v	2ρ	$3\chi_2$	$3\chi_1$	ρ	0
m_v	1	1	1	2	1
$\#\eta^{-1}(v)$	1	1	1	2	1

(b) $G = G_2$

v	2ρ	$3\chi_2$	$5\chi_1$	$3\chi_1+\chi_2$	$\chi_1+2\chi_2$	$4\chi_1$	$2\chi_1+\chi_2$	$2\chi_2$	$3\chi_1$	ρ	$2\chi_1$	χ_2	χ_1	0
m_v	1	1	1	2	1	2	3	2	3	2	2	2	1	1
$\#\eta^{-1}(v)$	1	1	1	2	0	1	0	1	0	2	0	1	1	1

Observe that, by the table for G_2, one can not expect equality in (*). Also there are components $V(v)$ in $V(\rho) \otimes V(\rho)$ (for G_2), such that v is not in the image of η. □

The following proposition shows that $V(\overline{\lambda + w\mu})$ occurs 'for the first time' in $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$, provided λ and μ both are regular. More precisely, we have:

(2.13) **Proposition.** *Assume that* $\lambda, \mu \in D$ *are both regular, i.e.,* $\lambda(\alpha_i^\vee), \mu(\alpha_i^\vee) \geqq 1$, *for all the simple co-roots* $\alpha_i^\vee$. *Fix* $w \in W$. *Then the* $\mathfrak{g}$*-module* $V(\overline{\lambda + w\mu})$ *does not occur in* $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{v\mu})$, *for any* $v < w$.

Proof. By Theorem (2.2), Corollary (2.3), and Propositions (2.4) and (2.9); it suffices to show that the map σ': $\mathbb{C}_\lambda \otimes V_w(\mu) \to V(\overline{\lambda + w\mu})$, constructed in § 2.7, reduces to the zero map on $\mathbb{C}_\lambda \otimes V_v(\mu) \hookrightarrow \mathbb{C}_\lambda \otimes V_w(\mu)$. Further since $\mathbb{C}_\lambda \otimes V_v(\mu)$ is $U(\mathfrak{n})$-cyclic; generated by the vector $1_\lambda \otimes e_{v\mu}$, it suffices to show that $\sigma'(1_\lambda \otimes e_{v\mu}) = 0$:

Now, since $v < w$, we can assume without loss of generality that $v \to w$, i.e., $l(w) = l(v) + 1$ and there exists a (positive) root β, such that $v_\beta \cdot v = w$ (where v_β denotes the reflection corresponding to β, i.e., $v_\beta(\chi) = \chi - \langle \chi, \beta^\vee \rangle \beta$, for any $\chi \in \mathfrak{h}^*$). As in Joseph [J_1, § 3.7], $X_\beta^{\langle v\mu, \beta^\vee \rangle} e_{w\mu} = e_{v\mu}$ (upto a non-zero scalar multiple). Observe that $\langle v\mu, \beta^\vee \rangle > 0$; by [BGG; Corollary 2.3(ii)] and the assumption that μ is dominant regular. So finally it suffices to show that $X_\beta^{\langle v\mu, \beta^\vee \rangle} e_{\overline{\lambda + w\mu}} = 0$:

But, since λ is dominant regular, $\langle v\mu, \beta^\vee \rangle \geqq k'_\beta + 1$, where (as in § 2.7)

$$\begin{aligned} k'_\beta &= 0, && \text{if } \langle \lambda + w\mu, \beta^\vee \rangle \geqq 0 \\ &= -\langle \lambda + w\mu, \beta^\vee \rangle, && \text{otherwise.} \end{aligned}$$

This proves the proposition. □

Since $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$ is a $\mathfrak{g}$ (in particular a T)-module, its formal character (over T) ch $(U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu}))$ makes sense. The following theorem determines the character. It can be thought of as a '$\mathfrak{g}$-analogue' of the Demazure character formula.

Let us denote by $R(T) := \mathbb{Z}[X(T)]$ the group algebra of the character group $X(T)$, i.e., $R(T)$ is the representation ring of the torus T. The ring $R(T)$ has a conjugation; given by $\overline{e^\chi} = e^{-\chi}$, for any $e^\chi \in X(T)$. Recall the definition of the Demazure operators D_w: $R(T) \to R(T)$ (for any $w \in W$) (in the form convenient for our purposes) from [A_2; § 4] or [Ku; § 3].

(2.14) **Theorem.** *For any integral weights* λ, μ *(not necessarily dominant) and any* $w \in W$, *we have:*

$$\chi(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu)) = \bar{D}_{w_0}(e^\lambda \cdot D_w(e^\mu)),$$

where w_0 *is the longest element of* W; $\chi(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu))$ *denotes* $\sum_p (-1)^p \operatorname{ch} H^p(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu))$; *and* $\bar{D}_w(e^\chi)$ *denotes* $\overline{D_w(e^\chi)}$.

In particular, for any $\lambda, \mu \in D$ *and* $w \in W$;

$$\operatorname{ch}(U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})) = D_{w_0}(e^\lambda \cdot D_w(e^\mu)).$$

Proof. By the Leray spectral sequence, for the canonical map π_w: $\tilde{X}_w \to G/B$, we get (see the proof of Theorem 2.2):

$$\chi(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu)) = \sum_{p,q} (-1)^{p+q} \operatorname{ch} H^p(G/B, \mathscr{M}_w(q)),$$

where $\mathscr{M}_w(q)$ is the locally free sheaf on G/B associated to the B-module $M_w(q)$ $:=\mathbb{C}_{-\lambda}\otimes H^q(X_w, \mathscr{L}_w(\mu))$. (As in the proof of Theorem (2.2) the line bundle $\mathscr{L}_w(\mu)$ is the restriction of $\mathscr{L}(\mu)$ to X_w.)

Now by the Demazure character formula (see, e.g., [A_2; Theorem (4.3)] or [S; Theorem (3)]):

$$\begin{aligned}\chi(\tilde{X}_w, \mathscr{L}_w(\lambda\boxtimes\mu)) &= \sum_q (-1)^q \sum_p (-1)^p \operatorname{ch} H^p(G/B, \mathscr{M}_w(q)) \\ &= \sum_q (-1)^q \bar{D}_{w_0}(\overline{\operatorname{ch} M_w(q)}) \\ &= \bar{D}_{w_0}(e^\lambda \cdot \overline{\chi(X_w, \mathscr{L}_w(\mu))}) \\ &= \bar{D}_{w_0}(e^\lambda \cdot D_w(e^\mu)).\end{aligned}$$

(Notice that our conventions are somewhat different from Andersen's; in that he calls the roots of B as negative. This has the effect that, in our notation, $\chi(X_w, \mathscr{M}^*) = \bar{D}_w(\operatorname{ch} M)$, for any B-module M.)

Second part of the theorem follows from Theorem (1.5)(b) together with Proposition (2.9). □

(2.15) *Remark.* As a particular case of the above theorem (taking $w=w_0$), we recover the following result due to Brauer [K_1; § 4.8]:

$$\operatorname{ch}(V(\lambda)\otimes V(\mu)) = D_{w_0}(e^\lambda \cdot (D_{w_0} e^\mu)), \qquad \text{for } \lambda, \mu \in D.$$

3. Identification of $\mathfrak{n}$-homology with the cohomology of $\tilde{X}_w$

For any $\mu\in D$ and $w\in W$, recall the definition of the Schubert module $V_w(\mu)$ from § 0. The Lie algebra homology $H_*(\mathfrak{n}, V_w(\mu))$ being a $\mathfrak{b}$-module (since $\mathfrak{n}$ is an ideal in $\mathfrak{b}$ and $V_w(\mu)$ is a $\mathfrak{b}$-module), we can talk of its weight space decomposition. With these notations, we prove the following:

(3.1) **Theorem.** *For any integral weight λ (not necessarily dominant), any $\mu\in D$, and any $w\in W$; we have a vector space isomorphism (for all $p\geqq 0$):*

$$[H_p(\mathfrak{n}, V_w(\mu))]_{-\lambda} \approx [H^p(\tilde{X}_w, \mathscr{L}_w(\lambda\boxtimes\mu))]^G,$$

where $[\ ]_{-\lambda}$ denotes the $-\lambda$-th weight space and $[\ ]^G$ denotes the G-invariants.

Proof. The Leray spectral sequence, corresponding to the map $\pi\colon \tilde{X}_w \to G/B$, has (see the proof of Theorem (2.2)):

$$E_2^{p,q} = H^p(G/B, \mathscr{M}_w(q)),$$

where $\mathscr{M}_w(q)$ is the locally free sheaf, associated to the B-module $M_w(q)$ $:=\mathbb{C}_{-\lambda}\otimes H^q(X_w, \mathscr{L}_w(\mu))$. (As in the proof of Theorem (2.2), $\mathscr{L}_w(\mu)$ stands for the restriction of the line bundle $\mathscr{L}(\mu)$ to X_w.) But by the cohomology vanishing theorem [RR; Theorem (2), § 3] (since μ is dominant), $H^q(X_w, \mathscr{L}_w(\mu))=0$, unless

$q=0$. Hence $E_2^{p,q}=0$, unless $q=0$. In particular, the spectral sequence degenerates at the E_2-term itself. So we get:

$$(\mathrm{I}_3) \qquad H^p(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu)) \approx H^p(G/B, \mathscr{M}_w),$$

where $\mathscr{M}_w$ is the locally free sheaf associated to the B-module $\mathbb{C}_{-\lambda} \otimes H^0(X_w, \mathscr{L}_w(\mu))$. Now by Bott's theorem (2.1):

$$H^p(G/B, \mathscr{M}_w) \approx \sum_{\theta \in D} V(\theta) \otimes [H^p(\mathfrak{n}, V(\theta)^* \otimes M_w)]^{\mathfrak{h}}$$

$$\approx \sum_{\theta \in D} V(\theta) \otimes [H^p(\mathfrak{n}, (V(\theta) \otimes \mathbb{C}_\lambda \otimes V_w(\mu))^*)]^{\mathfrak{h}}$$

(since by the Demazure character formula $H^0(X_w, \mathscr{L}_w(\mu))^* \approx V_w(\mu)$), i.e.,

$$(\mathrm{I}_4) \qquad H^p(G/B, \mathscr{M}_w) \approx \sum_{\theta \in D} V(\theta) \otimes [(H_p(\mathfrak{n}, V(\theta) \otimes V_w(\mu)) \otimes \mathbb{C}_\lambda)^*]^{\mathfrak{h}}$$

(since for any $\mathfrak{n}$-module N, $H_p(\mathfrak{n}, N)^* \approx H^p(\mathfrak{n}, N^*)$, and $\mathfrak{n}$ acts trivially on $\mathbb{C}_\lambda$).

Hence, by combining (I_3) and (I_4), we get:

$$H^p(\tilde{X}_w, \mathscr{L}_w(\lambda \boxtimes \mu)) \approx \sum_{\theta \in D} V(\theta) \otimes [H_p(\mathfrak{n}, V(\theta) \otimes V_w(\mu))_{-\lambda}]^*.$$

Now collecting the isotypical components in both the sides, corresponding to the trivial $\mathfrak{g}$-module $V(0)$, and observing that $H_p(\mathfrak{n}, V(\theta) \otimes V_w(\mu))$ is finite dimensional, we get the theorem. □

(3.2) *Remarks.* (a) If we specialize the above theorem to the case when $w=w_0$, we can easily deduce Kostant's theorem on $\mathfrak{n}$-cohomology (with weight space structure as well) [K_2; Theorem (5.14)] from the Borel-Weil-Bott theorem [K_2; Theorem (6.4)] and conversely, i.e., we can deduce Borel-Weil-Bott theorem from Kostant's theorem.

(b) One can easily formulate and prove an analogue of the above Theorem (3.1) for the Borel subgroup B replaced by any parabolic subgroup P.

(c) Our interest in proving the above theorem arose mainly from a recent paper of Joseph [J_2], where he has initiated the study of $H_*(\mathfrak{n}, V_w(\mu))$. We believe that our Theorem (3.1) can be used to study $H_*(\mathfrak{n}, V_w(\mu))$. We hope to come back to this question in a subsequent work.

(3.3) *Remark.* I announced to a group of mathematicians (which included O. Mathieu) at TIFR, on August 13, 1987, that I proved the PRV conjecture (Theorem (2.10)) using Theorems (1.5); (2.1); and (2.5) as the main ingredients. Even though Mathieu did not have a proof of the PRV conjecture at that time, he told me the next day that he also has obtained a proof. I have not seen his paper till now, nor do I have any idea of how different his proof is from mine.

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