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Extension of the Category $\mathcal{O}^g$ and a Vanishing Theorem for the Ext Functor for Kac–Moody Algebras

SHRAWAN KUMAR

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts and Tata Institute of Fundamental Research, Bombay, India

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INTRODUCTION

We extend the equivalence relation $\approx$ (resp. the category $\mathcal{O}^g$) (defined in [DGK]) from K^g to an, in general, much larger subset $K^{\text{w.g.}} \subset \mathfrak{h}^*$ (resp. to the category $\mathcal{O}^{\text{w.g.}}$), where w.g. stands for "weakly good". We denote the extended equivalence relation by $\approx^\circ$. We prove that the equivalence classes in $K^{\text{w.g.}}$ under $\approx^\circ$ have description exactly similar to that for the equivalence classes in K^g under $\approx$, given by Deodhar–Gabber–Kac. We also prove that their category decomposition theorem for $\mathcal{O}^g$ can be extended to $\mathcal{O}^{\text{w.g.}}$. We further prove that $\text{Ext}^*_{\mathfrak{g}}(M, N) = 0$, if M and $N \in \mathcal{O}$ (resp. in $\mathcal{O}^{\text{w.g.}}$) are of inequivalent $\sim$ (resp. $\approx^\circ$) types.

Now we describe the contents in more detail. *Unless otherwise stated,* $\mathfrak{g}$ *will denote an arbitrary symmetrizable Kac–Moody Lie algebra with its Cartan subalgebra* $\mathfrak{h}$.

Section 1. Fix λ, $\mu \in \mathfrak{h}^*$. Let us say that λ is related to μ if the irreducible module $L(\lambda)$ is a subquotient of the Verma module $M(\mu)$ and denote by $\sim$ the equivalence relation thus generated. Bernstein–Gelfand–Gelfand, using central character theory, gave a necessary and sufficient condition for λ to be related to μ, in the case when $\mathfrak{g}$ is finite dimensional [BGG]. Later Jantzen [J] simplified their proof, by making use of the Shapovalov bilinear form. An important aspect (specially for the infinite dimensional Lie algebraists) of Jantzen's simplification was that it did not use central character theory. Later Kac–Kazhdan generalized Jantzen's arguments to prove an analogue of the BGG theorem for arbitrary $\mathfrak{g}$ [KK]. It is clear from the description of Kac–Kazhdan [KK, Sect. 4, Theorem 2] that the equivalence classes under $\sim$ (due to the existence of imaginary roots in the infinite dim case) are fairly complicated to describe, e.g., it is no more true

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that $\lambda \sim \mu$ implies that there exists a $w \in W$ such that $w * \lambda = \mu$ ($w * \lambda$ is defined to be $w(\lambda + \rho) - \rho$). Subsequently, Deodhar-Gabber-Kac showed [DGK, Sect. 5] that if we take $\mu \in K^g$ (the Tits cone) $\subset \mathfrak{h}^*$ and take λ related to μ then λ is again in K^g and there exists $w \in W$ such that $w * \lambda = \mu$. (Thus the highest weights of irreducible subquotients of $M(\mu)$, $\mu \in K^g$, are given exactly by what one would expect if central character theory were available.)

The principal aim in the first section (of this paper) is to show that there is a subset $K^{\text{w.g.}}$, defined in Section (1.3), of $\mathfrak{h}^*$ ($K^{\text{w.g.}} \supset K^g$), such that if $\mu \in K^{\text{w.g.}}$ and $\lambda \in \mathfrak{h}^*$ is related to μ then $\lambda \in K^{\text{w.g.}}$ and there exists $w \in W(\mu)$ satisfying $w * \lambda = \mu$ ($W(\mu)$ is defined in Section (1.8)). From this we deduce that if we define an equivalence relation $\approx^\circ$ in $K^{\text{w.g.}}$ (Sect. (1.6)) (analogous to the definition of $\approx$ in K^g, given by [DGK, Sect. 5]), then (for $\lambda, \mu \in K^{\text{w.g.}}$) $\lambda \approx^\circ \mu$ iff there exists $w \in W(\mu)$ satisfying $w * \lambda = \mu$. This is content of our Theorem (1.7) and its Corollary (1.8). Our proof (of Theorem (1.7)) is along the lines of Jantzen [J] or Kac–Kazhdan [KK]. But we would like to mention a sort of "Localization" which is implicit in our proof. I believe that this idea of localization may be of interest elsewhere, e.g., in studying the nonsymmetrizable case.

Further, $K^{\text{w.g.}}$ is the complement (in $\mathfrak{h}^*$) of a union of (at the most) countably many hyperplanes (e.g., in the affine, including twisted affine, case $K^{\text{w.g.}}$ is complement of a single hyperplane), whereas K^g is, in general, only a "half" space. See Lemma (1.5).

We further introduce the category $\mathcal{O}^{\text{w.g.}}$ (similar to $\mathcal{O}^g$ introduced by DGK) as the full subcategory of $\mathcal{O}$ consisting of all those modules M such that all the irreducible subquotients of M have highest weights $\in K^{\text{w.g.}}$. We extend the category decomposition theorem [DGK, Theorem 5.7] to the whole of $\mathcal{O}^{\text{w.g.}}$. This is our Corollary 2.13(a). Introducing $K^{\text{w.g.}}$ has two immediate consequences. One; we show (in Proposition (1.10)) that in the affine (including twisted affine) case $K^{\text{w.g.}}$ is stable under $\sim$ and hence if $\lambda, \mu \in K^{\text{w.g.}}$ then $\lambda \sim \mu$ iff $\lambda \approx^\circ \mu$. In particular, if $\lambda, \mu \in K^g$ and $\lambda \sim \mu$ then $\lambda \approx \mu$. This settles a question [DGK, Remark 5.5] in affirmative, for the affine case. We show, by an explicit example, that this is false in general. Our counter-example is in the case of hyperbolic rank 2 Lie algebras. See Example (1.12) and the Remarks (1.13). Another application (though of similar nature) is in proving a vanishing theorem (Corollaries 2.13(b)) in homology (which was conjectured in [Ku]).

Section 2. The main result here is Theorem (2.2). This asserts that, for all $n \geqslant 0$, $\operatorname{Ext}^n_{\mathfrak{g}}(M, N) = 0$ if $M, N \in \mathcal{O}$ (resp. $\in \mathcal{O}^{\text{w.g.}}$) are of inequivalent $\sim$ (resp. $\approx^\circ$) types (see Definition (2.1)). This theorem, for the particular value of $n = 1$, is one of the main results of [DGK].

Their decomposition theorems (Theorems (4.2) and (5.7)) will prove that (under the assumptions of our theorem on M and N) $\mathrm{Ext}^n_{\mathcal{O}}(M, N)$ (resp. $\mathrm{Ext}^n_{\mathcal{O}^g}(M, N)) = 0$. ($\mathrm{Ext}_{\mathcal{O}}$ denotes the Ext functor in the category $\mathcal{O}$, as defined by Buchsbaum [B].) But we do not know if, for arbitrary $\mathfrak{g}$, the following holds:

(1) $\mathrm{Ext}^n_{\mathcal{O}}(M, N) \approx \mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{h})}(M, N)$ for $M, N \in \mathcal{O}$ and

(2) $\mathrm{Ext}^n_{\mathcal{O}^g}(M, N) \approx \mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{h})}(M, N)$ for $M, N \in \mathcal{O}^g$.

(For finite dim $\mathfrak{g}$, this (of course) is known to be true.) The validity of (1) and (2) would provide an alternative proof of our vanishing theorem (2.2). One of the difficulties is that the category $\mathcal{O}$ (in contrast with the finite dim case) does not have enough projectives, e.g., it can be seen (by using some results in [RW]) that, in the affine case, $L(-\rho)$ is not the image of any projective object in $\mathcal{O}$.

Theorem (2.2) is used to deduce the category decomposition result (Corollaries (2.13)(a)) for the category $\mathcal{O}^{\text{w.g.}}$. As another consequence (of Theorem 2.2)), we deduce some vanishing theorems for $H_*(\mathfrak{g}, M)$ and $H^*(\mathfrak{g}, M)$ (Corollaries 2.13(b)). In particular, one can immediately deduce the vanishing of $H_*(\mathfrak{g}, L(\lambda))$ and $H^*(\mathfrak{g}, L(\lambda))$, for $L(\lambda)$ an integrable highest weight module with $\lambda \neq 0$ (a result due to M. Duflo) and the vanishing of $H_*(g, M)$ and $H^*(\mathfrak{g}, M)$, for $M \in \mathcal{O}$ provided the Casimir acts as an automorphism on M. The vanishing of $H_*(\mathfrak{g}, M)$, in this case, is due to [Ku, Theorem 1.2].

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NOTATIONS

Unless otherwise stated $\mathfrak{g}$, throughout, will denote an arbitrary symmetrizable Kac–Moody Lie algebra/$\mathbb{C}$, $U(\mathfrak{g})$ its universal enveloping algebra and W will denote its Weyl group, as defined in [K, Chap. 3]. *We will use the same notations as in* [Ku, Section 0], in particular, recall that, σ is an invariant nondegenerate symmetric bilinear form on $\mathfrak{h}^*$; for a vector space V over $\mathbb{C}$, V^* will denote the full dual $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$; for a left $U(\mathfrak{g})$-module M, by M' we will mean the right $\mathfrak{g}$-module with the underlying space being the same as M and the action being $m \cdot a = T(a) \cdot m$ for $m \in M$ and $a \in U(\mathfrak{g})$, where T is the unique anti-automorphism of $U(\mathfrak{g})$ which is -1 on $\mathfrak{g}$. Modules will be left unless explicitly stated. The symbol $\otimes$ without a subscript will mean tensor product over $\mathbb{C}$ and $\mathbb{N}$ will denote the set of positive integers. For a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{a}$, and (left) $\mathfrak{g}$-modules M, N; by $\mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{a})}(M, N)$ (resp. $\mathrm{Tor}^{(\mathfrak{g},\mathfrak{a})}_n(M', N)$) we will mean $\mathrm{Ext}^n(M, N)$ (resp. $\mathrm{Tor}_n(M', N)$) with respect to the pair (of rings)

$(U(\mathfrak{g}), U(\mathfrak{a}))$, as defined by Hochschild in [H]. As is customary, $\mathrm{Ext}^n_{(\mathfrak{g},0)}$ (resp. $\mathrm{Tor}_n^{(\mathfrak{g},0)}$) will be abbreviated to $\mathrm{Ext}^n_{\mathfrak{g}}$ (resp. $\mathrm{Tor}_n^{\mathfrak{g}}$). We further define (as usual)

$$H_n(\mathfrak{g}, \mathfrak{a}, M) = \mathrm{Tor}_n^{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}, M) \approx \mathrm{Tor}_n^{(\mathfrak{g},\mathfrak{a})}(M', \mathbb{C})$$

and

$$H^n(\mathfrak{g}, \mathfrak{a}, M) = \mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{a})}(\mathbb{C}, M).$$

We also fix $\rho \in \mathfrak{h}^*$, satisfying $\rho(h_i) = 1$ for all the simple coroots h_i, $1 \leqslant i \leqslant l$. For $\lambda \in \mathfrak{h}^*$, we denote by $M(\lambda)$ the Verma module (for $\mathfrak{g}$) with highest weight λ and $L(\lambda)$ the (unique) irreducible quotient of $M(\lambda)$.

1. An Extension of the Category $\mathcal{O}^g$

(1.1) *A bilinear form on $U(\mathfrak{g})$ with values in $S(\mathfrak{h})$, given by Shapovalov.* Since $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, one obtains the decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes S(\mathfrak{h}) \otimes U(\mathfrak{n})$, where $S(\mathfrak{h}) = U(\mathfrak{h})$ is the symmetric algebra of $\mathfrak{h}$. In particular, $U(\mathfrak{g})$ can be expressed as a direct sum of two subspaces

$$U(\mathfrak{g}) = S(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}). \qquad (\mathrm{I}_1)$$

Let Γ (resp. Γ^+) be the lattice (resp. semigroup) $\subseteq \mathfrak{h}^*$, generated by the simple roots $\{\alpha_1, ..., \alpha_l\}$, i.e.,

$$\Gamma = \sum_{i=1}^{l} \mathbb{Z}\alpha_i$$

and

$$\Gamma^+ = \sum_{i=1}^{l} \mathbb{Z}^+\alpha_i,$$

where $\mathbb{Z}^+$ is the set of nonnegative integers.

The Γ-gradation on $\mathfrak{g}$, of course, induces a Γ-gradation on $U(\mathfrak{g})$. For any $\beta \in \Gamma$, we denote by $U(\mathfrak{g})_\beta$ the βth graded component of $U(\mathfrak{g})$.

We denote by H: $U(\mathfrak{g}) \to S(\mathfrak{h})$, the projection of $U(\mathfrak{g})$ on the $S(\mathfrak{h})$-factor, given by (I_1).

Shapovalov has introduced [S, Sect. 2] a symmetric bilinear form A on $U(\mathfrak{g})$ with values in $S(\mathfrak{h})$, defined by

$$A(x, y) = H(\theta(x) \cdot y), \qquad \text{for} \quad x, y \in U(\mathfrak{g}),$$

where θ is the unique involutive anti-automorphism of $U(\mathfrak{g})$ satisfying $\theta(e_i) = f_i$ and $\theta|_{\mathfrak{h}} = \mathrm{Id}$.

Clearly $A(U(\mathfrak{g})_{\beta_1}, U(\mathfrak{g})_{\beta_2})=0$ if $\beta_1 \neq \beta_2$.

For $\beta \in \Gamma^+$, we denote by A_β the restriction of A to $U(\mathfrak{n}^-)_{-\beta} \otimes U(\mathfrak{n}^-)_{-\beta}$, where $U(\mathfrak{n}^-)_{-\beta} = U(\mathfrak{n}^-) \cap (U(\mathfrak{g})_{-\beta})$.

The following basic theorem is due to Shapovalov [S, Sect. 2] in the finite dimensional case and due to Kac–Kazhdan [KK, Theorem 1] in the infinite dimensional case.

(1.2) THEOREM. *Let $\mathfrak{g}$ be any symmetrizable Kac–Moody Lie algebra. Fix $\beta \in \Gamma^+$. We denote by* $\det A_\beta^-$ *the determinant of the symmetric bilinear form A_β^-: $U(\mathfrak{n}^-)_{-\beta} \times U(\mathfrak{n}^-)_{-\beta} \to S(\mathfrak{h})$, with respect to some $\mathbb{C}$-basis of $U(\mathfrak{n}^-)_{-\beta}$ (so that* $\det A_\beta^-$ *is determined only up to a (nonzero) constant complex multiple). Then*

$$\det A_\beta^- = \prod_{\alpha \in \Delta_+} \prod_{n=1}^{\infty} \left[h_\alpha + \rho(h_\alpha) - n \frac{\sigma(\alpha, \alpha)}{2} \right]^{P(\beta - n\alpha)},$$

where P denotes the Kostant partition function and for $\alpha = \sum_{i=1}^{l} n_i \alpha_i$, $h_\alpha = \sum_{i=1}^{l} n_i (\sigma(\alpha_i, \alpha_i)/2)\, h_i$. Further the roots $\alpha \in \Delta_+$, in the above expression of $\det A_\beta^-$, *are taken as many times as their multiplicities.*

(1.3) DEFINITION (An extension of Tits cone). For any $\alpha \in \Delta_+^{\mathrm{im}}$ (Δ_+^{im} denotes the set of all the positive imaginary roots), let us consider the set

$$S_\alpha = \left\{ \lambda \in \mathfrak{h}^* : \sigma(\lambda + \rho, \alpha) = \frac{\sigma(\alpha, \alpha)}{2} \right\}$$

and define $S = \bigcup_{\alpha \in \Delta_+^{im}} S_\alpha$. We now define $K^{\mathrm{w.g.}} = \mathfrak{h}^* \backslash S$ (w.g. stands for weakly good).

We define a shifted action of W on $\mathfrak{h}^*$ as follows. For $w \in W$ and $\lambda \in \mathfrak{h}^*$, put

$$w * \lambda = w \cdot (\lambda + \rho) - \rho$$

(We reserve the notation $w*$ for this shifted action, to distinguish it from the usual action of W on $\mathfrak{h}^*$ denoted by $w \cdot \lambda$.)

It can be easily seen that

(a) $K^{\mathrm{w.g.}} \supset K^g$ (K^g is defined in [DGK, Sect. 5])

(b) S (and hence $K^{\mathrm{w.g.}}$) is stable under the shifted W-action.

Now we define a full subcategory $\mathcal{O}^{\mathrm{w.g.}}$ of $\mathcal{O}$ such that

$$\mathcal{O} \supset \mathcal{O}^{\mathrm{w.g.}} \supset \mathcal{O}^g$$

and $\mathcal{O}^{\mathrm{w.g.}}$ shares many important properties of $\mathcal{O}^g$.

(1.4) DEFINITION. We define a full subcategory $\mathcal{O}^{\mathrm{w.g.}}$ of $\mathcal{O}$ consisting of all those modules $M \in \mathcal{O}$ such that all the irreducible subquotients of M (also called components of M) have highest weights $\in K^{\mathrm{w.g.}}$.

Since $K^{\mathrm{w.g.}}$ contains K^g, $\mathcal{O}^{\mathrm{w.g.}}$ contains $\mathcal{O}^g$ as a subcategory ($\mathcal{O}^g$ is defined in [DGK, Sect. 5]). In fact, we will see, in the next lemma, that $\mathcal{O}^{\mathrm{w.g.}}$ is "much" larger than $\mathcal{O}^g$ in general.

(1.5) LEMMA. *A comparison between $K^{\mathrm{w.g.}}$ and K^g is shown in Table* I.

Proof. (a) Follows from [DGK, Proposition 5.2].

(b) The assertion about K^g follows from [DGK, Proposition 5.2; K, Proposition 6.3, p. 66].

The assertion about $K^{\mathrm{w.g.}}$ follows from the description of the imaginary roots, as in [K, Theorem 5.6, p. 53] and the description of the form σ on $\mathfrak{h}^*$ as given in [K, Chap. 6].

(c) The assertion about K^g follows from [K, Exercise 5.20, p. 60], since in this case there are no null roots. The assertion about $K^{\mathrm{w.g.}}$ follows from [M, Sect. 3, Theorem 1] (see Example (1.12)). ∎

Analogous to the definition of the equivalence relation $\approx$ in K^g, as given in [DGK, Sect. 5], we define an equivalence relation $\approx^\circ$ in $K^{\mathrm{w.g.}}$ as follows.

(1.6) DEFINITION. Let $\lambda, \mu \in K^{\mathrm{w.g.}}$. We first define $\lambda \precapprox^\circ \mu$ if the irreducible module $L(\lambda)$ (with highest weight λ) is a subquotient of the Verma module $M(\mu)$. Now define $\approx^\circ$ as the equivalence relation in $K^{\mathrm{w.g.}}$, generated by the relation $\precapprox^\circ$. More explicitly, $\lambda \approx^\circ \mu$ if there exists a finite chain (for some $n > 0$) $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_n = \mu$, with all $\lambda_i \in K^{\mathrm{w.g.}}$ and such that, for all $0 \leqslant i < n$, $\{\lambda_i, \lambda_{i+1}\}$ satisfies

$$\lambda_i \precapprox^\circ \lambda_{i+1} \qquad \text{or} \qquad \lambda_{i+1} \precapprox^\circ \lambda_i.$$

The "concrete" description of the relation $\approx$ in K^g, as given in [DGK, Proposition 5.6], can be extended to $K^{\mathrm{w.g.}}$. This is one of the main motivations behind introducing $K^{\mathrm{w.g.}}$ and the category $\mathcal{O}^{\mathrm{w.g.}}$. More precisely we have the following.

(1.7) THEOREM. *Let $\lambda_0 \in K^{\mathrm{w.g.}}$ and $\mu_0 \in \mathfrak{h}^*$ with $\lambda_0 \neq \mu_0$. Then $L(\mu_0)$ is a subquotient of $M(\lambda_0)$ if and only if there exist positive real roots $\phi_1, \ldots, \phi_n$ $(n \geqslant 1)$ and elements $\lambda_1, \ldots, \lambda_n = \mu_0 \in \mathfrak{h}^*$ satisfying, for all $1 \leqslant i \leqslant n$,*

(1) $\sigma(\lambda_{i-1} + \rho, \phi_i^\vee) \in \mathbb{N}$ *and*

(2) $\gamma_{\phi_i} * \lambda_{i-1} = \lambda_i$,

TABLE I[a]

	$\mathfrak{g}$	K^g	$K^{\text{w.g.}}$
(a)	Finite dimensional	$\mathfrak{h}^*$	$\mathfrak{h}^*$
(b)	Indecomposable affine (including twisted affine)	$\{\lambda \in \mathfrak{h}^*: \operatorname{Re} \sigma(\lambda+\rho, \delta) > 0$ or $\operatorname{Re} \sigma(\lambda+\rho, \delta) = 0$ and $\operatorname{Im} \sigma(\lambda+\rho, \delta) > 0\}$	$\mathfrak{h}^* \setminus \left\{ \sum_{i=0}^{l} \mathbb{C}\alpha_i - \rho \right\}$
(c)	Symmetric hyperbolic rank 2 Case, i.e., $\mathfrak{g} = \mathfrak{g}(H_a)$ where $H_a = \left(\begin{smallmatrix} 2 & -a \\ -a & 2 \end{smallmatrix}\right)$ and $a \geqslant 3$. (If $a \leqslant 2$ then $\mathfrak{g}(H_a)$ is either finite dim. or affine and hence covered above)	$K^g \cap \mathfrak{h}^*_{\mathbb{R}} = -\rho + \{x\alpha_1 + y\alpha_2:$ $x, y \in \mathbb{R}$ and $((a+\sqrt{a^2-4})/2)\, \bar{y} \leqslant x \leqslant ((a-\sqrt{a^2-4})/2)\, y\}$	$\mathfrak{h}^* \setminus \bigcup_{(m,n) \in Z_a} S_{(m,n)}$, where $S_{(m,n)} = \{\lambda \in \mathfrak{h}^*: \lambda(h_1)m + \lambda(h_2)n = m^2 - amn + n^2 - m - n\}$ and $Z_a = \{(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+: m+n > 0$ and $m^2 - amn + n^2 \leqslant 0\}$.

[a] Re (resp. Im) denotes the real (resp. imaginary) part and δ is as defined in [K, Chap. 5].

where, for any real root ϕ, $\phi^\vee$ denotes $2\phi/\sigma(\phi, \phi)$ and γ_ϕ denotes the reflection (through ϕ) $\gamma_\phi(\beta) = \beta - \sigma(\beta, \phi^\vee)\phi$. Of course $\gamma_\phi \in W$.

Since $K^{\text{w.g.}}$ is W-stable (under the shifted action), we get that $\mu_0 \in K^{\text{w.g.}}$. In particular, the Verma module $M(\lambda)$, with highest weight $\lambda \in K^{\text{w.g.}}$, is in the category $\mathcal{O}^{\text{w.g.}}$.

Proof. The proof makes essential use of the Shapovalov bilinear form and its determinant. We could have deduced the theorem from [KK, Theorem 2], but we decided to give a complete proof (not using [KK, Theorem 2], though our proof is quite similar) to bring out a "localization" implicit in our proof. (We localize the Shapovalov determinant in $K^{\text{w.g.}}$.) I feel that this idea of localization may be useful elsewhere, e.g., in studying the nonsymmetrizable case.

Fix any $\lambda \in \mathfrak{h}^*$ and consider the Verma module $M(\lambda)$. Of course,

$\operatorname{ch} M(\lambda) = e^\lambda \sum_{\eta \in \Gamma^+} P(\eta)\, e^{-\eta}$, where $\operatorname{ch} M(\lambda)$ denotes the character of $M(\lambda)$. $\quad (\mathrm{I}_2)$

Fix, once and for all, a $z \in \mathfrak{h}^*$ such that $\sigma(z, \alpha) \neq 0$, for any $\alpha \in \Gamma^+ \setminus \{0\}$. Embed $S(\mathfrak{h}) \subsetneq \mathbb{C}[t] \otimes S(\mathfrak{h})$ by $f \mapsto 1 \otimes f$. Recall, from (1.1), the definition of Shapovalov bilinear form $A\colon U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to S(\mathfrak{h}) \subsetneq \mathbb{C}[t] \otimes S(\mathfrak{h})$ and denote by A^- the restriction of A to $U(\mathfrak{n}^-) \otimes U(\mathfrak{n}^-)$. Let $ev_{\lambda + tz}\colon C[t] \otimes S(\mathfrak{h}) \to \mathbb{C}[t]$ denote the evaluation map, defined by

$$f \otimes \bar{h}_1 \cdots \bar{h}_n \mapsto f \cdot (\lambda(\bar{h}_1) + tz(\bar{h}_1)) \cdot \cdots \cdot (\lambda(\bar{h}_n) + tz(\bar{h}_n)),$$

for $f \in \mathbb{C}[t]$ and any $\bar{h}_i \in \mathfrak{h}$.

Let $\hat{A}_\lambda$ denote the composite map $ev_{\lambda + tz} \circ A^- \circ \psi$

$$M(\lambda) \otimes M(\lambda) \xrightarrow{\psi} U(\mathfrak{n}^-) \otimes U(\mathfrak{n}^-) \xrightarrow{A^-} \mathbb{C}[t] \otimes S(\mathfrak{h}) \xrightarrow{ev_{\lambda+tz}} \mathbb{C}[t],$$

where ψ is induced by the inverse of the map: $U(\mathfrak{n}^-) \to M(\lambda)$, defined by $f \mapsto f \cdot v_\lambda$ (where v_λ is some fixed nonzero highest weight vector of $M(\lambda)$).

Jantzen has defined a filtration $M(\lambda) = M^0(\lambda) \supset M^1(\lambda) \supset M^2(\lambda) \supset \cdots$ (by $\mathfrak{g}$-modules) of $M(\lambda)$ by $M^i(\lambda) = \{v \in M(\lambda)\colon \hat{A}_\lambda(v, w)$ is divisible by t^i for any $w \in M(\lambda)\}$. Then $M^1(\lambda)$ is the maximal proper submodule of $M(\lambda)$.

For any $\beta \in \Gamma^+$, one has [J]

$$\operatorname{ord}_t(\operatorname{ev}_{\lambda+tz} \det A_\beta^-) = \sum_{i \geq 1} \dim M^i_{\lambda-\beta}(\lambda), \qquad (\mathrm{I}_3)$$

where $\operatorname{ord}_t(Q)$ denotes the highest power of t which divides $Q(t)$, $\det A_\beta^- \in S(\mathfrak{h}) \subsetneq \mathbb{C}[t] \otimes S(\mathfrak{h})$ is defined in Theorem (1.2) and $M^i_{\lambda-\beta}(\lambda)$ denotes the $(\lambda - \beta)$th weight space of $M^i(\lambda)$.

From (I_2), (I_3), and Theorem (1.2), we obtain

$$\sum_{i\geq 1} \operatorname{ch} M^i(\lambda) = e^{\lambda} \sum_{\beta\in\Gamma^+} \operatorname{ord}_t(ev_{\lambda+tz} \det A_\beta^-)\, e^{-\beta}$$
$$= \sum_{\beta\in\Gamma^+} \sum_{(\alpha,n)\in D_\lambda} e^{\lambda} P(\beta-n\alpha)\, e^{-\beta},$$

where

$$D_\lambda = \left\{(\alpha, n)\in \Delta_+ \times \mathbb{N} : \sigma\left(\lambda+\rho-\frac{n}{2}\alpha, \alpha\right)=0\right\}.$$

So,

$$\sum_{i\geq 1} \operatorname{ch} M^i(\lambda) = \sum_{(\alpha,n)\in D_\lambda} \operatorname{ch} M(\lambda-n\alpha) \qquad (\mathrm{I}_4)$$

Taking $\lambda=\lambda_0\in K^{\text{w.g.}}$, we see (using [K, Proposition 5.5] and the definition of $K^{\text{w.g.}}$) that any $(\alpha,n)\in D_{\lambda_0}$ has the property that α is a real root and then (of course) n is unique (Localization). So we can express

$$\sum_{i\geq 1} \operatorname{ch} M^i(\lambda_0) = \sum \operatorname{ch} M(\lambda_0-n(\alpha)\,\alpha), \qquad (\mathrm{I}_5)$$

where the summation (in the right) runs over all those $\alpha\in\Delta_+^{\text{re}}$ satisfying $\sigma(\lambda_0+\rho,\alpha^v)$ is a positive integer (say) $n(\alpha)$.

Proof of the "only if" part of the theorem. Since $L(\mu_0)$ is a subquotient of $M(\lambda_0)$, of course, $\lambda_0-\mu_0\in\Gamma^+$. We prove the assertion by induction on $|\lambda_0-\mu_0|$. (For $\beta=\sum n_i\alpha_i\in\Gamma^+$, $|\beta|$ denotes $\sum n_i$.)

Since $M(\lambda_0)/M^1(\lambda_0)$ is the irreducible module $L(\lambda_0)$ and $\lambda_0\neq\mu_0$, we get, from (I_5), that $L(\mu_0)$ is a subquotient of $M(\lambda_0-n(\phi_1)\,\phi_1)$, for some $\phi_1\in\Delta_+^{\text{re}}$ and $n(\phi_1)\in\mathbb{N}$ satisfying $\sigma(\lambda_0+\rho,\phi_1^v)=n(\phi_1)$. Now $\gamma_{\phi_1} * \lambda_0 = \lambda_0-\sigma(\lambda_0+\rho,\phi_1^v)\,\phi_1=\lambda_0-n(\phi_1)\,\phi_1$ and hence $\lambda_0-n(\phi_1)\,\phi_1\in K^{\text{w.g.}}$. This proves, by induction, the only if part of the theorem.

The "if" part follows fairly easily from (I_5). ∎

The following corollary is an easy consequence of the above theorem.

(1.8) Corollary. *Let $\lambda,\mu\in K^{\text{w.g.}}$, then $\lambda\approx^\circ\mu$ if and only if there exists $w\in W(\lambda)$ such that $w*\lambda=\mu$, where $W(\lambda)\subset W$ is the group generated by all the reflections $\{\gamma_\phi\}$ for $\phi\in\Delta_+^{\text{re}}$ satisfying $\sigma(\lambda+\rho,\phi^v)\in\mathbb{Z}$. (Observe that $W(w*\lambda)=W(\lambda)$, for any $w\in W(\lambda)$.)*

(1.9) COROLLARY (of Corollary (1.8)). *$K^g \subset K^{\text{w.g.}}$ is stable under $\approx^\circ$, i.e., if $\lambda \in K^g$ and $\mu \in K^{\text{w.g.}}$ is such that $\mu \approx^\circ \lambda$ then $\mu \in K^g$. Moreover the equivalence relation $\approx^\circ$ restricted to K^g is the same as the equivalence relation $\approx$ on K^g, as defined in* [DGK, Sect. 5].

Proof. This follows immediately from Corollary (1.8) and the fact that K^g is W-stable under the shifted action of W. ∎

Recall the definition of an equivalence relation $\sim$ on $\mathfrak{h}^*$, as given in [DGK, Section 4].

(1.10) PROPOSITION. *Let $\mathfrak{g}$ be an affine (including twisted affine) Lie algebra. Let $\lambda \in K^{\text{w.g.}}$ and $\mu \in \mathfrak{h}^*$ such that $\lambda \sim \mu$. Then $\mu \in K^{\text{w.g.}}$ and hence $\lambda \approx^\circ \mu$. In particular (see Corollary (1.9)), if $\lambda \in K^g$ and $\mu \in \mathfrak{h}^*$ with $\lambda \sim \mu$ then $\mu \in K^g$ and $\lambda \approx \mu$.*

(1.11) *Remark.* This settles a question, due to Deodhar–Gabber–Kac [DGK, Remark 5.5], in the affirmative, for the affine Lie algebras. In the next example, we will show that Proposition (1.10) is false in general. We give a counterexample in the case of rank 2 hyperbolic Lie algebras, associated to symmetric Cartan matrices $H_a = \left(\begin{smallmatrix} 2 & -a \\ -a & 2 \end{smallmatrix}\right)$ with any positive integer $a \geqslant 3$. More specifically, we show that there exist λ, μ in the interior of the dominant chamber (of the dual Cartan algebra associated to H_a) such that $\lambda \sim \mu$ but $\lambda \not\approx \mu$.

Proof of Proposition (1.10). In view of Theorem (1.7), it suffices to show that if $L(\lambda)$ is a subquotient of $M(\mu)$ then $\mu \in K^{\text{w.g.}}$.

Since $L(\lambda)$ is a subquotient of $M(\mu)$, we have $\mu - \lambda = \sum_{i=0}^{l} n_i \alpha_i$, for some $n_i \in \mathbb{Z}^+$. Now, from the description of $K^{\text{w.g.}}$, as given in Lemma (1.5), we see that $\mu \in K^{\text{w.g.}}$, proving the proposition.

(1.12) EXAMPLE. Fix a symmetric hyperbolic 2×2 Cartan matrix $H_a = \left(\begin{smallmatrix} 2 & -a \\ -a & 2 \end{smallmatrix}\right)$ with $a \geqslant 3$.

We have the following description of the bilinear form on $\mathfrak{h}^*$.

$$\sigma(\alpha_1, \alpha_1) = \sigma(\alpha_2, \alpha_2) = 2, \qquad \sigma(\alpha_1, \alpha_2) = -a.$$

Further $\rho = (\alpha_1 + \alpha_2)/(2-a)$ and, by [M, Sect. 3, Theorem 1],

$$\Delta_+^{\text{im}} = \{m\alpha_1 + n\alpha_2 : m, n \in \mathbb{Z}^+, m+n > 0 \text{ and } m^2 - amn + n^2 \leqslant 0\}. \quad (\mathrm{I}_6)$$

We know, by Lemma (1.5), that

$$S = \bigcup_{(m,n) \in Z_a} S_{(m,n)},$$

where $S_{(m,n)} = \{\lambda \in \mathfrak{h}^*: \lambda(h_1)m + \lambda(h_2)n = m^2 - mna + n^2 - m - n\}$ and Z_a is as defined in Lemma (1.5), case (c).

For any $\lambda \in S_{(m,n)}$, by (I_4), we have

$$\lambda \sim \lambda - (m\alpha_1 + n\alpha_2). \tag{I_7}$$

For any $x \in \mathbb{R}$, denote $\lambda_x = x(\alpha_1 + \alpha_2) \in \mathfrak{h}^*$. We want to pick $\beta \in \mathfrak{h}^*$, satisfying:

(1) $\beta = m\alpha_1 + n\alpha_2 \in \Delta_+^{\mathrm{im}}$

(2) $\sigma(\lambda_x + \rho, \beta) = \sigma(\beta, \beta)/2$ and

(3) $\sigma(\lambda_x - \beta, \alpha_i) > 0$ for $i = 1, 2$.

(1), (2), and (3) together are equivalent to the following:

(a) $m, n \in \mathbb{Z}^+$, $m + n > 0$ and $m^2 - amn + n^2 \leqslant 0$

(b) $x = (-m^2 - n^2 + mna + m + n)/(m+n)(a-2)$ and

(c) $x < (an - 2m)/(a-2)$ and also $x < (am - 2n)/(a-2)$.

For any choice of $x \in \mathbb{R}$ and $m, n \in \mathbb{Z}^+$, satisfying (a), (b), and (c), we have, by (I_7), $\lambda_x \sim \lambda_x - (m\alpha_1 + n\alpha_2)$ and also $\lambda_x \sim \lambda_x - (n\alpha_1 + m\alpha_2)$ (since conditions (a), (b), and (c) are symmetric in m and n). Further, by (c), $\lambda_x - (m\alpha_1 + n\alpha_2)$ and $\lambda_x - (n\alpha_1 + m\alpha_2)$ both belong to the interior C^0 of the dominant chamber ($C^0 = \{\lambda \in \mathfrak{h}^*: \sigma(\lambda, \alpha_i) > 0$ for all $i\}$). In particular $\lambda_x - (m\alpha_1 + n\alpha_2) \sim \lambda_x - (n\alpha_1 + m\alpha_2)$ and if $m \neq n$ (so that $\lambda_x - (m\alpha_1 + n\alpha_2) \neq \lambda_x - (n\alpha_1 + m\alpha_2)$), there does not exist any $w \in W$ such that $w * (\lambda_x - (m\alpha_1 + n\alpha_2)) = \lambda_x - (n\alpha_1 + m\alpha_2)$ (since both of them belong to the dominant chamber) and hence $\lambda_x - (m\alpha_1 + n\alpha_2) \not\approx \lambda_x - (n\alpha_1 + m\alpha_2)$.

So it suffices to choose $x \in \mathbb{R}$ and $m \neq n \in \mathbb{Z}^+$ satisfying (a), (b), and (c). Of course, if we pick (m, n), the choice of x is uniquely determined by (b). We denote this x by $x_{(m,n)}$. Hence it remains to choose $m \neq n \in \mathbb{Z}^+$ satisfying (a) and (c) with x replaced by $x_{(m,n)}$. Also there are infinitely many such pairs (m, n), e.g., the pair $(m, n) = (k, k+1)$ does satisfy (a) and (c) for any integer $k \geqslant 7$.

(1.13) *Remarks.* (1) As in the above example, we fix $k = k_0 \geqslant 7$ and take $(m, n) = (k_0, k_0 + 1)$. If we now choose (e.g.) $a = 2k_0 - 1$ and $x = x_{(k_0, k_0+1)}$ then λ_x is integral and hence $\lambda_x - (k_0\alpha_1 + (k_0+1)\alpha_2)$ and $\lambda_x - ((k_0+1)\alpha_1 + k_0\alpha_2)$ are both dominant integral elements which are $\sim$ related but are not $\approx$ related.

(2) The above example can be suitably modified to give counter examples in the case of all the rank 2 (i.e., even in the nonsymmetric case) hyperbolic Kac–Moody algebras.

2. A VANISHING THEOREM FOR THE EXT FUNCTOR

As in [DGK, 4], we recall (resp. make) the following

(2.1) DEFINITION. Let Λ (resp. $\Lambda^{\text{w.g.}}$) be an equivalence class of $\mathfrak{h}^*$ under the equivalence relation $\sim$ (resp. an equivalence class of $K^{\text{w.g.}}$ under $\approx^\circ$). A module $M \in \mathcal{O}$ (resp. $M \in \mathcal{O}^{\text{w.g.}}$) is said to be of type Λ (resp. $\Lambda^{\text{w.g.}}$) iff all the irreducible subquotients of M have highest weights $\in \Lambda$ (resp. $\in \Lambda^{\text{w.g.}}$).

We denote by $\mathcal{O}_\Lambda$ (resp. $\mathcal{O}^{\text{w.g.}}_{\Lambda^{\text{w.g.}}}$) the full subcategory of the category $\mathcal{O}$ (resp. $\mathcal{O}^{\text{w.g.}}$) consisting of those modules $M \in \mathcal{O}$ (resp. $\mathcal{O}^{\text{w.g.}}$) such that M is of type Λ (resp. $\Lambda^{\text{w.g.}}$).
($K^{\text{w.g.}}$, $\mathcal{O}^{\text{w.g.}}$ and $\approx^\circ$ are defined in Sections (1.3), (1.4), and (1.6), respectively.)
One of the main results of this section is the following.

(2.2) THEOREM. *Let $\mathfrak{g}$ be any symmetrizable Kac–Moody Lie algebra. Then*

(a) *Let M, N be two $\mathfrak{g}$-modules in the category $\mathcal{O}$, such that M (resp. N) is of type Λ_M (resp. Λ_N). If $\Lambda_M \neq \Lambda_N$ then*

$$\operatorname{Ext}^n_{\mathfrak{g}}(M, N) = 0, \qquad \text{for all} \quad n \geqslant 0.$$

(b) *Let M, N be two $\mathfrak{g}$-modules in the category $\mathcal{O}^{\text{w.g.}}$ such that M (resp. N) is of type $\Lambda^{\text{w.g.}}_M$ (resp. $\Lambda^{\text{w.g.}}_N$). If $\Lambda^{\text{w.g.}}_M \neq \Lambda^{\text{w.g.}}_N$, then again*

$$\operatorname{Ext}^n_{\mathfrak{g}}(M, N) = 0, \qquad \textit{for all} \quad n \geqslant 0.$$

(2.3) *Remark.* The vanishing of $\operatorname{Ext}^n_{\mathfrak{g}}(M, N)$, as in part (a) of the above theorem, for the particular value of $n = 1$, is one of the main results of Deodhar–Gabber–Kac [DGK, Theorem 4.5]. Their decomposition theorem [DGK, Theorem 4.2] (resp. Theorem (5.7)) is clearly equivalent to the vanishing of $\operatorname{Ext}^n_{\mathcal{O}}(M, N)$ (resp. $\operatorname{Ext}^n_{\mathcal{O}^g}(M, N)$) for all $n \geqslant 0$, if M and N are of unequal types in $\mathcal{O}$ (resp. $\mathcal{O}^g$), whereas our theorem implies an even stronger vanishing result.

To prove the theorem, we need the following lemmas.

(2.4) LEMMA. *For any $\mathfrak{g}$-modules X, Y and any Lie subalgebra $\mathfrak{a}$ (including $\mathfrak{a} = 0$) of $\mathfrak{g}$, we have*

$$\operatorname{Ext}^n_{(\mathfrak{g},\mathfrak{a})}(X, Y^*) \approx [\operatorname{Tor}^{(\mathfrak{g},\mathfrak{a})}_n(X^t, Y)]^* \qquad \textit{for all} \quad n \geqslant 0.$$

(See the "Notations" for Y^* and X^t.)

Lemma (2.4) is well known and (in any case) easy to prove.

(2.5) DEFINITION. A $\mathfrak{g}$-module X is called a $(\mathfrak{g}, \mathfrak{h})$ module if (as an $\mathfrak{h}$-module) it is direct sum of finite dimensional $\mathfrak{h}$-modules.

(We do not demand that the isotypical components (under the $\mathfrak{h}$-action) are finite dimensional.)

The following lemma (I believe) is due to M. Duflo.

(2.6) LEMMA. *Let X, Y be $(\mathfrak{g}, \mathfrak{h})$ modules. Then* $\operatorname{Ext}^n_{\mathfrak{g}}(X, Y^*) \approx \operatorname{Ext}^n_{\mathfrak{g}}(X, Y^v)$, *for all $n \geqslant 0$, where Y^v denotes the $\mathfrak{h}$-semisimple part of Y^*. (More specifically, if we write $Y = \sum_{\beta \in \mathfrak{h}^*} Y_\beta$, where $Y_\beta = \{y \in Y: h \cdot y = \beta(h) \cdot Y\}$, then $Y^v = \sum_{\beta \in \mathfrak{h}^*} Y^*_\beta$.)*

Proof. Consider the cochain complex $C = \sum_{n \geqslant 0} C^n$ (with standard d), where $C^n = \operatorname{Hom}_{\mathbb{C}}(\Lambda^n(\mathfrak{g}) \otimes X, Y^*) \approx \operatorname{Hom}_{\mathbb{C}}(\Lambda^n(\mathfrak{g}) \otimes X \otimes Y, \mathbb{C})$. Of course (by definition) the cohomology of C is $\operatorname{Ext}_{\mathfrak{g}}(X, Y^*)$. Since $\mathfrak{g}$, X, and Y are all $(\mathfrak{g}, \mathfrak{h})$ modules, $\Lambda^n(\mathfrak{g}) \otimes X \otimes Y$ is again a $(\mathfrak{g}, \mathfrak{h})$-module, for any $n \geqslant 0$. Write $\Lambda^n(\mathfrak{g}) \otimes X \otimes Y = \sum_{\beta \in \mathfrak{h}^*} M^n_\beta$, where M^n_β is the isotypical component (with respect to the $\mathfrak{h}$-action) of $\Lambda^n(\mathfrak{g}) \otimes X \otimes Y$ corresponding to $\beta \in \mathfrak{h}^*$. Clearly $C^n = \prod_{\beta \in \mathfrak{h}^*} (M^n_\beta)^*$ and, for every $\beta \in \mathfrak{h}^*$, $\sum_n (M^n_\beta)^*$ is a subcomplex of C. For any $f \in C^n$, denote by f_β its component in the $(M^n_\beta)^*$ factor.

Now $C_{\text{res}} = \sum C^n_{\text{res}}$, where $C^n_{\text{res}} = \operatorname{Hom}_{\mathbb{C}}(\Lambda^n(\mathfrak{g}) \otimes X, Y^v)$, is a subcomplex of C and its cohomology is $\operatorname{Ext}_{\mathfrak{g}}(X, Y^v)$. Further $f \in C^n$ belongs to C^n_{res} if and only if, for any $v \in \Lambda^n(\mathfrak{h}) \otimes X$, $f_\beta(v)$ is zero (as an element of Y^*) for all but finitely many β's. In particular, $\sum_n (M^n_0)^* \subset C_{\text{res}}$.

Now, we want to prove that for any $f \in C^n$ (resp. $f \in C^n_{\text{res}}$) satisfying

(1) $df = 0$,

(2) $f_0 = 0$,

there exists a $g \in C^{n-1}$(resp. C^{n-1}_{res}) satisfying

(1) $dg = f$,

(2) $g_0 = 0$.

Of course, this would prove the lemma.

We now prove the existence of g. For any $\beta \neq 0 \in \mathfrak{h}^*$, fix a $h(\beta) \in \mathfrak{h}$ satisfying $\beta(h(\beta)) = 1$. Since $di_h + i_h d = L_h$, for any $h \in \mathfrak{h}$ (L denotes the representation of $\mathfrak{h}$ on C), we get $di_{h(\beta)}(f_\beta) + i_{h(\beta)} d(f_\beta) = f_\beta$. But since $f = \sum_{\beta \neq 0} f_\beta$ and $df = \sum_{\beta \neq 0} df_\beta = 0$, we have for all $\beta \neq 0 \in \mathfrak{h}^*$, $di_{h(\beta)}(f_\beta) = f_\beta$, i.e., $d(\sum_{\beta \neq 0} i_{h(\beta)}(f_\beta)) = f$. So the element $g = \sum_{\beta \neq 0} i_{h(\beta)}(f_\beta)$ does the job. (Observe that if $f \in C^n_{\text{res}}$ then g again is in C^{n-1}_{res}.) ∎

Define a twisted $\mathfrak{g}$-module structure on any $\mathfrak{g}$-module (M, π) by $\pi \circ \omega$, where ω is the (unique) $\mathbb{C}$-linear involution of $\mathfrak{g}$ defined by $\omega(e_i) = -f_i$ for all $1 \leqslant i \leqslant l$ and $\omega(h) = -h$, for all $h \in \mathfrak{h}$. We denote by M^{ω} the vector space M, under the ω-twisted $\mathfrak{g}$-module structure.

(2.7) *Remark.* For any $M \in \mathcal{O}$, the $\mathfrak{g}$-module $(M^{v})^{\omega}$ is nothing but M^{σ}, as defined in [DGK, Sect. 4].

(2.8) LEMMA. *$H_n(\mathfrak{g}, M) \approx H_n(\mathfrak{g}, M^{\omega})$, for all $n \geqslant 0$ and any (left) $\mathfrak{g}$-module M.*

Proof. We have the following commutative diagram,

$$\begin{array}{ccc} \Lambda^n(\mathfrak{g}) \otimes M & \xrightarrow{\partial} & \Lambda^{n-1}(\mathfrak{g}) \otimes M \\ \Big\downarrow \wr & & \Big\downarrow \wr \\ \Lambda^n(\mathfrak{g}) \otimes M^{\omega} & \xrightarrow{\partial} & \Lambda^{n-1}(\mathfrak{g}) \otimes M^{\omega} \end{array}$$

where the horizontal maps are the standard chain maps corresponding to the Lie algebra $\mathfrak{g}$ with coefficients (resp.) in M and M^{ω}. The vertical maps are defined by $x_1 \wedge \cdots \wedge x_n \otimes m \to \omega(x_1) \wedge \cdots \wedge \omega(x_n) \otimes m$, for $x_1,..., x_n \in \mathfrak{g}$ and $m \in M$, similarly the other vertical map.

This immediately proves the lemma.

(2.9) COROLLARY. *For any $M \in \mathcal{O}$, we have $H^n(\mathfrak{g}, M) \approx [H_n(\mathfrak{g}, M^{\sigma})]^*$.*

To prove, combine Lemmas (2.4), (2.6), and (2.8).

(2.10) LEMMA. *For any $(\mathfrak{g}, \mathfrak{h})$ module M and any $\mu \in \mathfrak{h}^*$*

$$\operatorname{Tor}_n^{(\mathfrak{g},\mathfrak{h})}(M', M(\mu)) \approx [H_n(\mathfrak{n}, M)]_{-\mu},$$

where $[H_n(\mathfrak{n}, M)]_{-\mu}$ denotes the $-\mu$ eigenspace (under the canonical action of $\mathfrak{h}$) of $H_n(\mathfrak{n}, M)$.

Proof. Take a $(\mathfrak{b}, \mathfrak{h})$-free resolution of the (left) $\mathfrak{b}$-module $\mathbb{C}_\mu$ ($\mathfrak{n}$ acts trivially on $\mathbb{C}_\mu$ and $\mathfrak{h}$ acts on $\mathbb{C}_\mu \approx \mathbb{C}$ by the character μ)

$$\cdots \to F_1 \to F_0 \to \mathbb{C}_\mu \to 0. \tag{S$_1$}$$

Since $U(\mathfrak{g})$ is right $U(\mathfrak{b})$-free ($U(\mathfrak{b})$ acting on $U(\mathfrak{g})$ by right multiplication) (by the Poincaré–Birkhoff–Witt theorem) we get an exact sequence (S_2), by tensoring (S_1) with $U(\mathfrak{g})$,

$$\cdots \to U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} F_1 \to U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} F_0 \to M(\mu) \to 0. \tag{S$_2$}$$

We have the following commutative diagram,

$$\begin{array}{ccccccccc}
\cdots \to & M' \underset{U(\mathfrak{b})}{\otimes} F_n & \to \cdots \to & M' \underset{U(\mathfrak{b})}{\otimes} F_1 & \to & M' \underset{U(\mathfrak{b})}{\otimes} F_0 \\
& \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\cdots \to & M' \underset{U(\mathfrak{g})}{\otimes} \left(U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} F_n \right) & \to \cdots \to & M' \underset{U(\mathfrak{g})}{\otimes} \left(U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} F_1 \right) & \to & M' \underset{U(\mathfrak{g})}{\otimes} \left(U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} F_0 \right)
\end{array}$$

where the vertical maps (defined by $m \otimes a \to m \otimes (1 \otimes a)$, for $m \in M'$ and $a \in F_n$) are all isomorphisms.

Observe that $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} F_n \approx U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U(\mathfrak{b}) \otimes_{U(\mathfrak{h})} F_n^0)$ (for some $\mathfrak{h}$-module F_n^0), since by assumption F_n is $(\mathfrak{b}, \mathfrak{h})$-free. But $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U(\mathfrak{b}) \otimes_{U(\mathfrak{h})} F_n^0) \approx U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} F_n^0$ and hence $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} F_n$ is $(\mathfrak{g}, \mathfrak{h})$-free. Hence

$$\operatorname{Tor}_n^{(\mathfrak{g},\mathfrak{h})}(M', M(\mu)) \approx \operatorname{Tor}_n^{(\mathfrak{b},\mathfrak{h})}(M', \mathbb{C}_\mu). \tag{I_8}$$

By definition, $\operatorname{Tor}_n^{(\mathfrak{b},\mathfrak{h})}(M', \mathbb{C}_\mu)$ is nth homology of the standard complex $\Lambda(\mathfrak{b}/\mathfrak{h}) \otimes (M \otimes \mathbb{C}_\mu)/\mathfrak{h} \cdot (\Lambda(\mathfrak{b}/\mathfrak{h}) \otimes (M \otimes \mathbb{C}_\mu))$. But since $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ (as a vector space) and $\mathfrak{h}$ acts reductively on M and $\mathfrak{b}$, we get that $\operatorname{Tor}^{(\mathfrak{b},\mathfrak{h})}(M', \mathbb{C}_\mu)$ is the homology of the complex $[\Lambda(\mathfrak{n}) \otimes M \otimes \mathbb{C}_\mu]^{\mathfrak{h}}$. Further since $\mathfrak{n}$ acts trivally on $\mathbb{C}_\mu$,

$$\operatorname{Tor}_n^{(\mathfrak{b},\mathfrak{h})}(M', \mathbb{C}_\mu) \approx [H_n(\mathfrak{n}, M)]_{-\mu}. \tag{I_9}$$

(I_8) and (I_9), together, prove the lemma. ∎

Now we are ready to prove the theorem.

(2.11) *Proof of Theorem* (2.2)(a).

$$\begin{aligned}
\operatorname{Ext}_{\mathfrak{g}}^n(M, N^\sigma) &\approx \operatorname{Ext}_{\mathfrak{g}}^n(M, (N^\omega)^v) && (\text{since } N^\sigma = (N^v)^\omega = (N^\omega)^v) \\
&\approx \operatorname{Ext}_{\mathfrak{g}}^n(M, (N^\omega)^*) && (\text{by Lemma (2.6)}) \\
&\approx [\operatorname{Tor}_n^{\mathfrak{g}}(M', N^\omega)]^* && (\text{by Lemma (2.4)})
\end{aligned}$$

Hence,

$$\operatorname{Ext}_{\mathfrak{g}}^n(M, N^\sigma) \approx [\operatorname{Tor}_n^{\mathfrak{g}}((M^\omega)', N)]^* \qquad (\text{by Lemma (2.8)}). \tag{I_{10}}$$

Since $(N^\sigma)^\sigma \approx N$, by [DGK, Proposition 4.6], replacing N by N^σ in (I_{10}), we get

$$\operatorname{Ext}_{\mathfrak{g}}^n(M, N) \approx [\operatorname{Tor}_n^{\mathfrak{g}}((M^\omega)', N^\sigma)]^*. \tag{I_{11}}$$

Since Tor commutes with direct limits in both the variables and if N is of type Λ_N then (clearly) N^σ is also of type Λ_N (see [DGK, Proposition (4.6)]), we need to prove the vanishing of $\operatorname{Tor}_n^{\mathfrak{g}}((M^\omega)', N)$, for

M and N finitely generated $\mathfrak{g}$-modules. By [GL, Lemma 4.4], for any $M \in \mathcal{O}$, there exists a (possibly infinite) increasing filtration $(0) = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$, of submodules of M such that $\bigcup_i M_i = M$ and M_i/M_{i-1} is a highest weight module, for all $i \geqslant 1$.

Further for any $M_1 \subset M_2$ and $N_1 \subset N_2$ there are exact sequences

$$\cdots \to \operatorname{Tor}_n^{\mathfrak{g}}(M_1^t, N_2) \to \operatorname{Tor}_n^{\mathfrak{g}}(M_2^t, N_2) \to \operatorname{Tor}_n^{\mathfrak{g}}((M_2/M_1)^t, N_2) \to \operatorname{Tor}_{n-1}^{\mathfrak{g}}(M_1^t, N_2) \to \cdots \tag{S_3}$$

and

$$\cdots \to \operatorname{Tor}_n^{\mathfrak{g}}(M_2^t, N_1) \to \operatorname{Tor}_n^{\mathfrak{g}}(M_2^t, N_2) \to \operatorname{Tor}_n^{\mathfrak{g}}(M_2^t, N_2/N_1) \to \operatorname{Tor}_{n-1}^{\mathfrak{g}}(M_2^t, N_1) \to \cdots \tag{S_4}$$

Hence we can assume that M (resp. N) is quotient of a Verma module $M(\lambda)$ (resp. $M(\mu)$). Write

$$0 \to K \to M(\lambda) \to M \to 0$$

and

$$0 \to K^1 \to M(\mu) \to N \to 0.$$

Assume, by induction, that for all $k < n$, $\operatorname{Tor}_k^{\mathfrak{g}}((M^{\omega})^t, N) = 0$, for any M and $N \in \mathcal{O}$ such that M is of type Λ_M and N is of type Λ_N with $\Lambda_M \neq \Lambda_N$. (Of course, the induction starts at $n = 0$.) Using the exact sequences (S_3) and (S_4) and the induction hypothesis, it suffices to show that $\operatorname{Tor}_n^{\mathfrak{g}}((M(\lambda)^{\omega})^t, M(\mu)) = 0$ for $\lambda, \mu \in \mathfrak{h}^*$ and $\lambda \not\sim \mu$.

By Lemma (2.10), $\operatorname{Tor}_n^{(\mathfrak{g},\mathfrak{h})}((M(\lambda)^{\omega})^t, M(\mu)) \approx [H_n(\mathfrak{n}, M(\lambda)^{\omega})]_{-\mu}$. But since $M(\lambda)$ is $U(\mathfrak{n}^-)$-free, $M(\lambda)^{\omega}$ is $U(\mathfrak{n})$-free and hence $H_n(\mathfrak{n}, M(\lambda)^{\omega})$ is 0 unless $n = 0$ and $H_0(\mathfrak{n}, M(\lambda)^{\omega}) \approx M(\lambda)^{\omega}/\mathfrak{n} \cdot M(\lambda)^{\omega}$. Further, $M(\lambda)^{\omega}/\mathfrak{n} \cdot M(\lambda)^{\omega}$ is one dimensional (over $\mathbb{C}$) $\mathfrak{h}$-module with character $-\lambda$ and hence $[H_0(\mathfrak{n}, M(\lambda)^{\omega})]_{-\mu} = 0$ (since $\lambda \neq \mu$). Thus we have established that $\operatorname{Tor}_n^{(\mathfrak{g},\mathfrak{h})}((M(\lambda)^{\omega})^t, M(\mu)) = 0$, for all $n \geqslant 0$.

By Hochschild-Serre [HS, Sect. 6], there is a spectral sequence with $E^2_{p,q} = H_p(\mathfrak{g}, \mathfrak{h}, M(\lambda)^{\omega} \otimes M(\mu)) \otimes H_q(\mathfrak{h})$ and converging to the homology $H_*(\mathfrak{g}, M(\lambda)^{\omega} \otimes M(\mu))$. (Although in [HS], it is proved for cohomology and under the restriction that $\mathfrak{g}$ is finite dimensional, it can be easily adopted to our situation, since $\mathfrak{h}$ acts reductively on $\mathfrak{g}$ as well as $M(\lambda)^{\omega} \otimes M(\mu)$.) Hence the vanishing of $\operatorname{Tor}_n^{(\mathfrak{g},\mathfrak{h})}((M(\lambda)^{\omega})^t, M(\mu)) \approx H_*(\mathfrak{g}, \mathfrak{h}, M(\lambda)^{\omega} \otimes M(\mu))$ implies the vanishing of $\operatorname{Tor}_*^{\mathfrak{g}}((M(\lambda)^{\omega})^t, M(\mu))$ and hence part (a) of the theorem follows.

Proof of part (b) is exactly similar. We just need to observe that for any $\lambda \in K^{\text{w.g.}}$ and $\mu \in \mathfrak{h}^*$ such that $L(\mu)$ is a subquotient of $M(\lambda)$ then μ is again $\in K^{\text{w.g.}}$, which (of course) follows from Theorem (1.7).

(2.12) *Remark.* The above proof (with suitable and obvious modifications) shows that under the assumptions of Theorem (2.2) ((a) or (b)) on M and N, we also have $\mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{h})}(M, N) = 0$, for all $n \geqslant 0$.

As (more or less) immediate consequences of the theorem, we get

(2.13) COROLLARIES. (a) *Let $M \in \mathcal{O}^{\text{w.g.}}$. Then there exists a unique family $\{M_{\Lambda^{\text{w.g.}}}\}$ of submodules of M such that*

(i) $M_{\Lambda^{\text{w.g.}}} \in \mathcal{O}^{\text{w.g.}}_{\Lambda^{\text{w.g.}}}$ *and*

(ii) $M = \sum_{\Lambda^{\text{w.g.}}} M_{\Lambda^{\text{w.g.}}}$.

This is generalization of [DGK, Theorem 5.7] to a (generally) much larger category $\mathcal{O}^{\text{w.g.}}$.

(b) *Let M be any module in the category $\mathcal{O}$ (resp. $\mathcal{O}^{\text{w.g.}}$) such that it has no irreducible subquotients $L(\mu)$ with $\mu \sim 0$ (resp. $\mu \approx^{\circ} 0$), then for all $n \geqslant 0$*

(1) $H_n(\mathfrak{g}, M) = H_n(\mathfrak{g}, \mathfrak{h}, M) = 0$ *and*

(2) $H^n(\mathfrak{g}, M) = H^n(\mathfrak{g}, \mathfrak{h}, M) = 0$.

One important example of such a M is any highest weight module with highest weight $\lambda \nsim 0$. If $\lambda \in K^{\text{w.g.}}$, we just demand that $\lambda \not\approx^{\circ} 0$, i.e., λ is not of the form $\omega\rho - \rho$, for any $w \in W$.

The following two special cases of (b) are of particular interest:

(b_1) *Let $L(\lambda)$ be the integrable highest weight module (of course λ, in this case, is dominant integral). Then if $\lambda \neq 0$*

$$H_n(\mathfrak{g}, L(\lambda)) = H_n(\mathfrak{g}, \mathfrak{h}, L(\lambda)) = H^n(\mathfrak{g}, L(\lambda)) = H^n(\mathfrak{g}, \mathfrak{h}, L(\lambda)) = 0,$$

for all $n \geqslant 0$.

More generally, let $L(\lambda)$ and $L(\mu)$ be integrable highest weight modules with $\lambda \neq \mu$ then, for all $n \geqslant 0$,

$$\mathrm{Ext}^n_{(\mathfrak{g},\mathfrak{h})}(L(\lambda), L(\mu)) = \mathrm{Ext}^n_{\mathfrak{g}}(L(\lambda), L(\mu)) = 0.$$

This result is due to M. Duflo (unpublished). See also [Ku, Theorem 1.7].

(b_2) *Let $M \in \mathcal{O}$ be such that the Casimir operator acts as an automorphism on M, then*

$$H_n(\mathfrak{g}, M) = H_n(\mathfrak{g}, \mathfrak{h}, M) = H^n(\mathfrak{g}, M) = H^n(\mathfrak{g}, \mathfrak{h}, M) = 0, \qquad \textit{for all} \quad n \geqslant 0.$$

The vanishing of $H_*(\mathfrak{g}, M)$ in this case is one of the main results of Kumar [Ku, Theorem 1.2].

Proof. (a) is an easy consequence of Theorem (2.2)(b). The details are similar to [DGK, Proof of Theorem 4.2, pp. 105–106].

(b) Since $H_n(\mathfrak{g}, M) = \mathrm{Tor}_n^{\mathfrak{g}}(M', \mathbb{C})$ and $[\mathrm{Tor}_n^{\mathfrak{g}}(M', \mathbb{C})]^* \approx \mathrm{Ext}_{\mathfrak{g}}^n(M, \mathbb{C})$ (by Lemma (2.4)) and further $H^n(\mathfrak{g}, M) = \mathrm{Ext}_{\mathfrak{g}}^n(\mathbb{C}, M)$, (1) and (2) of (b) follow immediately from Theorem (2.2) and Remark (2.12).

To prove (b_1); just observe that $L(\lambda) \in \mathcal{O}^{\text{w.g.}}$ and $\lambda \not\sim^\circ 0$. (For otherwise $\lambda = \omega\rho - \rho$ for some $w \neq e$ and λ is dominant, which is not possible.) The vanishing of $\mathrm{Ext}^n(L(\lambda), L(\mu))$ follows in the same way, from Theorem (2.2)(b).

To prove (b_2), observe that if $\lambda \sim \mu \in \mathfrak{h}^*$ and $V(\lambda)$ and $V(\mu)$ are highest weight modules (of highest weights λ and μ, respectively) then the Casimir operator acts on $V(\lambda)$ and $V(\mu)$ by the same scalar (and the scalar is $\sigma(\lambda+\rho, \lambda+\rho) - \sigma(\rho, \rho)$). In particular, if the Casimir acts as an automorphism on M then M cannot have any irreducible subquotient $L(\mu)$ with $\mu \sim 0$.

(2.14) *Remark.* Corollary (b) part (1) was conjectured in [Ku, Conjecture 1.10]. In fact, it was conjectured there that $H_*(\mathfrak{g}, V(\lambda)) = 0$, where $V(\lambda)$ is any highest weight module (with highest weight λ) provided $\lambda \neq w\rho - \rho$, for any $w \in W$. In view of Proposition (1.10), we have proved that this is indeed true in the affine (including twisted affine) case. It is true in general if $\lambda \in K^{\text{w.g.}}$. But we show, by the following example, that for general $\mathfrak{g}$ there exists $\lambda \in \mathfrak{h}^*$ and λ not of the form $w\rho - \rho$ for any $w \in W$, such that $H_*(\mathfrak{g}, V(\lambda)) \neq 0$. Of course, any such $\lambda \sim 0$ and $\mathfrak{g}$ is neither finite dimensional nor affine.

(2.15) Example. Suppose we have a $\lambda \neq 0 \in \mathfrak{h}^*$ such that the trivial module $\mathbb{C}$ is a subquotient of the Verma module $M(\lambda)$, i.e., there exist $\mathfrak{g}$-modules M and N satisfying $M \subsetneq M(\lambda)$ and we have an exact sequence

$$0 \to N \to M \to \mathbb{C} \to 0.$$

(Of course any such λ can not be of the form $w\rho - \rho$, for any $w \in W$.)

Now $\mathrm{Ext}_{\mathfrak{g}}^0(\mathbb{C}, M(\lambda)/N) = H^0(\mathfrak{g}, M(\lambda)/N) \approx [M(\lambda)/N]^{\mathfrak{g}}$. But $M(\lambda)/N$ contains $M/N \approx \mathbb{C}$ as a submodule and hence $\mathrm{Ext}_{\mathfrak{g}}^0(\mathbb{C}, M(\lambda)/N) \neq 0$. We construct a similar example in homology.

We have the following exact sequence ($M(\lambda)$, M, and N are as before)

$$0 \to \mathbb{C} \to M(\lambda)/N \to M(\lambda)/M \to 0.$$

This gives rise to the long exact homology sequence

$$\cdots \to H_1(\mathfrak{g}, M(\lambda)/M) \to H_0(\mathfrak{g}, \mathbb{C}) \to H_0(\mathfrak{g}, M(\lambda)/N) \\ \to H_0(\mathfrak{g}, M(\lambda)/M) \to 0.$$

From this we get that $H_1(\mathfrak{g}, M(\lambda)/M) \neq 0$, for otherwise, $\mathbb{C} \approx H_0(\mathfrak{g}, \mathbb{C}) \subsetneq H_0(\mathfrak{g}, M(\lambda)/N) \approx M(\lambda)/(\mathfrak{g} \cdot M(\lambda) + N) = 0$ (since $\lambda \neq 0$). A contradiction!

So it suffices to find $\lambda \in \mathfrak{h}^*$, $\lambda \neq 0$ such that the trivial module $\mathbb{C}$ is subquotient of $M(\lambda)$.

Again we consider symmetric hyperbolic 2×2 Cartan matrix $H_a = \begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}$ with $a \geqslant 3$.

Assume that there exists $(m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with $m + n > 0$ and

$$m + n + m^2 + n^2 = mna. \tag{$*$}$$

In this case, using (I_4 (of the previous section)), we can easily see that $\mathbb{C} = L(0)$ is a subquotient of $M(m\alpha_1 + n\alpha_2)$. (Observe that $m\alpha_1 + n\alpha_2 \in \Delta_+^{im}$ by (I_6).)

We come to the existence of m, n, and a satisfying ($*$). For example, for $a = 3$, at least the pairs $(m, n) = (2, 2), (2, 3), (3, 2)$ satisfy ($*$). For $a = 4$, the pairs $(m, n) = (1, 1), (1, 2), (2, 1)$ satisfy ($*$).

I do not know if there are infinitely many (m, n) and a satisfying ($*$).

(2.16) *Remark.* Many results in this paper, stated for the pair $(\mathfrak{g}, \mathfrak{h})$, can be extended to the pair $(\mathfrak{g}, \mathfrak{g}_S + \mathfrak{h})$, where $S \subset \{1, ..., l\}$ is a subset of finite type and $\mathfrak{g}_S$ is the (finite dimensional) Lie algebra corresponding to the submatrix $(a_{ij})_{i,j \in S}$. Since the extension is fairly straightforward to state and prove, we leave it to the interested reader.

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