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A GENERALIZATION OF THE CONNER CONJECTURE AND TOPOLOGY OF STEIN SPACES DOMINATED BY $\mathbb{C}^n$

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Dedicated to the memory of Professor B. V. Singbal

INTRODUCTION

THE principal aim of this paper is to study the topology of normal Stein spaces V^n (of complex dim n), which are images of (topologically) contractible normal Stein spaces W^n under a proper analytic map. We derive two results (theorems (2) and (7)) on the topology of V. The first result asserts that V is Q-acyclic as well as Z_p-acyclic for all those primes p, which are co-prime to deg f. Further V is simply connected, provided we assume that V is the homotopy type of a finite CW complex. (Of course, as is well known, any affine algebraic variety is the homotopy type of a finite CW complex.) This is an improvement of Gurjar's main results in [4 (b) and (c)]. He assumes that each irreducible component of the singular locus of V is of codimension at least $n/2+1$. The second (and main) result asserts that $H^n(V^n, Z) = 0$. (It may be mentioned that the vanishing of $H_n(V^n, Z)$ is, actually, a triviality. From the universal coefficient theorem, the vanishing of $H^n(V^n, Z)$ is equivalent to the vanishing of $H_{n-1}(V^n, Z)$.) This, in particular, gives that V^n is actually contractible for $n \leq 3$ (theorem 14). Using a result of Stallings this gives that, for $n = 3$, if we assume V to be smooth then V is diffeomorphic with $\mathbb{R}^6$ (corollary 15). Further, for any fixed prime $p \neq 2$ (see corollary 6), we give an example (example (13)) of a normal Stein space V^n (where $n = 2(j_1 + j_2) + 2$, for any positive integers j_1, j_2 with j_2 even and satisfying $j_2 \geq j_1 + 2$), which is the image of a contractible normal Stein space under a proper analytic map of degree p, such that $H^{2j_2-1}(V^n, Z_p) \neq 0$. Actually, I believe it should be possible to give such examples in dim $n = 4$ and that too in the algebraic category.

Finally, we use a result of Srinivas (see the Appendix) to conclude that, in case V^n is smooth affine algebraic variety and $W = \mathbb{C}^n$ with f a proper polynomial map then for $n \leq 4$, $CH^2(V^n) = 0$. Also, $CH^1(V^n) = \text{Pic}(V^n) = 0$, for any n, (due to Gurjar) and $CH^n(V^n) = 0$, for any n, which is easy to see.

Our theorem (2) is an easy consequence of a 'transfer' homomorphism in the theory of transformation groups, whereas theorem (7) is proved by making essential use of classical Smith theory. Due to lemma (5), we are led to the following question.

"Let X be a G-space (G a finite group) and K a subgroup of G, such that X/K is Z (or Z_p) acyclic. What can we say about the topology of X/G?"

We prove theorem (8) and its corollary (11), which, together, assert that; under a 'mild' restriction, if i_0 is the largest integer with $H^{i_0}(X, \Lambda) \neq 0$ and K is a subgroup of G with $H^{i_0}(X/K, \Lambda) = 0$ then $H^i(X/G, \Lambda) = 0$ for all $i \geq i_0$, where $\Lambda = Z$ or Z_p. We give an example (example (12)), to show that this result is sharp.

My most sincere thanks are due to R. V. Gurjar. His question led to various questions answered in this paper and with whom I had many helpful conversations. My thanks are also due to Spencer Bloch, M. V. Nori, R. R. Simha and A. A. Suslin for some useful conversations. Finally, I sincerely thank the referee for his comments which helped to bring the paper to its present form.

NOTATIONS

All the spaces, considered in this paper, are paracompact. *Cohomologies would mean sheaf cohomologies.* A space X is called Λ-acyclic (for a module Λ) if $H^0(X, \Lambda) \approx \Lambda$ and 0 otherwise. For a finite group G, $|G|$ will denote its order. For a G-space X, $X^G = \{x \in X: gx = x \text{ for all } g \in G\}$.

The transfer map, in the theory of transformation groups, gives the following.

PROPOSITION 1. *Let X be a G-space, where G is a finite group. Let K be a subgroup of G and p be any prime or $p = \infty$. In case $p < \infty$, we further assume that p is coprime to $|G|/|K|$.*

Then, the canonical map $H^i(X/G, Z_p) \to H^i(X/K, Z_p)$ is injective for all $i \geq 0$. In particular, if X/K is Z_p-acyclic then so is X/G.

(For $p < \infty$, Z_p denotes the field of order p and Z_∞ denotes the field of rational numbers Q.)

Proof. It suffices to recall that there exists a 'transfer' homomorphism $\tilde{\mu}$ from $H^i(X/K, Z_p) \to H^i(X/G, Z_p)$, such that the composite map $\tilde{\mu} \circ \pi_2^*: H^i(X/G, Z_p) \to H^i(X/G, Z_p)$ is multiplication by $|G|/|K|$ (π_2 is the orbit map: $X/K \to X/G$). For a proof see, e.g. [2; chapter III, proposition 2.4].

We use the above proposition to deduce the following improvement of Gurjar's main results in [4 (b) and (c)].

THEOREM 2. *Let $f: W^n \to V^n$ be a proper analytic map from a (topologically) contractible and normal Stein space W (of complex dim n) onto a normal Stein space V. Then*

(a) *$H^i(V, Z_p) = 0$ for all $i > 0$ and for $p = \infty$ or any prime p which is coprime to deg f.*

(b) *If we further assume that V is the homotopy type of a finite CW complex, then V is simply connected and its analytic Picard group is 0.*

Remarks 3. (a) As is well known, any affine algebraic variety is the homotopy type of a finite CW complex.

(b) If, in the above theorem, W and V are assumed to be normal affine algebraic varieties with f being a polynomial map, then Pic $(V) = 0$ provided Pic $(W) = 0$. (Pic (W) is known to be zero, if W is smooth. See, Gurjar [4 (a); theorem 1].)

The proof of this remark is given in Gurjar [4 (a)].

(c) In the above theorem, if we further assume that V is smooth and $n \geq 3$ then V is simply connected at ∞.

In fact, any simply connected smooth Stein space of dim $n \geq 3$ is simply connected at ∞. This follows easily from Andreotti and Frankel's proof of the Lefschetz theorem on hyperplane sections. See, e.g., [7; Part I, §7].

Proof (*of the theorem*) 4. (a) is an immediate consequence of propostion (1), in view of the next lemma.

To prove (b); we imitate the interesting argument due to Gurjar [4 (a), (b) and (c)]. Let $\tilde{V} \xrightarrow{\pi} V$ be the simply connected cover of V and let $\tilde{f}: W \to \tilde{V}$ be some lift of f. Gurjar [4 (a); lemma 3.1 and remark following it] proves that $\pi_1(V)$ is finite. He tacitly assumes (in his lemma (3.1)) that W and V are normal. Now, $\tilde{V}$ can be given the structure of a normal Stein space so that $\tilde{f}: W \to \tilde{V}$ is a proper (and hence open and hence surjective) morphism. Applying our theorem (2), part (a), to the map $\tilde{f}$, we get that $\chi(\tilde{V}) = 1$, where χ denotes the Euler–Poincaré characteristic. But $\chi(\tilde{V}) = |\pi_1(V)| \chi(V)$, and hence V is simply connected. It is here we are using that V is the homotopy type of a finite CW complex.

Since the analytic Picard group of $V \approx H^2(V, Z)$, its triviality follows.

LEMMA 5. *Given any two irreducible and normal Stein spaces W, V and a proper surjective analytic map $f\colon W \to V$, there exists an irreducible and normal Stein space X with an analytic map $\psi\colon X \to W$, a finite group G acting as analytic automorphisms on X and a subgroup K of G, such that the map ψ factors through X/K inducing isomorphism from X/K onto W and the map $f \circ \psi$ factors through X/G inducing isomorphism from X/G onto V.*

Proof can be found in Holmann [6; §4, pp. 339–340].

COROLLARY 6. *Under the hypothesis of theorem (2) if the map $f\colon W \to V$ is a Galois cover (i.e. the function field $\mathbb{C}(W)$ is Galois over $\mathbb{C}(V)$) (e.g. if $\deg f = 2$), then V is Z-acyclic (in fact contractible).*

This corollary is immediate, in view of the above lemma and [3 (b); chapter III, theorem 7.12].

We come to the following main result of this paper.

THEOREM 7. *If a normal Stein space V^n (of complex* dim n*) is the image of a (topologically) contractible normal Stein space W^n under a proper analytic map, then*

$$H^n(V^n, Z) = 0.$$

(One important example of such a W^n is, of course, $\mathbb{C}^n$.)

In fact, for this theorem to hold good, the contractibility of W^n could be relaxed to the requirement that $H^n(W^n, Z) = 0$.

Proof. This follows easily from the following theorem (more precisely its corollary (11)) coupled with lemma (5) and the following result due to Hamm [5].

"Any Stein space V^n (of complex dim n) is homotopy type of a n-dimensional CW complex".

THEOREM 8. *Let X be a G-space (G a finite group) of finite cohomological dimension and p be a prime, such that $H^i(X, Z_p) = 0$ for all $i > i_0$ ($i_0 \geq 0$ is fixed). Let K be a subgroup of G with $H^{i_0}(X/K, Z_p) = 0$. We further assume that $H^i(X_s(K, p), Z_p) = 0$ for all $i \geq i_0$, where $X_s(K, p) = \{x \in X\colon p$ divides the order of $I_x/I_x \cap L$, where I_x denotes the isotopy at x (with respect to the G-action) and L is the largest normal subgroup of G contained in $K\}$. Then*

$$H^i(X/G, Z_p) = 0 \text{ for all } i \geq i_0.$$

Remarks 9. (a) The above theorem for $p = \infty$ is trivially true, in view of proposition (1).

(b) The fact, that $H^i(X/G, Z_p) = 0$ for $i > i_0$, follows easily from the classical Smith theory.

Proof. The proof makes essential use of the classical Smith theory. Fix a G-stable closed subspace Y of X. Let $\mathscr{A} = \mathscr{A}(X, Y)$ denote the sheaf over X/G, with stalks defined by

$$\begin{aligned} \mathscr{A}_{\mathbf{x}} &= H^0(\pi^{-1}\mathbf{x}, Z_p) \quad \text{for} \quad \mathbf{x} \in X/G \backslash Y/G \\ &= 0 \qquad\qquad\qquad \text{for} \quad \mathbf{x} \in Y/G \end{aligned}$$

(π is the orbit map: $X \to X/G$). Clearly, G acts on the sheaf $\mathscr{A}$ and let $\mathscr{A}^G$ denote the sheaf of G-invariants in $\mathscr{A}$. From the Leray spectral sequence (see Bredon [3(a)]) for the map π,

$H^*(X/G, \mathscr{A}) \approx H^*(X, Y, Z_p)$. In fact, more generally $H^*(X/G, \mathscr{A}^H) \approx H^*(X/H, Y/H, Z_p)$, for any subgroup H of G.

In view of proposition (1), we can assume that p divides $|G|/|K|$. Let P' be a (normal) subgroup of a p-group $P \subset G$, such that $|P|/|P'| = p$. Fix an element $g \in P \backslash P'$, and let $\sigma = \sum_{k=1}^{p} g^k$ and $\tau = 1 - g$ be the elements in $Z_p[G]$ (group ring of G). Since G acts on the sheaf $\mathscr{A}$, $Z_p[G]$ also acts on $\mathscr{A}$. Let $\tilde{\sigma}$ (resp $\tilde{\tau}$) denote the sheaf morphism: $\mathscr{A}^{P'} \to \mathscr{A}$ induced by the action of σ (resp τ) on $\mathscr{A}$. ($\tilde{\sigma}$ is nothing but the 'transfer' homomorphism at the sheaf level and $\tilde{\sigma}(\mathscr{A}^{P'}) \subset \mathscr{A}^P$.) This gives rise to the following sheaf sequences.

$$(\mathrm{S}_1)\ldots \qquad \tilde{\tau}(\mathscr{A}^{P'}) \overset{i_1}{\hookrightarrow} \mathscr{A}^{P'} \overset{\tilde{\sigma}}{\to} \mathscr{A}^{P}, \quad \text{and}$$

$$(\mathrm{S}_2)\ldots \qquad \mathscr{A}^{P} \overset{i_2}{\hookrightarrow} \mathscr{A}^{P'} \overset{\tilde{\tau}}{\to} \tilde{\tau}(\mathscr{A}^{P'})$$

where i_1 and i_2 are the canonical inclusions.

It can be seen that both the sheaf sequences (S_1) and (S_2) are short exact (i.e. with zeroes at both ends) provided $Y \supset X_s(K, p)$. From now on, we assume that $Y = X_s(K, p)$ and we abbreviate $X_s(K, p)$ by X_s. They ((S_1) and (S_2)) give rise to the following long exact cohomology sequences:

$$(\mathrm{E}_1)\ldots \qquad \ldots \to H^{j+2k+1}(X/P', X_s/P', Z_p) \overset{\tilde{\sigma}^*}{\to} H^{j+2k+1}(X/P, X_s/P, Z_p) \overset{\delta}{\to} H^{j+2k+2}(X/G, \tilde{\tau}(\mathscr{A}^{P'})) \to \ldots$$

$$(\mathrm{E}_2)\ldots \qquad \ldots \to H^{j+2k}(X/P', X_s/P', Z_p) \overset{\tilde{\tau}^*}{\to} H^{j+2k}(X/G, \tilde{\tau}(\mathscr{A}^{P'})) \overset{\delta}{\to} H^{j+2k+1}(X/P, X_s/P, Z_p) \to \ldots$$

From (E_1) and (E_2) we get the following inequalities:

$$\dim H^{j+1+2k}(X/P, X_s/P) \leq \dim H^{j+1+2k}(X/P', X_s/P') + \dim H^{j+2+2k}(X/G, \tilde{\tau}(\mathscr{A}^{P'}))$$

and

$$\dim H^{j+2k}(X/G, \tilde{\tau}(\mathscr{A}^{P'})) \leq \dim H^{j+2k}(X/P', X_s/P') + \dim H^{j+1+2k}(X/P, X_s/P).$$

Summing the above two inequalities, over all $k \geq 0$, we get (after cancellation) the fundamental inequality (using the fact that X is of finite cohomological dimension):

$$(\mathrm{I})\ldots \qquad \dim H^j(X/G, \tilde{\tau}(\mathscr{A}^{P'})) \leq \sum_{d \geq 0} \dim H^{j+d}(X/P', X_s/P').$$

Now, let P_0 be a p-Sylow subgroup of K and choose a filtration of subgroups $P_0 \subset P_1 \subset \ldots \subset P_l$, where P_l is a p-Sylow subgroup of G and P_m is (normal) in P_{m+1} of index p for all $0 \leq m < l$. We have the following commutative diagram of successive transfer homomorphisms:

$$\begin{array}{ccccccccc} \mathscr{A}^{P_0} & \to & \mathscr{A}^{P_1} & \to & \ldots & \to & \mathscr{A}^{P_l} & \to & \mathscr{A}^{G} \\ & \searrow & & & & & & \nearrow & \\ & & & & \mathscr{A}^{K} & & & & \end{array}$$

This gives rise to the following commutative diagram in cohomology:

$$(\mathrm{D})\ldots \qquad \begin{array}{ccccccc} H^{i_0}(X/P_0, X_s/P_0) & \to & \ldots & \to & H^{i_0}(X/P_l, X_s/P_l) & \to & H^{i_0}(X/G, X_s/G) \\ & \searrow & & & & \nearrow & \\ & & & H^{i_0}(X/K, X_s/K) & & & \end{array}$$

Further we isolate the following.

LEMMA 10. *The transfer homomorphism:*
$H^{i_0}(X/P_m, X_s/P_m) \to H^{i_0}(X/P_{m+1}, X_s/P_{m+1})$ *is surjective for all* $0 \le m < l$.

Proof. By assumption $H^i(X_s, Z_p) = 0$ for all $i \ge i_0$ and further we know, from classical Smith theory (which asserts that if X is any space of finite cohomological dimension such that $H^i(X, Z_p) = 0$ for all $i > i_0$ and a finite group G acts on X then $H^i(X/G, Z_p) = 0$ for all $i > i_0$), that $H^i(X/P_m, Z_p) = 0$ for all $i > i_0$. Hence $H^i(X/P_m, X_s/P_m) = 0$, for all $i > i_0$. Now, using the inequality (I), we get that $H^{i_0+1}(X/G, \tilde{\tau}(\mathscr{A}^{P_m})) = 0$. This, in turn, (using the cohomology sequence E_1) gives the surjectivity. □

Proof of the theorem (*continued*). Moreover, the transfer homomorphism: $H^{i_0}(X/P_l, X_s/P_l) \to H^{i_0}(X/G, X_s/G)$ is surjective, since $|G/P_l|$ is coprime to p (see the proof of proposition 1). Hence, from commutativity of (D), $H^{i_0}(X/K, X_s/K) \to H^{i_0}(X/G, X_s/G)$ is a surjection. Further, we have the following commutative diagram.

$$\begin{array}{ccccc} H^{i_0}(X/K, X_s/K) & \xrightarrow{\gamma_1} & H^{i_0}(X/K) & & \\ \downarrow & & \downarrow & & \\ H^{i_0}(X/G, X_s/G) & \xrightarrow{\gamma_2} & H^{i_0}(X/G) & \to & H^{i_0}(X_s/G) \end{array}$$

(Here the horizontal maps are restriction homomorphisms and the vertical maps are the transfer homomorphisms.) Since $H^{i_0}(X_s, Z_p) = 0$ (by assumption), γ_2 is surjective and hence the right vertical map is also surjective. But $H^{i_0}(X/K) = 0$ (by assumption) and hence $H^{i_0}(X/G) = 0$, proving the theorem. □

The following corollary follows easily by using the Bockstein sequence.

COROLLARY 11. *Let X be a G-space of finite cohomological dimension , satisfying $H^i(X, Z) = 0$ for all $i > i_0$ and K be a subgroup of G with $H^{i_0}(X/K, Z) = 0$. Further, we assume that $H^i(X_s(K, p), Z_p) = 0$ for all $i \ge i_0$ and for all the primes p dividing $|G|/|K|$. Then, we have $H^i(X/G, Z) = 0$ for all $i \ge i_0$.*

Proof. By the above theorem, $H^i(X/G, Z_p) = 0$ for all the primes p and all $i \ge i_0$. Considering the Bockstein long exact sequence, corresponding to the coefficient sequence $0 \to Z \xrightarrow{\times p} Z \to Z_p \to 0$, we have the surjective homomorphism $H^i(X/G, Z) \xrightarrow{\times p} H^i(X/G, Z)$. Since this is true for all the primes p, we have that

$$H^i(X/G, Z) \xrightarrow{\times |G|/|K|} H^i(X/G, Z) \text{ is a surjection.}$$

Now, let $\tilde{\mu}: H^i(X/K, Z) \to H^i(X/G, Z)$ be the transfer homomorphism. As the composite map $\tilde{\mu} \circ \pi_2^*$ is multiplication by $|G|/|K|$ ($\pi_2: X/K \to X/G$ is the orbit map), $\tilde{\mu}$ is surjective for all $i \ge i_0$. This proves the corollary.

We give below an example to show that our Theorem (8) is sharp. In fact, we give an example of a compact smooth oriented $G(= Z_p \rtimes (Z_2 \oplus Z_2))$-manifold M^{i_0} of real dim i_0, such that $M^{i_0}/Z_2 \oplus Z_2$ is contractible but $H^{i_0-3}(M^{i_0}/G, Z_p) \ne 0$. From this example we extract another example of a $Z_p \rtimes Z_2$-space X (X is a finite CW complex), with $H^{i_0}(X, Z_p) \ne 0$, $H^i(X, Z_p) = 0$ for $i > i_0$, such that X/Z_2 is contractible and $H^{i_0-1}(X/Z_p \rtimes Z_2, Z_p) \ne 0$. (Examples of a compact smooth oriented $Z_p \rtimes Z_2$-manifold M^{i_0}, such that M^{i_0}/Z_2 is Z_p-acyclic (not Z-acyclic) and $H^{i_0-1}(M^{i_0}/Z_p \rtimes Z_2, Z_p) = 0$, are much easier to give.)

Example 12. We denote the Euclidean open (resp. closed) disc in $\mathbb{R}^k$ (around 0) of radius ε by $\mathring{D}_\varepsilon^k$ (resp. D_ε^k) and S_ε^k will denote the Euclidean sphere in $\mathbb{R}^{k+1}$ of radius ε. We write D^k for

D_1^k and similarly for $\mathring{D}_1^k$ and S_1^k. We will denote the map: $\mathbb{R}^{k+1}\backslash\{0\} \to S^k$, given by a $\to a/\|a\|$, by η.

Let us fix a (finite) prime $p \neq 2$ and integers $j_1, j_2 \geq 1$, satisfying $j_2 \geq j_1 + 2$ and j_2 is even.

Consider the manifold $N = N_{j_1, j_2} = \{S^{2(j_1+j_2)} \backslash e_1(S^{2j_1} \times D_\varepsilon^{0^{2j_2}})\} \bigcup_{S^{2j_1} \times S_\varepsilon^{2j_2-1}} D^{2j_1+1} \times S_\varepsilon^{2j_2-1}$,

where e_1 is the embedding of $S^{2j_1} \times D_\varepsilon^{2j_2}$ into $S^{2(j_1+j_2)}$ given by $(a, b) \mapsto \eta(a, b)$. Consider the embedding e_2: $D_\varepsilon^{2j_1+1} \times S^{2j_2-1} \to S^{2(j_1+j_2)}$, defined by $e_2(t, z_1, \ldots, z_{j_1}, w_1, \ldots, w_{j_2}) = \eta(t, z_1, \ldots, z_{j_1-1}, w_1, z_{j_1} + w_1^p, w_2, \ldots, w_{j_2})$ where $t \in \mathbb{R}$ and all $z_i, w_i \in \mathbb{C}$. We construct

$$M = M_{j_1, j_2} = \{N_{j_1, j_2} \backslash e_2(D_\varepsilon^{0^{2j_1+1}} \times S^{2j_2-1})\} \bigcup_{S_\varepsilon^{2j_1} \times S^{2j_2-1}} S_\varepsilon^{2j_1} \times D^{2j_2}.$$

It is easy to see (using long exact sequences corresponding to appropriate excisive couples) that $H^{0, 2j_1+2j_2}(M, Z) \approx Z$; $H^{2j_1+1, 2j_2}(M, Z) \approx Z_p$ and 0 otherwise.

We define a group action on M as follows. Let β: $\tilde{Z}_2 \to \mathrm{Aut}(Z_p)$ be the nontrivial homomorphism ($\tilde{Z}_2$ is nothing but a copy of Z_2). Consider the group $G = Z_2 \oplus (Z_p \rtimes_\beta \tilde{Z}_2)$. Let θ_1 (resp. θ_2) denote the generator of Z_2 (resp. $\tilde{Z}_2$). We define

$$\begin{array}{lll} \theta_1 & & (-t, z_1, \ldots, z_{j_1}, w_1, \ldots, w_{j_2}) \\ \theta_2 & (t, z_1, \ldots, z_{j_1}, w_1, \ldots, w_{j_2}) & (t, z_1, \ldots, z_{j_1}, w_1, \bar{w}_2, \ldots, \bar{w}_{j_2}) \\ \rho & & (t, z_1, \ldots, z_{j_1}, w_1, \rho w_2, \ldots, \rho w_{j_2}) \end{array}$$

where ρ is a p-th root of unity and $\bar{w}_2$ denotes the complex conjugate of w_2. The embeddings e_1 and e_2 being G-equivariant, we get a G-action on the manifold M. Let $K = Z_2 \oplus \tilde{Z}_2$.

It can be shown (with some work) that M/K is contractible. Further $H^{2j_2-1}(M/G, Z_p) \neq 0$ and $H^i(M/G, Z_p) = 0$ for $i > 2j_2 - 1$. So, taking $j_1 = 1$, it gives the desired example.

Now we define $X = M_{j_1, j_2}/Z_2$. Clearly the group $Z_p \rtimes \tilde{Z}_2$ acts on the space X. It can be easily seen that $H^{2j_2}(X, Z_p) \neq 0$ and $H^i(X, Z_p) = 0$ for all $i > 2j_2$. (Of course, $X/\tilde{Z}_2$ is contractible and $H^{2j_2-1}(X/Z_p \rtimes \tilde{Z}_2, Z_p) \neq 0$.) □

We can appropriately 'thicken' the above example to get the following.

An analytic example 13. Fix a (finite) prime $p \neq 2$ (see Corollary 6) and integers $j_1, j_2 \geq 1$ with $j_2 \geq j_1 + 2$ and j_2 is even. We give an example of normal Stein spaces $W = W_{j_1, j_2}$ and $V = V_{j_1, j_2}$, of complex dim $2(j_1 + j_2) + 2$, with a proper analytic map f of degree p from W onto V such that W is contractible and $H^{2j_2-1}(V, Z_p) \neq 0$. This answers the question (asked by Gurjar) of whether all such V are contractible in the negative.

We have defined two manifolds $N = N_{j_1, j_2}$ and $M = M_{j_1, j_2}$ in the previous section. We embed N in $\mathbb{R}^{2(j_1+j_2)+1}$ by embedding $D^{2j_1+1} \times S_\varepsilon^{2j_2-1}$ in $\mathbb{R}^{2(j_1+j_2)+1}$, under the map $(t, z, w) \mapsto (t, z, w)/\sqrt{(1+\varepsilon^2)}$ ($(t, z) \in D^{2j_1+1}$ and $w \in S_\varepsilon^{2j_2-1}$) and taking the canonical embedding of $S^{2(j_1+j_2)}$. These two patch up to give an embedding μ_1 of N in $\mathbb{R}^{2(j_1+j_2)+1}$. Now embed $S_\varepsilon^{2j_1} \times D^{2j_2}$ in $\mathbb{R}^{2(j_1+j_2)+2}$ under the map $(t, z_1, \ldots, z_{j_1}, w_1, \ldots, w_{j_2}) \mapsto$

$$\left[\frac{(t, z_1, \ldots, z_{j_1-1}, w_1, z_{j_1} + w_1^p, w_2, \ldots, w_{j_2})}{\left(1 + t^2 + \left\{\sum_{i=1}^{j_1-1} |z_i|^2\right\} + |z_{j_1} + w_1^p|^2\right)^{\frac{1}{2}}}, 1 - \|w\|^2\right] \in \mathbb{R}^{2(j_1+j_2)+1} \times \mathbb{R}$$

$$((t, z_1, \ldots, z_{j_1}) \in S_\varepsilon^{2j_1} \text{ and } (w_1, \ldots, w_{j_2}) \in D^{2j_2}) \text{ where } \|w\|^2 = \sum_{i=1}^{j_2} |w_i|^2.$$

The embedding μ_1 of N in $\mathbb{R}^{2(j_1+j_2)+1} \times 0 (\subsetneq \mathbb{R}^{2(j_1+j_2)+2})$ and the above embedding of $S_\varepsilon^{2j_1} \times D^{2j_2}$ patch up to give an embedding μ_2 of M in $\mathbb{R}^{2(j_1+j_2)+2}$.

G acts as (linear) isometries on $\mathbb{R}^{2(j_1+j_2)+2}$ by

$$(t, z_1, \ldots, z_{j_1}, w_1, \ldots, w_{j_2}, r) \begin{cases} \xrightarrow{\theta_1} (-t, z_1, \ldots, z_{j_1}, w_1, \ldots, w_{j_2}, r) \\ \xrightarrow{\theta_2} (t, z_1, \ldots, z_{j_1}, w_1, \bar{w}_2, \ldots, \bar{w}_{j_2}, r) \\ \xrightarrow{\rho} (t, z_1, \ldots, z_{j_1}, w_1, \rho w_2, \ldots, \rho w_{j_2}, r) \end{cases}$$

for $t, r \in \mathbb{R}$ and $z_1, \ldots, z_{j_1}, w_1, \ldots, w_{j_2} \in \mathbb{C}$. Extend the action of G to $\mathbb{C}^{2(j_1+j_2)+2}$, by complex linearity. It is trivially seen that the embedding μ_2 of M in $\mathbb{R}^{2(j_1+j_2)+2}$ is G-equivariant. See also [3(b); chapter VI, theorem 4.1].

Let $X_\varepsilon = \{x \in \mathbb{C}^{2(j_1+j_2)+2}: d(x, M) < \varepsilon\}$ (d is the usual Hermitian metric). Then X_ε is G-stable (action of G being isometric) and is a Stein manifold for some small ε. See [9; corollary 4.2]. Identifying X_ε with a nhd of zero section in the normal bundle of M, we get a G-equivariant deformation of X_ε onto M.

Define $W = X_\varepsilon / Z_2 \oplus \tilde{Z}_2$ and $V = X_\varepsilon / G$. W and V are normal Stein spaces (since they are orbit spaces of a normal Stein space under finite group actions). We take, for f, the orbit map from W onto V. (Of course, W and V are homotopy type of finite CW complexes.)

By the previous example W is contractible and $H^{2j_2-1}(V, Z_p) \neq 0$. □

By specializing our results to the case when $n \leq 3$, we get the following.

THEOREM 14. *Let $f: W^n \to V^n$ be a proper analytic map from a (topologically) contractible and normal Stein space W (of complex* dim *n) onto a normal Stein space V. We further assume that $n \leq 3$, then V also is contractible (we do not* a priori *assume that V is the homotopy type of a finite CW complex).*

Proof. Let $\tilde{V}$ be the simply connected cover of V and let $\tilde{f}$ be a lift of f i.e.

$$\begin{array}{ccc} & & \tilde{V} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \pi \\ W & \xrightarrow[f]{} & V \end{array}$$

Since $H^2(\tilde{V}, Z)$ injects into $H^2(\tilde{V}, Q)$ and $H^2(\tilde{V}, Q)$ is 0 (due to theorem (2)), we have $H^2(\tilde{V}, Z) = 0$. Applying theorem (7) to $\tilde{V}$, we get $\tilde{V}$ is Z-acyclic (in cohomology). This gives that $H_i(\tilde{V}, Z_p) = 0$ for all $i \geq 1$ and any prime p or ∞. But then $\mathrm{Tor}_1^Z(H_i(\tilde{V}, Z), A) = 0$ for all Z-modules A and hence $H_i(\tilde{V}, Z)$ is a flat Z-module. Considering the exact sequence $0 \to Z \to Q \to Q/Z \to 0$, we get $H_i(\tilde{V}, Z) \hookrightarrow H_i(\tilde{V}, Z) \otimes_Z Q = H_i(\tilde{V}, Q) = 0$. So $\tilde{V}$ is contractible. As a consequence $\pi_1(V)$ acts fixed point freely on a finite dimensional contractible space $\tilde{V}$. This is possible only if $\pi_1(V) = \{1\}$, proving that V is contractible.

COROLLARY 15. *In the situation of the above theorem, if $n = 3$ and V is a Stein manifold then V is diffeomorphic with $\mathbb{R}^6$.*

This follows from a theorem of Stallings [10; theorem 5.1], which states that a smooth contractible real manifold M^n (without boundary), which is simply connected at ∞, is diffeomorphic with $\mathbb{R}^n$ for $n \geq 5$. □

Remark 16. In the case when $W = \mathbb{C}^2$ and V^2 is a smooth affine algebraic variety with f a proper polynomial map from $\mathbb{C}^2$ onto V, Miyanishi [8(a)] has shown that V is algebraically $\mathbb{C}^2$. In fact, recently [8(b)] he has proved that if we assume that V is just normal, (instead of being smooth) then $V \approx \mathbb{C}^2/G$, where G is a 'small' finite subgroup of $GL(2, \mathbb{C})$.

Srinivas proves the following theorem. We include a proof of this in the appendix.

THEOREM 17. *Let V be a quasi-projective smooth variety over $\mathbb{C}$, such that $CH^2(V)$ (the Chow group) is N-torsion for some fixed integer $N > 0$ and $H^3(V, Q) = 0$. Then* $CH^2(V) \subsetneq H^4(V, Z)$. □

Now let V^n be any smooth affine algebraic variety which is the image of $\mathbb{C}^n$ under a proper polynomial map f, then it is trivial to see that $CH^i(V)$ is N-torsion for all $i > 0$ (where $N = \deg f$). We have the following.

COROLLARY 18. *Let V^n be as above. Then*

$$(1)\ CH^1(V) = CH^n(V) = 0$$
$$(2)\ CH^2(V) = 0 \text{ for } n \leq 4.$$

Proof. Combine theorem (17) with theorems (2) and (7) and with the fact that $CH^n(V^n)$ is always torsion free. □

Finally, we add the following.

Remark 19. I believe it should be possible to give examples of 4 dim normal affine algebraic varieties V such that V is the image of a contractible normal affine algebraic variety W under a proper polynomial map and such that V is not contractible (and hence $H_2(V, Z) \neq 0$). Of course, such an example will show that theorem (7), together with theorem (2), is sharp.

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APPENDIX

(V. Srinivas)

THEOREM. *Let V be a smooth quasi-projective variety over $\mathbb{C}$ such that $CH^2(V)$ is N-torsion for some integer $N > 0$ and $H^3(V, \mathbb{Q}) = 0$. Then $CH^2(V) \subset H^4(V, \mathbb{Z})$.*

Proof. We use the following results of Bloch and Ogus [1]. Let $X/\mathbb{C}$ be a smooth variety and let $\mathscr{H}^q_{\mathbb{Z}} = \mathscr{H}^q_{\mathbb{Z}}(X)$ denote the sheaf on X, for the Zariski topology, associated to the presheaf $U \to H^q(U, \mathbb{Z})$ (singular cohomology). Then Bloch–Ogus construct a flasque resolution

$$0 \to \mathscr{H}^q_{\mathbb{Z}} \to i_{\eta *} H^q(\mathbb{C}(\eta)) \to \bigoplus_{x \in X^1} (i_x)_* H^{q-1}(\mathbb{C}(x)) \to \dots$$

$$\dots \to \bigoplus_{x \in X^{q-1}} (i_x)_* H^1(\mathbb{C}(x)) \to \bigoplus_{x \in X^q} (i_x)_* \mathbb{Z} \to 0.$$

Here X^i is the set of generic points of subvarieties of condimension i and η is the generic point of X. If $x \in X^i$ and $Z \subset X$ the corresponding subvariety of codimension i then, by definition,

$$H^p(\mathbb{C}(x)) = \lim_{U \subset Z} H^p(U, \mathbb{Z})$$

where U runs over all non-empty Zariski open sets in Z. Finally, $(i_x)_* G$ denotes the direct image on X of the constant sheaf G on Z, for any abelian group G. Since flasque sheaves are acyclic for the Zariski topology, this resolution can be used to compute the cohomology groups $\mathscr{H}^p(X, \mathscr{H}^q_{\mathbb{Z}})$; in particular $H^p(X, \mathscr{H}^q_{\mathbb{Z}}) = 0$ if $p > q$.

Bloch and Ogus also show that there is a spectral sequence (the 'coniveau' spectral sequence) with

$$E_2^{p,q} = H^p(X, \mathscr{H}^q_{\mathbb{Z}}) \Rightarrow H^{p+q}(X, \mathbb{Z}).$$

Since $E_2^{p,q} = 0$ for $p > q$, we get an exact sequence

$$0 \to E_\infty^{0,3} \to E_2^{0,3} \to E_2^{2,2} \to E_\infty^{2,2} \to 0$$

yielding an exact sequence

$$\text{(E)} \dots \qquad H^3(X, \mathbb{Z}) \to H^0(X, \mathscr{H}^3_{\mathbb{Z}}) \to H^2(X, \mathscr{H}^2_{\mathbb{Z}}) \to H^4(X, \mathbb{Z}).$$

If X is projective, Bloch–Ogus show that $H^p(X, \mathscr{H}^p_{\mathbb{Z}}) = A^p(X)$, the group of condimension p cycles modulo algebraic equivalence, and $E_2^{p,p} \to H^{2p}(X, \mathbb{Z})$ is the cycle map.

LEMMA 1. *For any smooth variety $X/\mathbb{C}$, $\mathscr{H}^3_{\mathbb{Z}}$ is torsion free (so that $H^0(X, \mathscr{H}^3_{\mathbb{Z}})$ is torsion free).*

Proof. We have to show that for any $N > 0$, the map induced by multiplication by N on $\mathscr{H}^3_{\mathbb{Z}}$ is injective.

From the exact sequence

$$\dots \to \mathscr{H}^i_{\mathbb{Z}} \to \mathscr{H}^i_{\mathbb{Z}/N} \to \mathscr{H}^{i+1}_{\mathbb{Z}} \xrightarrow{\times N} \mathscr{H}^{i+1}_{\mathbb{Z}} \to \dots$$

it suffices to prove that $\mathscr{H}^2_{\mathbb{Z}} \to \mathscr{H}^2_{\mathbb{Z}/N}$ is surjective. If $\pi\colon X_{an} \to X$ is the identity map on X, where X_{an} is the associated analytic space, the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X_{an}} \to \mathcal{O}^*_{X_{an}} \to 0$$

$$f \to e^{2\pi i f}$$

yields a sequence of sheaves on X

$$0 \to \mathbb{Z} \to \pi_* \mathcal{O}_{X_{an}} \to \pi_* \mathcal{O}^*_{X_{an}} \to \mathscr{H}^1_{\mathbb{Z}} \to \dots$$

The map $\mathcal{O}_X^* \to \pi_* \mathcal{O}_{X_{an}}^* \to \mathcal{H}_{\mathbb{Z}}^1$ yields a diagram

$$\begin{array}{ccccc} \mathcal{O}_X^* \otimes \mathcal{O}_X^* & \longrightarrow & \mathcal{H}_{\mathbb{Z}}^1 \otimes \mathcal{H}_{\mathbb{Z}}^1 & \xrightarrow{\text{cup}} & \mathcal{H}_{\mathbb{Z}}^2 \\ \downarrow{\scriptstyle \text{symbol}} & & & & \downarrow \\ \mathcal{K}_{2,X} & \longrightarrow\!\!\!\!\rightarrow & \mathcal{K}_{2,X} \otimes \mathbb{Z}/N & \xrightarrow[\sim]{\alpha} & \mathcal{H}_{\mathbb{Z}/N}^2 \end{array}$$

Here $\mathcal{K}_{2,X}$ is the sheaf associated to the presheaf $U \to K_2(H^0(U, \mathcal{O}_X))$, where K_2 is the functor defined by Milnor [4]; the symbol is also defined in [4]. The map α is the composite

$$\mathcal{K}_{2,X} \otimes \mathbb{Z}/N \xrightarrow{\beta} \mathcal{H}_{et}^2(\mu_N^{\otimes 2}) \xrightarrow{\simeq} \mathcal{H}_{\mathbb{Z}/N}^2$$

where β is the Galois symbol (see Bloch [2; Chapter 5], for example) and $\mathcal{H}_{et}^2(\mu_N^{\otimes 2})$ is the sheaf associated to $U \to H_{et}^2(\mu_N^{\otimes 2})$. We identify $\mathcal{H}_{et}^2(\mu_N^{\otimes 2})$ with $\mathcal{H}_{\mathbb{Z}/N}^2$ via the isomorphism $H_{et}^2(U, \mu_N^{\otimes 2}) \cong H_{et}^2(U, \mathbb{Z}/N) \cong H^2(U, \mathbb{Z}/N)$, where $\mu_N^{\otimes 2} \cong \mathbb{Z}/N$ on U (since $\mu_N \subset \mathbb{C}$) and we use the comparison theorem for etale and singular cohomology for the second isomorphism.

By a result of Stein [5], the map 'symbol' is surjective, while β is an isomorphism (hence so is α) by results of Mercurjev-Suslin [3]. This proves the surjectivity of $\mathcal{H}_{\mathbb{Z}}^2 \to \mathcal{H}_{\mathbb{Z}/N}^2$, proving the lemma.

We now consider the exact sequence (E)

$$H^3(V, \mathbb{Z}) \xrightarrow{f} H^0(V, \mathcal{H}_{\mathbb{Z}}^3) \xrightarrow{g} H^2(V, \mathcal{H}_{\mathbb{Z}}^2) \to H^2(V, \mathbb{Z})$$

as above.

By hypothesis $H^3(V, \mathbb{Q}) = 0$, so that $H^3(V, \mathbb{Z})$ is torsion. Since $H^0(V, \mathcal{H}^3)$ is torsion free, $f = 0$ and so g is injective. We claim that $H^2(V, \mathcal{H}_{\mathbb{Z}}^2) \cong CH^2(V)$. This implies that $H^2(V, \mathcal{H}_{\mathbb{Z}}^2)$ is torsion, so that $g = 0$, proving the theorem. The claim follows from

Lemma 2. *For any smooth quasi-projective variety $X/\mathbb{C}$. $H^p(X, \mathcal{H}_{\mathbb{Z}}^p) \simeq A^p(X)$.*

Assuming the lemma, we see that $CH^2(V)$ maps onto $H^2(V, \mathcal{H}_{\mathbb{Z}}^2)$ and the kernel is the group of cycles algebraically equivalent to 0 modulo rational equivalence. But this group is divisible [1; lemma (7.10)] and is a subgroup of the N-torsion group $CH^2(V)$, we deduce that $CH^2(V) \cong A^2(V) \cong H^2(V, \mathcal{H}_{\mathbb{Z}}^2) \subset H^4(V, \mathbb{Z})$, proving the theorem.

Proof of lemma 2. The lemma is proved in [1] (theorem (7.3)) if X is projective. If X is quasi-projective, let Y be a smooth projective variety such that $X \subset Y$ is an open immersion. From the Bloch–Ogus resolution for $\mathcal{H}^p$ on Y, we have a diagram

$$\begin{array}{ccccc} \bigoplus_{x \in Y^{p-1}} H^1(C(x)) & \longrightarrow & \bigoplus_{x \in Y^p} \mathbb{Z} & \longrightarrow & H^p(Y, \mathcal{H}_{\mathbb{Z}}^p) \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{x \in X^{p-1}} H^1(C(x)) & \longrightarrow & \bigoplus_{x \in X^p} \mathbb{Z} & \longrightarrow & H^p(X, \mathcal{H}_{\mathbb{Z}}^p) \to 0 \end{array}$$

The first two vertical arrows are the projections induced by the inclusions $X^{p-1} \subset Y^{p-1}$, $X^p \subset Y^p$. Thus

$$\ker (H^p(Y, \mathcal{H}_{\mathbb{Z}}^p) \to H^p(X, \mathcal{H}_{\mathbb{Z}}^p))$$

is generated by codimension p cycles on Y supported on $Y - X$. Thus $[Z] = 0$ in $H^p(X, \mathcal{H}_{\mathbb{Z}}^p)$ for some condimension p cycle Z on X iff the closure $\bar{Z}$ of Z in Y satisfies

$$[\bar{Z}] = [Z'] \text{ in } H^p(Y, \mathcal{H}_{\mathbb{Z}}^p)$$

for some $Z' \subset Y - X$. Thus $(\bar{Z} - Z')$ is algebraically equivalent to 0 on Y i.e. Z is algebraically equivalent to 0 on X (this is the definition of algebraic equivalence on the open variety X). This proves the lemma.

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