RATIONAL HOMOTOPY THEORY OF FLAG VARIETIES
ASSOCIATED TO KAC-MOODY GROUPS

By
Shrawan Kumar
Mathematical Sciences Research Institute,
Berkeley, CA
and
Tata Institute of Fundamental Research,
Colaba, BOMBAY (INDIA)

Introduction
This paper is a sequel to my earlier paper "Geometry of Schubert cells and cohomology of Kac-Moody Lie-algebras". It uses many results from the paper, just mentioned, in an essential manner.

Let g be a Kac-Moody Lie-algebra and let \(\mathfrak{p}_X\) be a parabolic subalgebra of finite type. Let G be the algebraic group (in general infinite dimensional), in the sense of Duflo and Zong, associated with g (called a Kac-Moody algebraic group) and let \(P_X\) be the parabolic subgroup of finite type of G, associated with \(\mathfrak{p}_X\). One of the principal aims of this paper is to study the rational homotopy theory of the flag varieties \(G/P_X\). We prove that \(G/P_X\) is a "formal" space in the sense of rational homotopy theory. Further, we explicitly determine the minimal models of the flag varieties \(G/B\). We also prove that the Lie-algebras cohomology, with trivial coefficients, \(H^i_q(\mathfrak{g})\) (resp. \(H^i_q(\mathfrak{g},\mathfrak{r})\)) is isomorphic, as graded algebras, with singular cohomology, \(H^i(G,\mathbb{C})\) (resp. \(H^i(G/P_X,\mathbb{C})\)) and the isomorphism is explicitly given by an integration map. \(q^1\) denotes the commutator subalgebra of \(g\) and \(r_X\) is the reductive part of \(\mathfrak{p}_X\).

Now we describe the contents of this paper in more detail.

Chapter (1) is devoted to recalling various definitions and well known elementary facts from Kac-Moody theory. We fix notations to be used throughout the paper.

Chapter (1). Main result of this section is theorem (1.6). This

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states that $H^1(g, r, \mathfrak{g})$ (resp. $H^1(g_1, \mathfrak{g}_1)$) is isomorphic with $H^1(G/L, \mathfrak{g})$ (resp. $H^1(G/L, \mathfrak{g}_1)$), as graded algebra and moreover this isomorphism is explicitly given by an integration map. In particular, this gives a "complete" description of the cohomology algebra of the loop algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1^{1, \infty}$ and its central extension (the affine algebra), for any finite dimensional semi-simple Lie-algebra $\mathfrak{g}_0$. Kac-Peterson also claim to have proved that $H^1(g_1)$ is isomorphic with $H^1(G, \mathfrak{g})$.

Their proofs have not yet appeared, but presumably, it is very different from ours. As more or less immediate corollaries (1.3) we deduce that $H^1(g)$ and $H^1(g_1)$ are both Lie-algebras: for a finite dimensional simple Lie-algebra $\mathfrak{g}_0$, $H^1(\mathfrak{g}_0 \oplus \mathfrak{g}_1^{1, \infty})$ is one dimensional; $H^1(\mathfrak{g}_1)$ is always 0 for any symmetrizable Kac-Moody Lie-algebras and hence, in particular, the standard map $g_1 \rightarrow g_0 \oplus \mathfrak{g}_1^{1, \infty}$ (where $g$ is the affine Lie-algebra associated with the finite dimensional simple Lie-algebra $\mathfrak{g}_0$) is a universal central extension. A similar result is true in the twisted affine case. Universality of this central extension is originally due to H. Garland, R. Wilson and V. Chari.

Chapter 2. One of the main results of this section is theorem (2.2), which states that the DGA (differential graded algebra) $C(\mathfrak{g}, \mathfrak{r}, \mathfrak{a})$ is formal (in the sense of rational homotopy theory). Our proof of this is similar to one of the proofs given by Deligne-Griffiths-Morgan and Sullivan for the formality of compact Kähler manifolds, but there is one essential difference in that the usual Hodge decomposition for Kähler manifolds is replaced by the "Hodge decomposition" with respect to the disjoint operators $d$ and $\mathfrak{g}$ developed in [Ku1]. This theorem, coupled with a technical lemma (Lemma 2.6), gives rise to theorem (2.7) which states that $G/P_X$ is a formal space (where $P_X$ is any standard parabolic of $G$ of finite type).

So that, complete rational homotopy information of $G/P_X$ can be derived from the cohomology algebra $H^*(G/P_X)$. Also, in particular, all the necessary products of any order are zero over $\mathcal{G}$. As a second application of theorem (2.2), we prove that the Leray-Serre spectral sequence in cohomology corresponding to the fibration $G \rightarrow G/B$ degenerates at $E_2$ over $\mathcal{G}$. In fact, recently, Kac-Peterson have proved a far reaching result that this spectral sequence degenerates
at $E_2$ even over $\mathbb{Z}/p\mathbb{Z}$, for any prime $p$.

In Chapter 3, we explicitly determine the minimal models for the flag varieties $G/B$ (or any symmetrizable Kac-Moody group $G$). We also determine the Lie-algebra structure (under Whitehead product) on $\pi_*(G/B) \otimes \mathbb{Q}$. See Theorem (3.8) for the complete description.

After this work was done, I learnt from Victor Kac that Theorem (2.7) was observed by P. Deligne (using the machinery of $\ell$-adic cohomology) in a private communication to him. My very sincere thanks are due to Dale Peterson for many helpful conversations. I thank Helsuke Hironaka, Victor Kac, James R. Munkres, Leslie D. Saper and Pradeep Shukla for some helpful conversations.
0. Preliminary and Notations

(9.1) Definitions.

(a) A symmetrizable generalized Cartan matrix \( A = (a_{ij})_{i,j \in I \subseteq K} \) is a matrix of integers satisfying \( a_{ii} = 2 \) for all \( i \), \( a_{ij} \leq 0 \) if \( i \neq j \). \( A \) is symmetric for some diagonal matrix \( D = \text{diag} (d_1, \ldots, d_I) \) with \( d_i > 0 \) if \( i \in \Phi \).

(b) Choose a triple \((h, \nu, \chi)\), unique up to isomorphism, where \( h \) is a vector space over \( C \) of dimension co-rank \( A \), \( \nu = \{ \eta_i \}_{i \in I} \subset h^* \) and \( \chi \) are linearly independent indexed sets satisfying \( A_{ij} \eta_i = \eta_j \). The Kac-Moody algebra \( g = g(A) \) is the Lie algebra over \( C \), generated by \( h \) and the symbols \( \eta_i \) and \( f_i \) \((i \in I)\) with the defining relations \( [h, \eta] = 0 \); \([h, f] = a_{ij} [h, \eta_j] \) and so \((1_{16i} \eta_j) \) for all \( i, j \in I \). 

\( h \) is canonically embedded in \( g \).

(9.2) Root space decomposition \( \{ \eta, f \} \). There is available the root space decomposition \( g = h \oplus \bigoplus_{\alpha \in \Delta} g_{\alpha} \), where \( g_{\alpha} = \{ x \in g \mid [h, x] = a_{\alpha} x \), for all \( h \in h \} \) and \( \Delta = \{ \alpha \in h^* \mid \text{such that } g_{\alpha} \not= 0 \} \). Moreover \( \Delta = \Delta_+ \cup \Delta_- \).

where \( \Delta_+ \subset \Delta \), \( \eta \in \Delta_+ \), \( \eta \in \mathbb{Z}_+ \) (the non-negative integers) for all \( i \) and \( \Delta_- = -\Delta_+ \). Elements of \( \Delta_+ \) (resp. \( \Delta_- \)) are called positive (resp. negative) roots.

(9.3) Parabolics. We fix a subset \( X \) (including \( X = \emptyset \)) of \( \{1, \ldots, I\} \) of finite type, i.e., the submatrix \( A_X = (a_{ij})_{i,j \in I_X} \) is a classical Cartan matrix of finite type. There is a natural injection \( g_X = g(A_X) \hookrightarrow g(A) \). Defins \( A_X^+ \) (resp. \( A_X^- \)) \( = \Delta_+ \cap \bigoplus \mathbb{Z} \alpha_i \) (resp. \( \Delta_- \cap \bigoplus \mathbb{Z} \alpha_i \)) where \( \alpha_i \) is linear span of \( \chi_{16i} \).

Define the following Lie-subalgebras. \( \pi = \bigoplus_{\alpha \in \Delta_+} g_{\alpha} \) \( = u = g_X \).

\[ r = \sum_{\alpha \in \Delta_+} g_{\alpha}, \quad r = r_X + h \) and \( \rho = \rho_X = r + u \). Of course \( r \) is a reductive algebra. \( \rho \) is called the \( F \)-parabolic.
subalgebra \( \mathfrak{a} \) for finite dimensionality of \( q_{\lambda} \) defined by \( X \). If \( X = \mathbb{N} \), the associated parabolic \( \rho = h + n \) is the Borel subalgebra. If \( A \) itself is a finite type (i.e., \( A \) is a classical Cartan matrix), then the \( \mathbb{R} \)-parabolic subalgebras are precisely the parabolic subalgebras of \( q \) containing the Borel subalgebra \( h + n \).

9.4 Weyl group \([K]\).
There is a Weyl group \( W \subset \text{Aut}(\mathfrak{h}) \) generated by the reflections \( C_{\lambda} \in \text{End}(\mathfrak{h}) \), associated to the Lie-algebra \( \mathfrak{g} \). \( W \) is a Coxeter system, hence we can talk of the lengths of elements of \( W \).

\( W \) preserves \( \mathfrak{A} \). \( \mathfrak{A}^{\alpha} \) is defined to be \( W \cdot \omega \) and \( \mathfrak{A}^{\alpha} = \mathfrak{A} \setminus \mathfrak{A}^{\alpha} \). For \( \alpha \in \mathfrak{a}^{\mathbb{R}} \), \( \dim \mathfrak{a}_{\alpha} = 1 \) and \( \mathfrak{a} \cap \mathfrak{a} = \{0, \omega \} \).

9.5 Cartan involution. There is a \( \mathbb{R} \)-linear unique involution \( \omega \) of \( \mathfrak{g} \) defined by \( \omega([h]) = -[h] \) for all \( h \in \mathfrak{h} \) and \( \omega([x]) = -[x] \) for all \( x \in \mathfrak{g} \). It is easy to see that \( \omega \) leaves \( q(\mathfrak{g}) \) (the 'real points' of \( \mathfrak{g} \)) stable.

Further, there is a unique \textit{conjugate linear} involution \( \omega_{q} \) of \( q \) which coincides with \( \omega \) on \( q(\mathfrak{g}) \).

9.6 Algebraic group associated to a Kac-Moody Lie-algebra \( \mathfrak{g} \). \( \mathfrak{g} \) and \( [V, \theta] \). A \( g^{1} = [a, q(\mathfrak{a})] \) module \( (V, \theta) \) is called integrable, if \( \theta(a) \) is locally nilpotent whenever \( a \in \mathfrak{a} \). For \( a \in \mathfrak{a}^{\mathbb{R}} \). Let \( G^{*} \) be the free product of the additive groups \( \mathfrak{g}(a)^{q(\mathfrak{a})} \) with canonical inclusions \( 1_{q}: \mathfrak{g}_{q}^{1} \rightarrow G^{*} \). For any integrable \( g^{1} \)-module \( (V, \theta) \), define a homomorphism \( \theta^{*}: G^{*} \rightarrow \text{Aut}_{q}(V) \).

9.7 Let \( \mathfrak{N} \) be the intersection of all \( \text{Ker} \theta^{*} \). Let \( G = G^{*}/\mathfrak{N} \).

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Choose \( A_i \in L^i(\mathfrak{g}(k)) \), satisfying \( A(0) = a_i \) for all \( \lambda \in \mathfrak{h} \). There is an embedding \( \text{KER}_2 ; \text{page 162-165} \)

\[ G \rightarrow \text{A} = \left( \bigoplus_{\lambda \in \mathfrak{h}} L^i(\mathfrak{g}(k)) \right) \oplus \left( \bigoplus_{\lambda \in \mathfrak{h}} L^\lambda(\mathfrak{g}(k)) \right) \]

defined by \( (\sigma, \mu) = \left( \sum_{\lambda \in \mathfrak{h}} \mu(\lambda, \lambda) \left( \bigoplus_{\lambda \in \mathfrak{h}} L^i(\mathfrak{g}(k)) \right), \sum_{\lambda \in \mathfrak{h}} \sigma(\lambda, \lambda) \left( \bigoplus_{\lambda \in \mathfrak{h}} L^\lambda(\mathfrak{g}(k)) \right) \right) \).

Here \( L^i(\mathfrak{g}(k)) \) is the integrable highest weight module with highest weight \( i \). \( L^\lambda(\mathfrak{g}(k)) \) is the vector space \( L^\lambda \) regarded as a \( \mathfrak{g} \)-module under \( \mu^*(\mathfrak{g}(k)) = \mu^*(\mathfrak{g}(k)) \mu^*(\mathfrak{g}(k)) \) is a highest weight vector in \( L^\lambda \) and \( \mu^*(\mathfrak{g}(k)) \) is denoted \( \mu^*(\mathfrak{g}(k)) \) regarded as an element in \( L^\lambda \).

For "differentiating" \( \mu \), we get an embedding \( \mu^* : \mathfrak{g} \rightarrow \text{A} \). More explicitly \( \mu^*(x) = \left( \sum_{\lambda \in \mathfrak{h}} \lambda(\lambda) \left( \bigoplus_{\lambda \in \mathfrak{h}} L^i(\mathfrak{g}(k)) \right), \sum_{\lambda \in \mathfrak{h}} \lambda(\lambda) \left( \bigoplus_{\lambda \in \mathfrak{h}} L^\lambda(\mathfrak{g}(k)) \right) \right) \), for \( x \in \mathfrak{g} \).

\( \text{A} \) is endowed with a Hausdorff topology defined as follows.

A set \( V \subset \text{A} \) is open if and only if \( \forall \mathfrak{v} \in \text{A} \cap V \), for all the finite dimensional vector sub-spaces \( \mathfrak{v} \) of \( \text{A} \). Now, put the subspace (through \( \mu \)) topology on \( \mathfrak{g} \). \( \mathfrak{g} \) may be viewed as a, possibly infinite dimensional, affine algebraic group in the sense of {Šafarevič} [Saj] with Lie-algebra \( \mathfrak{g}^1 \). For a proof, see [KP, 94]. In [KP, 94], a priori a different topology is put on \( \mathfrak{g} \) but it can be seen that these two topologies, on \( \mathfrak{g} \), actually coincide.

(39) Recall, from (30.3), the conjugate linear involution \( \omega_0 \) of \( \mathfrak{g} \). On "integration" this gives rise to an involution \( \omega_0 \) of \( G \). Let \( K \) denote the fixed point set of this involution.

(30.3) The subgroup of \( \text{Aut}(\mathfrak{g}(k)) \) generated by the reflections \( x_i^\pm 1 \) is denoted by \( \text{W} \) (resp. \( \text{W}_k \)), where \( x_i^\pm 1 \) is \( \eta_i(\pm h_i) \), for all \( h \in \mathfrak{h} \). It is easy to see that, under the canonical identification \( x : \text{Aut} \mathfrak{h} \rightarrow \text{Aut} (\mathfrak{h}^1) \) (given by \( x(f)(h) = \mathfrak{h}^1 f(h) \)) for \( f \in \text{Aut} \mathfrak{h} ; f \in \mathfrak{h} \) and \( h \in \mathfrak{h} \). \( \text{W} \) corresponds with \( \text{W}_k \).

(30.3) In fact \( x(f)^\pm 1 = x(f) \), for all \( f \in x_i^\pm 1 \), from now on, we would identify \( \text{W} \) with \( \text{W} \) under \( x \) and use the same symbol \( W \) for both.

For each \( t \in \mathbb{G} \), there exists a unique homomorphism \( \phi_t : \text{SL}_2(\mathbb{G}) \rightarrow G \) satisfying \( \phi(t) = \exp(f_{t1}) = \exp(f_{t1}) \) (for all \( t \in \mathbb{G} \)). Define \( G_1 = \phi(t) \cdot \mathbb{G} \cdot \phi^{-1}(t) \). \( G_1 = \ldots \)
\( \delta \text{H} \rangle \langle K \rangle; \ N_i = \text{Normalizer of } H_i \text{ in } G; \ H = \text{the subgroup of } G \) generated by all \( H_i; \ K = \text{the subgroup of } G \) generated by all \( N_i \). There is an isomorphism \( \gamma : W \rightarrow N/H \) such that \( \gamma(y) \) is the coset \( N_i H \subset H \) mod \( H \). See [KP, 72]. We would, sometimes, identify \( W \) with \( N/H \) under \( \gamma \).

Put \( B = HU \) (\( U \) is defined in §10.6) and \( P = F_X = BWK \). Denote by \( R_X \) the subgroup \( K \cap P_X \). It is easy to see that the canonical inclusion \( K/R_X \rightarrow G/P_X \) is a (surjective) homomorphism. Use [KP, Theorem 4.6]. If \( C \subset G \) is given the subspace topology and topology on \( G \) is described in §10.6).

6.9. Bruhat decomposition ([KP], [KP], and [T]). Recall the definition of \( W^1_X \) from §10.4. \( W^1_X \) can be characterized as the set of elements of minimal length in the cosets \( W_X \) (\( w \in W \)) (each such coset contains a unique element of minimal length).

\( G \) can be written as disjoint union \( G = \bigcup_{w \in W^1_X} (U w^{-1} P_X) \) so that \( G/P_X = \bigcup_{w \in W^1_X} U w^{-1} P_X/P_X \).

(\( w \)) is an element of \( N \) satisfying \( w(w) \) and \( H = \gamma(w) \). In fact, we will choose \( w(w) \in N \cap K \), which is possible because \( K \subset H \).

\( G/P_X \) is a C-W complex with cells \( C = U w^{-1} P_X/P_X \) and \( \dim_{w} V_{w} = 2 \) length \( w \). (To interchange right and left cosets we have, in the expression of \( V_{w} \) \( w^{-1} \) instead of \( w(w) \) as in \( KP \).)

6.10. Notations. Throughout the paper, unless otherwise specifically stated, all the vector spaces will be over \( \mathbb{C} \) and linear maps would be \( \mathbb{C} \)-linear maps. For a vector space \( V \), \( A(V) \) denotes the exterior algebra and \( S(V) \) denotes the symmetric algebra.

For a \( \mathbb{C} \)-algebra pair \((q,r)\), \( C(q,r) \) denotes the standard co-chain complex associated to the pair \((q,r)\). See, e.g., [AM, 11]. For a topological space \( X \), \( C(X, U) \) will denote the usual singular co-chain complex of \( X \) with coefficients in \( U \). Unless otherwise stated, cohomologies would be with complex coefficients.
The symmetrizability assumption on the Kac-Moody Lie-algebras g(A) (i.e., A is symmetizable) would be implicitly assumed throughout the paper. By a Kac-Moody algebraic group, we mean a group G (as defined in §10.8), associated to some Kac-Moody Lie algebra g. The subgroup K (defined in §10.7) would be called the standard compact real form of G (though it is non-compact in general). By a standard parabolic of G, we would mean PX (defined in §10.86 for some X ∈ C(1, 1)). If, in addition, X is of finite type F_X would be called a standard parabolic of finite type.

When X = A, so that P_A = B, it is called the standard Borel subgroup of G.
1. An analogue of Cartan-deRham Theorem for Kac-Moody Groups

(1.1) Let \( g = \mathfrak{g}(\mathbb{A}) \) be a Kac-Moody Lie-algebra associated to a generalized Cartan matrix \( A = (a_{ij}) \) such that \( A \in \mathbb{C} \) and let \( X \subset \mathfrak{g}(\mathbb{A}) \) be a subset of finite type. There is associated to a group \( G \), its standard compact real form \( K \) and a standard parabolic subgroup \( P = P_X \) as described in \(|\#0.10|\).

(1.2) Definitions.

(a) We recall the definition of a smooth map from a finite dimensional smooth manifold \( M \to K \) or \( K/K_X \) from \(|\#0.31|\) \( K_X = K \cap P_X \).

Let \( f : M \to K \) be a continuous map. Consider the composite of the maps

\[
M \xrightarrow{\pi} K \hookrightarrow G \xrightarrow{i} \mathfrak{a} \quad (i \text{ is defined in } |\#0.6|).
\]

Since \( \text{int} f : M \to \mathfrak{a} \) is continuous, given any \( x_0 \in M \), there exists an open neighborhood \( N(x_0) \) of \( x_0 \) in \( M \) such that \( \text{int} f |N(x_0)| \subset \mathfrak{a} \), for some finite dimensional vector subspace \( \mathfrak{a} \) of \( \mathfrak{a} \). We say that \( f \) is smooth at \( x_0 \) if the restricted map \( \text{int} f |N(x_0)| \to \mathfrak{a} \) is smooth

(= \( C^n \)) in the usual sense. The map \( f \) itself is said to be smooth if \( f \) is smooth at all \( x_0 \in M \).

A map \( f : M \to K/K_X \) is said to be smooth if for any \( x_0 \in M \), there exists an open neighborhood \( N(x_0) \) of \( x_0 \) in \( M \) and a smooth lift \( \tilde{f} : N(x_0) \to K \) (i.e. \( \pi \circ \tilde{f} = f |N(x_0)| \)) where \( \pi \) is the canonical projection: \( K \to K/K_X \).

(b) By a smooth singular \( n \)-simplex in \( K \) (resp. \( K/K_X \)), we mean a continuous map \( f : \Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^n : t_0 \geq 0 \text{ and } \sum t_i \leq 1\} \to K \) (resp. \( f : \Delta^n \to K/K_X \)) such that there exists an open neighborhood \( N \) of \( \Delta^n \) in \( \mathbb{R}^n \) and a smooth map \( \pi|N : N \to K \) (resp. \( \pi|N : N \to K/K_X \)) extending \( f \).

Let us denote by \( \Delta^n_{\mathbb{N}}(K) \) (resp. \( \Delta^n_{\mathbb{N}}(K/K_X) \)), the free abelian group on the set of all the smooth singular \( n \)-simplices \( f \) in \( K \) (resp. in \( K/K_X \)).

Finally, denote \( \sum_{n \geq 0} \text{Hom}_{K}(\Delta^n_{\mathbb{N}}, \mathfrak{a}) \) (resp. \( \sum_{n \geq 0} \text{Hom}_{K}(\Delta^n_{\mathbb{N}}, \mathfrak{a}) \))
Let $M$ be a finite dimensional smooth manifold with a smooth map $f: M \to K$ resp. $f: M \to K^q$. Given $u \in C^0_q(A^1)$ resp. $u \in C^0_q(A^1, r_1)$, $r = r_f$ is defined in (10.3) and $r^1 = r \cap q_1$, we construct a smooth $q$-form $f^1 u$ on $M$ as follows.

Fix a $x_0 \in M$. Choose a local smooth lift $\tilde{f}: \text{N}(x_0) \to K$.

When $f: M \to K$, $\tilde{f}$ is, of course, $f$ itself.) Consider the map

$$i \circ L_{x_0} \circ \tilde{f}: \text{N}(x_0) \to \mathbb{A},$$

where $L_{x_0}$ is the left translation by $\tilde{f}(x_0)^{-1} x_0 \tilde{f}(x_0)^{-1}$; $\mathbb{A} \to K$. Define $\tilde{u} \circ \tilde{u}_0 = (i \circ L_{x_0} \circ \tilde{f})^* \tilde{u}_{x_0}$, where $\tilde{u}$ is any translation invariant $q$-form on $\mathbb{A}$ so that $\tilde{u}$ is given by $\tilde{u}_0 \in \operatorname{Hom}_K(A^q(A), C^0)$ satisfying $\tilde{u}_0[1]_{A^q(A)} = u$. ($q^1$ is identified as a subspace of $A \times K \times \mathbb{S}$ see (10.6).)

It is a routine checking, using the following facts, that $\tilde{f}^* u$ is well defined, i.e., if $\tilde{u}_0$ does not depend upon the particular choice of $x_0$, $\tilde{u}$ and further $(\tilde{f}^* u)$ is a smooth $q$-form on $M$.

Let $M$ be a (finite dim.) smooth manifold and $x_0 \in M$. Given two smooth maps $f, g: (M, x_0) \to (G, e)$ i.e. $\tilde{f} = i \circ f: M \to A$ is smooth and so is $\tilde{g}$, then the following are true.

1. The map $H^{-1}: (M, x_0) \to (G, e)$, defined by $H^{-1}(m) = \frac{\text{exp}(-H(m))}{\text{exp}(-H(x_0))}$ for all $m \in M$, is smooth and $d(t \cdot^{-1} x_0) = (df)x_0 = (dg)x_0$.

2. Fix any $s \in A$, then the map $f_{\ast s}: M \to A$, defined by $f_{\ast s}(m) = f(s \cdot s)x_0$ is smooth.

3. $(df)(x_0) \Gamma_{x_0}(M) \subset (\text{Ad})^1$.

4. Fix $a \in G$, then the map $a \cdot^{-1}: (M, x_0) \to (G, e)$, defined by
is smooth and for any $v \in T_{x_0}(M)$,

$$\delta s \Gamma s^{-1} x_0 v = \delta (\text{Ad } x_0 v),$$
where $x_0 \in q^1$ is the element satisfying $\delta x_0 v = i(x_0 v)$. (Ad: $G \to \text{Aut}(q^1)$ is defined in [KP2: 13].)

(1) and (2) are easy in view of [KP2: 14]. Dusa Peterson showed me proofs of (3) and (4).

(1.3) Integration map. We describe an 'integration' map

$$\int: C^1_c(K, E) \to C^0_c(q^1, E)$$
as follows.

$$\int u = \int_{\Delta^n} u^a, \text{ for } u \in C^0_c(q^1) \text{ and } f: \Delta^m \to K \text{ a}
simplex} \in A_n^c(K, \Sigma K)$$

Exactly similarly, we can define an integration map

$$\int: C^1_c(q^1, r^{-1}) \to C^0_c(K/K_X, E).$$

We have the following two technical lemmas.

(1.4) Lemma. The integration maps $\int: C^1_c(q^1) \to C^0_c(K, E)$

and $\int: C^1_c(q^1, r^{-1}) \to C^0_c(K/K_X, E)$ are both co-chain maps. Further they induce algebra homomorphisms in cohomology.

Proof. We would prove that $\int: C^1_c(q^1) \to C^0_c(K, E)$ is a co-chain map, which induces algebra homomorphism in cohomology. The proof of the analogous statement for $K/K_X$ is similar.

To prove that $\int$ is a co-chain map, in view of Stokes' theorem, it suffices to show that for any (finite dimensional) smooth manifold $M$ and a smooth map $f: M \to K$, we have, for any $u \in C^0_c(q^1)$,

$$\delta f^{ab} (u) = f^a (du).$$

Extend $v$ arbitrarily to an element $v_0$ of $\text{Hom}_{\mathbb{R}}(q^1, \mathbb{A})$, $q^1$ is canonically embedded in $\mathbb{A}$ via $i$. See (5.6). The embedding $\iota: K \to \mathbb{A}$ is $K$-equivariant (it acting on $K$ by left multiplication and of course $\mathbb{A}$ is a representation space for $K$).

Extend $v_0$ to a $K$-invariant form $\delta v_0$ on $\mathbb{A}$, though defined only on

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Since the representation map \( G \times \mathcal{A} \to \mathcal{A} \) is regular (see [KP, 4.4]), \( \Sigma_0 \) can further be extended to a smooth (in the obvious sense) \( \omega \)-form \( \Sigma_0 \) defined on whole of \( \mathcal{A} \). Of course, \( (i-\alpha)(\Sigma_0) = \text{diff}(\Sigma_0) \Sigma_0 \). Further, \( (i-\alpha)^{\Sigma_0} \) can be easily seen to be the form \( \Sigma_0 \). So, in view of \( \Sigma_0 \)-invariance of \( \Sigma_0 \) on \( (\mathbb{R}, t) \), it is enough to show that

\[
(\theta(\bar{\Sigma}_0)^{\Sigma_0}, x_0, \ldots, x_n) = (\delta(\Sigma_0), \ldots, \delta(x_n)), \quad \text{for all } t_0, \ldots, t_n \in \mathbb{R}.
\]

Fix any \( \alpha \) locally-finite elements \( x_0, \ldots, x_n \in \mathfrak{a}^* \). Consider the 1-parameter group of diffeomorphisms \( \theta(t) \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A} \), defined by \( \theta(t)x\colon= \exp(t\alpha)x \). It can be easily seen that the corresponding vector field \( \bar{X}_0 \) on \( \mathcal{A} \) is given by \( \bar{X}_0(x) = x_0 \). Now we would write \( \epsilon \) for \( \theta(\alpha) \).

\[
(\theta(\bar{X}_0)x_0, \ldots, x_n) = \sum_{i=0}^n (-1)^i \frac{\partial^n}{\partial x_0^i} \left( \theta(\bar{X}_0x_0, \ldots, x_n) \right) + \sum_{i<j} (-1)^i \frac{\partial^n}{\partial x_0^i} \left( \theta(\bar{X}_0, \ldots, x_j, \ldots, x_0, \ldots, x_n) \right)
\]

\[= \sum_{i<j} (-1)^i \left( \frac{\partial^n}{\partial x_0^i} \left( \theta(\bar{X}_0, \ldots, x_j, \ldots, x_0, \ldots, x_n) \right) \right) \cdot \left( \frac{\partial^n}{\partial x_0^j} \left( \theta(\bar{X}_0x_0, \ldots, x_n) \right) \right)
\]

But since, for all \( 0 \leq i, j \leq n, \)

\[
\exp(-t\alpha)x_j \exp(t\alpha)x_j = x_j,
\]

\[
\epsilon \exp(-t\alpha)x_j \exp(t\alpha)x_j = x_j,
\]

\[
(\epsilon^{\exp(-t\alpha)x_j})x_j = x_j,
\]

Also, \( x_jx_0 = \sum_{t=0}^\infty \frac{\exp(-t\alpha)x_j \exp(t\alpha)x_j - x_j}{\alpha} \).
Putting these in (9) we get

\[
\begin{align*}
(\delta_{ij})_{j=1}^{\infty} a_j &= \sum_{i=0}^{\infty} (-1)^{i+1} \sum_{j=0}^i \delta_{ij} a_{n_0} \ldots \delta_{ij} a_{n_i} \\
&= \sum_{j=0}^i (-1)^{i+1} \delta_{ij} a_{n_0} \ldots \delta_{ij} a_{n_i} \\
&= \sum_{j=0}^i (-1)^{i+1} \delta_{ij} a_{n_0} \ldots \delta_{ij} a_{n_i}
\end{align*}
\]

Since ad locally-finite elements in \(g^1\) span \(g^1\), this proves that \(f\) is
a co-chain map.

Now we prove that \( \int \) induces algebra homomorphism in cohomology.

Let \( \text{Sing}^c_\mathbb{R}(K) \) resp. \( \text{Sing}^d_\mathbb{R}(K) \) denote the simplicial set 
\( n \rightarrow \text{Sing}^n_\mathbb{R}(K) \) (resp. \( \text{Sing}^n_\mathbb{R}(K) \)) where \( \text{Sing}^n_\mathbb{R}(K) \) resp. \( \text{Sing}^n_\mathbb{R}(K) \) denotes the set of all the smooth (resp. continuous) singular 
\( n \)-simplices in \( K \) with the standard face and degeneracy maps. Let 
\( \Omega^p_{\text{diff}}(\text{Sing}^n_\mathbb{R}(K)) = \bigoplus_{i=0}^p \Omega^i_{\text{diff}}(\text{Sing}^n_\mathbb{R}(K)) \) denotes the piece-wise smooth de-Rham complex associated to the simplicial set \( \text{Sing}^n_\mathbb{R}(K) \).

where an element of \( \Omega^p_{\text{diff}}(\text{Sing}^n_\mathbb{R}(K)) \) is, by definition, a function \( \theta \) which assigns to each element of \( \text{Sing}^n_\mathbb{R}(K) \) \( n = 0, 1, 2, \ldots \) a complex

valued smooth \( p \)-form on \( \Delta^n \) (i.e. a \( p \)-form on \( \Delta^n \subset \mathbb{R}^n \)), which extends to a smooth \( p \)-form on an open neighborhood \( U \) of \( \Delta^n \), such that \( \theta \) commutes with the face and degeneracy operators. \( \Omega^p_{\text{diff}}(\text{Sing}^n_\mathbb{R}(K)) \) is made into a DGA (DGA is defined in §0.3(a)) under pointwise addition, multiplication and the usual differential of forms. Define a

DGA morphism \( \int: \Omega^p_{\text{diff}}(\text{Sing}^n_\mathbb{R}(K)) \rightarrow \mathbb{R} \) by \( \int \omega = \int \omega \), for \( \omega \in \Omega^p_{\text{diff}}(\text{Sing}^n_\mathbb{R}(K)) \) and for any smooth singular \( n \)-simplex \( \sigma: \Delta^n \rightarrow \mathbb{R} \).

There \( \int \) a canonical integration map \( \int: \bigoplus_{i=0}^p \Omega^i_{\text{diff}}(\text{Sing}^n_\mathbb{R}(K)) \rightarrow \mathbb{R} \), defined by

\[
\int \theta = \int \theta_n, \quad \text{for } \theta \in \Omega^p_{\text{diff}}(\text{Sing}^n_\mathbb{R}(K))
\]

and for any smooth singular simplex \( \sigma: \Delta^p \rightarrow \mathbb{R} \). (We denote the

integration map here by \( \int \) to distinguish it from our earlier integration map \( \int \).

By Stokes' theorem \( \int \) is a co-chain map. Further, it is known

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(see [57, 577] and our next lemma (1.9)) that
\[ \phi : \mathcal{C}^\infty_{-}(K) \longrightarrow \mathcal{C}^\infty_{0}(K) \]
induces algebra isomorphism in cohomology. Of course, by definition, \( \phi \eta \phi \) and hence the assertion that \( \phi \) induces algebra homomorphism in cohomology, follows.

The restriction map
\[ \gamma : \mathcal{C}^\infty_{K}(K) \longrightarrow \mathcal{C}^\infty_{K}(K) \]
induces isomorphism in cohomology, where \( \mathcal{C}^\infty_{K}(K) \) is the usual (continuous) singular co-chain complex with complex coefficients. A similar statement holds good with \( K \) replaced by \( K/K \) throughout.

**Proof.** For any \( n \geq 0 \), let \( \mathcal{A}_n^\infty \) be the sheaf on \( K \) associated with the presheaf (for any open set \( U \) in \( K \))
\[ 0 \longrightarrow \mathcal{A}_n^\infty \longrightarrow \mathcal{C}^\infty_n \bigcup \mathcal{C}^\infty_{n-1} \longrightarrow \mathcal{C}^\infty_n \longrightarrow \cdots \]
where \( \mathcal{A}_n^\infty \) denotes the free abelian group on the set of all the smooth singular simplices \( \sigma : \Delta^n \longrightarrow U \). There is clearly a sheaf sequence
\[ 0 \longrightarrow \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty_0 \longrightarrow \mathcal{C}^\infty_1 \longrightarrow \mathcal{C}^\infty_2 \longrightarrow \cdots \]
(\( \mathcal{E} \) denotes the constant sheaf on \( K \).

To prove the lemma, it suffices (see [Wa; Chapter 5]) to show that the above sequence (8) is exact and all the sheaves \( \mathcal{C}^\infty_n \) are fine sheaves.

(a) \( \mathcal{C}^\infty_n \) are fine sheaves. Choose a locally finite open cover \( \{U_\alpha\} \) of \( K \) (\( K \) being a paracompact space, this is possible).
Choose a (discontinuous) partition of unity \( \{\sigma_\alpha\} \) subordinate to the cover \( \{U_\alpha\} \) in which the functions \( \sigma_\alpha \) take values 0 or 1 only.
For each \( \alpha \) and \( n \), define an endomorphism \( e_{\alpha,n} \) of \( \mathcal{C}^\infty_n \) by setting
\[ (e_{\alpha,n}f)(\sigma) = \sigma_\alpha(0)\sigma(\sigma_{\alpha}f)(\sigma_{\alpha}(0)) \]
for \( f \in \mathcal{C}^\infty_n \) and \( \sigma \) a smooth singular \( n \)-simplex in \( U \) and \( f \in \mathcal{C}^\infty_n \).
This defines a partition of unity for all the sheaves \( \mathcal{C}^\infty_n \), concluding that they are fine.

(b) The sequence (8) is exact. We need to prove a Poincaré-type lemma. Write any element \( \eta \in \mathcal{A} \) as

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\[ a = \frac{1}{2} \sum_{i=1}^{n} \lambda_i \in \text{weights of } v_i^1(a) \left( \Lambda_i^A \right) \]
\[ + \frac{1}{2} \sum_{i=1}^{m} \mu_i \in \text{weights of } w_i^1(a) \left( \Lambda_i^B \right) \]

where \( v_i^1 (\text{resp. } w_i^1) \) denotes \( \lambda_i \) (resp. \( \mu_i \)) weight vector \( \in (\Lambda_i^A) \) (resp. \( \Lambda_i^B \)). Let \( N = \{ a \in A^\infty : v_i^1(a) \in \mathbb{R}^{|\Lambda_i^A|} \} \) and \( v_i^1(a) \in \mathbb{R}^{|\Lambda_i^A|} \) for any \( i \). Fix a smooth function \( \psi : [0, 1] \to [0, 1] \) satisfying \( \psi(0) = 0 \) for all \( t \leq 3/4 \) and \( \psi(1) = 1 \) for all \( t \geq 1 \). Define a contraction \( H : \mathbb{R} \times \mathbb{R} \to N \) by

\[ H(a) = \frac{1}{2} \sum_{i=1}^{n} \psi(t) \left( A_i - \lambda_i \right) v_i^1(a) \left( \Lambda_i^A \right) + \frac{1}{2} \sum_{i=1}^{m} \psi(t) \left( B_i + \mu_i \right) w_i^1(a) \left( \Lambda_i^B \right) \]

\[ + \frac{1}{2} \sum_{i=1}^{n} \psi(t) \left( 2t \right) A_i \left( a \right) \psi \left( 2t \right) \left( A_i \right) + \frac{1}{2} \sum_{i=1}^{m} \psi(t) \left( 2t \right) B_i \left( a \right) \psi \left( 2t \right) \left( B_i \right) \]

where \( v_i^1(a) \) (resp. \( w_i^1(a) \)) denotes \( \lambda_i \) (resp. \( \mu_i \)) weight vector \( \in (\Lambda_i^A) \) (resp. \( \Lambda_i^B \)). Let \( \lambda_i = \psi(t) (A_i - \lambda_i) \) and \( \mu_i = \psi(t) (B_i + \mu_i) \). Fix a smooth function \( \psi : [0, 1] \to [0, 1] \) satisfying \( \psi(0) = 0 \) for all \( t \leq 3/4 \) and \( \psi(1) = 1 \) for all \( t \geq 1 \). Define a contraction \( H : \mathbb{R} \times \mathbb{R} \to N \) by

\[ H(a) = \frac{1}{2} \sum_{i=1}^{n} \psi(t) \left( A_i - \lambda_i \right) v_i^1(a) \left( \Lambda_i^A \right) + \frac{1}{2} \sum_{i=1}^{m} \psi(t) \left( B_i + \mu_i \right) w_i^1(a) \left( \Lambda_i^B \right) \]

\[ + \frac{1}{2} \sum_{i=1}^{n} \psi(t) \left( 2t \right) A_i \left( a \right) \psi \left( 2t \right) \left( A_i \right) + \frac{1}{2} \sum_{i=1}^{m} \psi(t) \left( 2t \right) B_i \left( a \right) \psi \left( 2t \right) \left( B_i \right) \]

\[ \text{where } v_i^1(a) \left( \Lambda_i^A \right) \text{ and } w_i^1(a) \left( \Lambda_i^B \right) = a_i \psi(t) \left( A_i \right) \left( \Lambda_i^A \right) \]

\[ \text{where } A_i = \sum_{j=1}^{n} a_{ij} \text{ (resp. } \mu_i = -\lambda_i \text{) then } \text{ht} \cdot A_i - \lambda_i \text{ (resp. } \mu_i). \]

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\( \delta^* A \) is defined to be \( \Sigma \delta^* \cdot \psi(\delta) \) is defined to be \( \pi \lambda \)

\( H \) has the following properties.

(1) \( H \) is smooth (in the obvious sense).

(2) \( H \mathbb{R} \times (\mathbb{R} \cap \bar{N}) \subset \mathbb{R} \cap \bar{N} \).

(3) \( H(t,a) = a \), for all \( t \geq 1 \) and \( a \in \mathbb{R} \cap \bar{N} \).

(4) \( H(t,a) = (t,a) \), for all \( t \leq 1/4 \) and all \( a \in \mathbb{R} \).

Now we are ready to show that

\[
0 \rightarrow E \rightarrow C_c^0(\mathbb{R} \cap \bar{L}) \rightarrow C_c^1(\mathbb{R} \cap \bar{L}) \rightarrow \cdots
\]

is exact. It suffices to find a homotopy operator, i.e., a linear map

\[
h_p : C_c^p(\mathbb{R} \cap \bar{L}) \rightarrow C_c^{p-1}(\mathbb{R} \cap \bar{L}), \text{ for all } p \geq 1,
\]

satisfying

\[
d\circ h_p + h_p \circ d = \text{Id}.
\]

For a smooth singular simplex \( \sigma : \Delta^{p-1} \rightarrow \mathbb{R} \cap \bar{L} \), define a smooth singular simplex \( h_p \sigma : \Delta^p \rightarrow \mathbb{R} \cap \bar{L} \).

\[
\tilde{h}_p \sigma(t_1, \ldots, t_p) = \tilde{t}_1^{p-1} \tilde{t}_2 \cdots \tilde{t}_p
\]

for \( t_1, \ldots, t_p \neq 0 \)

\[
= 0 \text{ for } t_1, \ldots, t_p \leq 1/4.
\]

Now, define \( \partial_0 \tilde{h}_p \sigma = f_0 \tilde{h}_p \sigma \), for \( f \in C_c^0(\mathbb{R} \cap \bar{L}) \).

It is easy to see that \((\ast)\) is satisfied.

Since \( \mathbb{K} \) is homogeneous under right multiplication and also that, we can choose a co-final system \( \mathcal{C}(\mathbb{R}^+) \) of open neighborhoods of \( v(0) \) in \( \mathbb{K} \) such that \( H \mathbb{R} \times (\mathbb{K} \cap \bar{N}) \subset \mathbb{K} \cap \bar{N} \), we get that the sheaf sequence \((S)\) is exact.

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The case of $E/K_X$ is similar. We can define a similar smooth contraction of the open set $\cap_{w \in W} U^w \cap \mathcal{X}^1 \subset G/P_X$.

**Remark.** A contraction, similar to $H$ above, has earlier been used by Dad-Peterson to prove contractibility of $X$.

We come to the main theorem of this section.

(1.9) **Theorem.** Let $g = g(A)$ be the Kac-Moody Lie-algebra associated to a symmetrically generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ and let $X \subset \{1, \ldots, \varepsilon\}$ be a subset of finite type. Then the identification maps (defined in (1.3))

(a) \[ \int: C(g^1, r^1_X) \to C^{\omega}(K/K_X, E) \] and

(b) \[ \int: C(g^1) \to C^{\omega}(E, E) \]

both induce algebra isomorphisms in cohomology.

Recall that $g^1 = \{g_0, g_1\}; r^1_X = g_1 \cap r_X$ (where $r_X$ is defined in (9.21)) $Y$ is the standard compact real form of the Kac-Moody algebraic group $G$ associated to $g$ and $K_X = K \cap P_X$ ($P_X$ is the standard parabolic subgroup of $G$). See (10.10).

In particular, in view of lemma (1.5), the Lie-algebra cohomology $H^0(g^1, r^1_X)$ (resp. $H^0(g^1)$) is algebra isomorphic with the singular cohomology $H^0(K/K_X, E)$ (resp. $H^0(K, E)$). Also, by (Le 56), the canonical inclusion $C(g^1, r^1_X) \to C(g^1, r^1_X)$ induces isomorphism in cohomology.

**Proof.**

(a) By lemma (1.5) and the Bruhat decomposition (10.09),

\[ \dim H^0(K/K_X, E) = \dim H^0(K/K_X, E) = \# \text{elements of length } n/2 \]

in $W_{1/2}$. ($W_{1/2}$ is defined in (9.61). Also, by (Le 56) see also [Koij; 12.3]); \[ \dim H^0(g, r_X) = \# \text{elements of length } n/2 \text{ in } W_{1/2}. \]

Hence \[ \dim H^0(K/K_X, E) = \dim H^0(g, r_X). \] Since, by lemma (1.4), \[ \int \] induces algebra homomorphism, it suffices to show (for dimensional considerations) that the induced map $H^0(\int)$, $H^0(g, r_X) \to$
\[ H^{2n}(C^*_{\text{u}}(K/K^0, \mathbb{C})) \] is injective for all \( n \geq 0 \).

For any \( \omega \in \Omega^1 \) of length \( n \), \( U \omega \in \Omega^{2n-1} \) in \( G/P_X \) is an open cell (of real dim \( 2n \)) in \( G/P_X \) (i.e., homogeneous with \( \mathbb{C}^n \)). See (10.9). 

Further, this extends to a smooth singular simplex \( \sigma_{\omega} : 
\Delta^n \to G/P_X \in \mathcal{A}^{2n}(G/P_X) \), so that \( \mathcal{D}(\sigma_{\omega}) \) is a 2n-1 dim cycle in \( G/P_X^{2n-2} \). But since \( H_{2n-1}(G/P_X^{2n-2}) = 0 \), there exists a 2n-dim chain \( b_{\omega} \in \mathcal{G}^n(G/P_X^{2n-2}) \) such that \( \mathcal{D}(b_{\omega}) = \mathcal{D}(\sigma_{\omega}) \). In fact, we can further choose \( b_{\omega} \in \mathcal{A}^n(G/P_X) \).

By [Kn1; Theorem 4.5], there are "d-b harmonic" forms \( \left( \mathcal{L}_{\sigma_{\omega}} \right) \) such that
\[ \int_{\partial_R} s_{\omega} = \int_{\partial_R} \mathcal{L}(s_{\omega}) = s_{\omega} \] for \( \omega \in \Omega^1 \) with \( \partial(s) = \#(s) \). So
\[ \int_{\partial_R} \mathcal{L}(s_{\omega}) = \int_{\partial_R} s_{\omega} = s_{\omega} \] for \( s_{\omega} \in \Omega^1 \) with \( \partial(s) = \#(s) \). (Actually the integrand itself is 0, as \( s_{\omega} \) is a 2n-form and \( b_{\omega} \) is a chain in \( (G/P_X)^{2n-2} \). Since \( \left( \mathcal{L}_{\sigma_{\omega}} \right) \) is a d-b harmonic form in \( (G/P_X)^{2n-2} \), it is an \( \mathcal{A}^n \)-basis of \( H^n(\mathbb{R}^n, \mathbb{C}) \) by [Kn1; Thm. 13.1].

This immediately gives injectivity of \( H^n \). Hence (a) follows.

(b) There is a Hochschild-Serre filtration \( F^p = C^p(G \to K) \) with respect to the subalgebra \( K \), defined as follows. \( F^p_G = \{ g \in C^p(G) : \text{im}(\partial_{g}) = 0 \} \), whenever \( n-p+1 \) of the arguments \( \partial_j \) belong to \( K \), and \( F^p = \mathbb{C} \) if \( n = p \).

Also there is a Leray-Serre filtration \( G = (G_p) \) of \( C^*_{\text{u}}(K) \), associated with the filtration \( \mathfrak{n} : K \to K/T \) (where \( T = B \mathfrak{n}^0(K) \)), defined by \( G_p = \{ c \in C^*_{\text{u}}(K) : \text{im}(\partial)^0 = 0 \} \), where \( K^{p-1} \)

denotes \( \pi^{-1}(K/T)^{p-1} \) and \( \Delta(K^{p-1}) \) denotes the usual (continuous) singular chain complex of \( K^{p-1} \). (\( K/(T)^{p-1} \), as earlier, denotes \( n^{(p)} \)-th skeleton of \( K/T = G/B \) under the Bruhat decomposition.)

It is fairly easy to see that \( F^p \subset G_p \) for all \( p \).

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Let $E_r(f)$ and $E_r(g)$ be the spectral sequences associated with the filtrations $F$ and $G$ respectively. Since $\int$ preserves filtrations, it induces a map $E_r(f) \rightarrow E_r(g)$ for all $r$. 

By [85; 141], $E^2_{r, q}(f) \cong H^q(F^{p \cdot 2k}, \Omega^p)$ and $E^2_{r, q}(g) \cong H^q(G^{p \cdot 2k}, \Omega^p)$ and converging to the cohomology $H^q$ (1). (Although, in [85], this is proved under the assumption that $q$ is finite dimensional, it can easily be adapted to our situation since $h^k$ acts reducibly on $q$.) 

Further, we can similarly modify the proof of lemma (1.5) to give the following generalization (cf. lemma (1.5)). 

For any $p > 0$, the restriction map $\text{Hom}_{\text{alg}}(A \otimes \mathbb{C}^{p \cdot 2k}, \mathbb{C}) \rightarrow \text{Hom}_{\text{alg}}(A \otimes \mathbb{C}^{p \cdot 2k}, \mathbb{C})$ induces an isomorphism in cohomology. 

Using the above lemma, this gives that the filtration $G = G_\cdot$ is regular (and hence strongly convergent) in the sense of [CE; page 324] and also (by Lurie-Serre) $E^2_{r, q}(G) \cong H^q(K/T) \otimes H^r(T)$. 

By part (b) of this theorem $H^q(F) \cong H^q(G)$ for all $p > 0$. From this it is fairly easy to see that $E^2_{r, q}(F) \cong E^2_{r, q}(G)$ for all $p$ and $q$, and hence $H^q(F) \cong H^q(G)$ for all $r \geq 2$ and all $p, q \geq 0$. 

This completes the proof of part (b) as well.

(1.7) Remark. Kac-Peterson also claim to have proved that $H^\ast_{\text{Lie}}(K)$ is isomorphic with $H^\ast_{\text{alg}}(K)$, although their proofs have not yet appeared.

The following lemma is trivial to verify.

(1.8) Lemma. For any Lie algebra $g$ and a subalgebra $z$, $H^\ast_{\text{Lie}}(g) = H^\ast_{\text{Lie}}(g, z)$, for $z$ a central subalgebra $z$ of $g$ such that $z \subseteq z$.

Proof. In fact the co-chain complex $C(z, z)$ itself is isomorphic with $C(z, z, z)$.

(1.9) Corollaries.

(a) For any Kac-Moody Lie algebra $g$, $H^\ast_{\text{Lie}}(g)$ and $H^\ast_{\text{alg}}(g)$ are both Hopf algebras.

(b) Let $\varphi$ be a finite dimensional simple Lie-algebra and $\varphi$ be an automorphism of $\varphi$, of order $k$, induced by an automorphism of the Dynkin diagram (so that $k = 1, 2, 3$). Then
$H^2(q^0)$ is one dimensional, where

$$\phi^{(k)} = \sum_{m=0}^{k-1} a_m \otimes t^m$$

($a_m = \{ x \in g : [x] = 2^x(-1)^{m/2}x^{-1}x \}$).

(c) Let $g$ be any Kac-Moody Lie-algebra then $H^2(q^1) = 0$. In particular, let $g$ be the affine Lie-algebra associated to a finite dim. simple Lie-algebra $\mathfrak{g}_\mathbb{C}$ and an automorphism $\tau$ of $\mathfrak{g}_\mathbb{C}$, of order $k$, as in (b). Then the one dimensional central extension 0 $\rightarrow$ $\mathfrak{g}$ $\rightarrow$ $\mathfrak{g}(\tau)$ $\rightarrow$ 0 (see [WW, page 210]), $\mathfrak{g}$ $\mathfrak{g}$ is nothing but $\mathfrak{g}(\tau)$ in the notation of [WW,}, 193], $\mathfrak{g}$ $\mathfrak{g}$ is universal.

(d) $H^2(q) = \Lambda^2(q/q^1)$ and $H^2(q) = H^2(q) \otimes \Lambda^2(q/q^1)$, for any $q$.

Proof.
(a) Since $H^2(q^0) = H^2(K,K)$; $K$ is a topological group and $H^2(q) = H^2(q^0) \otimes \Lambda^2(q/q^1)$ by [E, Proposition 1.9], (a) follows.

(b) We prove (b) in the special case $\theta = 1$. The general case is exactly similar. Specializing theorem (1.8) (a) to the affine Lie-algebra $\mathfrak{g}$ associated with $\mathfrak{g}_\mathbb{C}$ and choosing an appropriate maximal parabolic $\mathfrak{p}_\mathbb{C}$, we get that $H^2(q^0,q^1+\mathfrak{g}_\mathbb{C})$ is one dimensional, since, from Bruhat-decomposition, $K/K$ can be easily seen to have only one cell in dim. 2.

By lemma (1.8), taking $a = k^1 + q_0$ and $z =$ centre of $q^1$, we get $H^2(q^0, q^1+\mathfrak{g}_\mathbb{C}) = H^2(q_0) \otimes \mathfrak{g}$ $\tau$ $\mathfrak{g}$ $\mathfrak{g}$ (1.2.4, i.e. this is available) and the fact that $H^1(q_0) = H^1(q_0) = 0$, we get (b).

(c) By theorem (1.9) (b), we have $H^2(q^1) = H^2(q)$. But, by [E, Theorem 4], $K$ is simply connected. Further, using the long exact homotopy sequence for the fibration $K \rightarrow H/T$, we get

$$0 \rightarrow \pi_2(K) \rightarrow \pi_2(H/T) \rightarrow \pi_2(T) \rightarrow 0 .$$

Since $\pi_2(K/T)$ and $\pi_2(T)$ are free abelian groups with equal rank, we get $\pi_2(K) = 0$. This gives $H^2(q^1) = 0$.

Now universality of the central extension follows immediately from (1.9) (b) together with standard facts on central extensions. See
(6) This follows easily from (5) and (KS16: Proposition 1.9).

\textbf{(1.10) Remark.} (1.9) (6) is due to the referees of [G]. See [G; 12]. (1.9) (c) in the affine case is, independently, due to Garland [G, Theorem (3.14)] and Vyjayanthi Chari (unpublished) and the twisted affine case is due to Wilson [W]. (1.9) (d) is strengthening of some results due to Berman [B].

\textbf{(1.11) Remark.} Union mixture of topological and geometric arguments, we show that, in general, the inclusion of the space of bi-invariant forms \( C_{\mathbb{R}}(\mathfrak{g}^\mathbb{R}^+) \to C(\mathfrak{g}^+) \) does not induce isomorphism in cohomology. The counterexample exists in any irreducible Kac-Moody Lie-algebras except in the case where it is a finite-dimensional Lie-algebras or \( \mathfrak{sl}(n) \).
2. Formality of Flat Varieties Associated to Kac–Moody Groups

We recall some, fairly known, definitions from rational homotopy theory. See, e.g., [DGMS]; [GM]; [Q]; [S1]; [S2].

(2.1) Definitions.

(a) A differential graded algebra \(A\) (abbreviated to DGA) is a graded algebra (over \(\mathbb{C}\)) \(A = \bigoplus_{\mathbb{Z}_{p > 0}} A_p\) with a differential (i.e. \(d^2 = 0\)) \(d : A \to A\) of degree +1, such that

1. \(A\) is graded commutative, i.e.,
\[ab = (-1)^{|a||b|}ba\] for \(a \in A^n\) and \(b \in A^m\).

2. \(d\) is a derivation, i.e.,
\[(da)b = (da)b + (-1)^{|a|}a(db)\] for \(a \in A^n\).

A is said to be connected if \(H^0(A)\) is the ground field \(\mathbb{C}\) and \(A\) is one-connected if, in addition, \(H^1(A) = 0\).

(b) A DGA \(\mu\) is a minimal differential algebra, if

(i) \(\mu\) is decomposable, i.e., \(d(\mu^n) \subset \mu^1 \cdot \mu^t\) (\(\mu^n\) denotes the augmentation ideal \(\sum_{p > 0} \mu_p\)).

(ii) \(\mu\) may be written as an increasing union of sub-DGAs \(\mu_0 = \mathbb{C} \subset \mu_1 \subset \mu_2 \subset \ldots\) with \(\mu_i \subset \mu_{i+1}\) an elementary extension for all \(i \geq 0\), i.e., \(\mu_{i+1}\) is a graded algebra of the form \(\mu_i \oplus \text{Sym} V_{d_i}\), for some \(d_i > 0\) (\(\text{Sym} V_{d_i}\) denotes the symmetric (resp. exterior) algebra on \(V_{d_i}\) if \(d_i\) is even (resp. odd). We assign grade \(d_i\) to elements of \(V_{d_i}\)) and such that \(d_{\mu_{i+1}} = d_i\) and \(d_{\mu_{i+1}}(V_{d_i}) \subset V_{d_i}\).

(c) A minimal model for a DGA \(A\) is a minimal
differential algebra $\mu_A$ together with a DGA homomorphism $\mu: \mu_A \rightarrow A$ such that $\mu$ induces isomorphism in cohomology.

An important fact is that every one connected DGA $A$, such that $H^i(A)$ is finite dimensional for all $i$, has a minimal model unique up to isomorphism. See [DGMS, Theorem 1.2 (a)].

In this paper, we would only consider one-connected DGA’s $A$ with the additional assumption that $H^i(A)$ is finite dimensional for all $i$. From now on, this would be our implicit assumption on DGA’s.

(iii) A minimal differential algebra $\mu$ is said to be formal if there is a DGA homomorphism $\mu: \mu \rightarrow H^*(\mu)$ inducing the identity on cohomology. ($H^*(\mu)$ is equipped with identically zero differential.)

(iv) The homotopy type of a DGA $A$ is a formal consequence of its cohomology if its minimal model is formal.

Now we can state one of the main theorems of this section.

(2.3) Theorem. Let $g = g(\alpha)$ be the Kac-Moody Lie-algebra associated to a symmetrizable generalized Cartan matrix $\Lambda = \{e_i\}_{i \in I \cup I}$ and let $X \subset \{1, \ldots, \alpha\}$ be a subset of finite type.

Then, the homotopy type of the DGA $C(\alpha, r)$ is formal consequence of its cohomology, where $r = r_X$ and $C(\alpha, r)$ are defined in §6.3 and §6.10 respectively.

Proof. Our proof of this theorem is similar to the first proof of formality of Kahler manifolds, given by Deligne-Griffiths-Morgan-Sullivan [DGMS: 56]. One essential difference, however, is that the Hodge decomposition with respect to the operators $d^\alpha$ (in the definition of $d^\alpha$ is replaced by the Hodge decomposition proved in [H2], for the ”disklike” operators $\alpha$ and $\beta$.

We need the following $d\alpha$ lemma.

(2.3) Lemma. Recall the definition of the operators $d\alpha$:

$C(\alpha, r) \rightarrow C(\alpha, r)$ for $[K_{\alpha}]: 53$. As in [DGMS], define the operator $d\alpha = (\alpha - \alpha)$ acting on $C(\alpha, r)$. Then we have

(1) In $d \cap \ker d\alpha \subset \ker (H^\alpha)$ and
Proof. Let \( \omega \in \text{Im } d \cap \text{Ker } d^c \). Since \( d = d' + d'' \)

\[ d'' \omega = 0 = d'' \omega \]

From the 'Hodge type decomposition' [Ko2; Theorem 3.13 and Remark 3.14] and disjointness of \((d',d'')\) [Ko2; Proposition 3.7], we get \( \omega \in \text{Im } d' \cap \text{Ker } S' \). Further, again by using [Ko2; Lemma (3.8), Theorem (3.13), Remark (3.14) and Lemma (3.35)], we get \( \text{Im } d' \subset \text{Im } S = \text{Im } d \oplus \text{Im } d'' \). Since, by assumption, \( \omega \in \text{Im } d \), we get \( \omega \in \text{Im } d' \). Write \( \omega = d'n \), for some \( n \in C(g,r) \). Express \( n = d' n_1 + 3' n_2 + 3_3 \), for some \( n_1, n_2 \in C(g,r) \) and \( n_3 \in \text{Ker } S' = \text{Ker } S'' \). This gives, on taking \( d'' \)

\[ d'' \omega = d'' (d'n_1 + 3' n_2 + 3_3) \quad \text{(since } n_3 \in \text{Ker } S') \]

Using \( d'' 3' = 3' d'' = 0 \) and \( d'' d'n_1 = d'' d'n_2 = 0 \) (see [Ko2; Lemma 3.1]), the identity \((1.2)\)) we get \( d'' n = -d'' n_1 - 3' d'n_2 \). So \( d'' \omega = d'' (d'n_1 + 3' n_2) = 0 \) (since \( d'' \omega = d'' \omega = 0 \), by \((1.2)\)). By disjointness of the pair \((d',d'')\) [Ko2; Proposition 3.7], \( d'' d'n_2 = 0 \). Putting this in \((1.2)\), we get the first part of this lemma. The second part follows exactly similarly.

\(2.4\) Proof of Theorem \(2.2\). Denote by \( H_{d'}(g,r) \) the \( d' \)
cohomology of the complex \( C(g,r) \) under \( d' \) and by \( Z_{d'}(g,r) \) the \( d' \) closed forms in \( C(g,r) \). Consider the diagram

\[ C(g,r) \xrightarrow{d'} Z_{d'}(g,r) \xrightarrow{\iota} H_{d'}(g,r), \]

where \( \iota \) is the canonical inclusion and \( \alpha \) the canonical projection.

Since \( d d' = -d' d \), \( Z_{d'}(g,r) \) is stable under \( d \). Moreover, by
the previous lemma, the differential induced by $d$ on $\mathbb{E}_d(g,r)$ is zero.

We prove that $i$ and $a$ both induce isomorphisms in cohomology, if we consider $C(g,r)$, $Z_d(g,r)$ and $H_d(g,r)$ as co-chain complexes under $d$.

(1) $a$ is surjective: Given $\omega \in Z_d(g,r)$, we need to show that there exists $a \in C(g,r)$ such that $\omega = d\alpha$ is closed. By $d\alpha$-lemma, $\omega = -d\alpha$, so $d\omega = d\alpha = 0$.

(2) $a$ is injective: We need to show that $im\ d \cap ker\ d \subset im\ Z_d(g,r)\backslash ker\ d$, which is immediate from $d\alpha$-lemma.

(3) $i$ is injective: We need to prove that $im\ d \cap ker\ Z_d(g,r) \subset ker\ d$. Use $d\alpha$-lemma.

(4) $i$ is surjective: We need to show that

$$im\ d + ker\ d \cap ker\ d = ker\ d.$$ 

By [Ku: Theorem 3.35, Remark 3.14 and Lemma 3.31], $ker\ S \subset ker\ d \cap ker\ d + im\ d = ker\ d$. This gives surjectivity of $i$.

Theorem follows, now, by choosing a minimal model $\mu$ for the DGA $\mathbb{E}_d(g,r)$. (Observe that $C(g,r)$, hence $Z_d(g,r)$, is non-connected and $\mathbb{E}(C(g,r))$ is finite dimensional for all $d$, by [Li: 16] or [Ku: Theorem 3.15].)

We recall the following

(2.3) Definition [Sp]: [DGLMS]. A polyhedron $Y$ (we assume, for simplicity, that $Y$ is simply connected and $\mathbb{H}(Y,g)$ is finite dimensional for all $g$) is said to be a formal consequence of its cohomology over $0$ (or a formal space over $0$) if the homotopy type of the DGA of $0$-polynomial
forms $E^r_Y$ (see [DGMS; 12] for the definition of $E^r_Y$) is a formal consequence of its cohomology.

The formality of $Y$ does not depend upon particular choice of simplicial structure on $Y$, in fact let $\mathbb{E}^r_Y$ denote the DGA of $\mathcal{O}$-polynomial forms on $Y$ with respect to some other triangulation of $Y$ then the minimal models of $E^r_Y$ and $\mathbb{E}^r_Y$ are isomorphic. This can be easily seen by taking a common subdivision. Now using [HaSi; Theorem 6.82 or [DGMS; 112], we see that the minimal models of $E^r_Y$ and $\mathbb{E}^r_Y$ are themselves isomorphic.

(2.9) Lemma: Minimal models of the DGA's $C_{(g,r)}$ and $E^r_{\mathbb{Q}}$ are isomorphic.

Proof: We have described the DGA $\mathcal{O}_p \cdot dR(\text{Sing}_{\mathcal{O}}(K/K_0))$ of piece-wise smooth forms associated to the simplicial set $\text{Sing}_{\mathcal{O}}(K/K_0)$ in the proof of Lemma (1.4). Further, we described an integration map $\int: \mathcal{O}_p \cdot dR(\text{Sing}_{\mathcal{O}}(K/K_0)) \rightarrow C_{\mathcal{O}}(K/K_0)$. Similarly, we can define $\mathcal{O}_p \cdot dR(\text{Sing}(K/K_0))$ associated to the simplicial set $\text{Sing}(K/K_0)$ and also $\mathcal{O}_p \cdot dR(\text{Sing}(K/K_0)) \rightarrow C_{\mathcal{O}}(K/K_0)$, where $\mathcal{O}_p$ consists of only polynomial forms in $C$ (with respect to the Starcentric co-ordinates on $\Lambda^n$). We have the following commutative diagram:

\[
\begin{array}{ccc}
\gamma & \mathcal{O}_p \cdot dR(\text{Sing}_{\mathcal{O}}(K/K_0)) & \mathcal{O}_p \cdot dR(\text{Sing}(K/K_0)) \\
\downarrow & \downarrow & \downarrow \\
\gamma & C_{\mathcal{O}}(K/K_0) & C(K/K_0) \\
\end{array}
\]

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where $\mathbb{P}^{n}_{\mathbb{P}^{n}_{\mathbb{P}^{n}}}$ is the space of $n$-polynomial forms on $K/K_X$ with respect to some fixed triangulation of $K/K_X$. $C_{\text{simp}}(K/K_X)$ is the simplicial co-chain complex of $K/K_X$, the maps $\alpha$, $\nu$, $\gamma$ are the canonical restrictions and the map $\eta$ is defined during the proof of lemma (1.4).

All the three horizontal maps induce algebra isomorphisms in cohomology. (See [94], [77] and our lemma (1.5).) By lemma (1.5) (resp. theorem (1.11)) $\nu$ (resp. $\eta$) induces algebra isomorphism in cohomology, $\alpha$, of course, induces algebra isomorphism in cohomology. Hence $\alpha_{\nu}$ and $\gamma_{\eta}$ both induce isomorphisms in cohomology, which proves the lemma.

As an immediate corollary of theorem (2.2), lemma (2.6) and [H18: Corollary 6.9], we get the following.

(2.7) Theorem. Let $G$ be a Kac-Moody algebraic group and let $P = P_X$ be a standard parabolic (of $G$) of finite type (see [10.19]) for terminology.

Then the space $G/P$ is a formal space over $\mathfrak{g}$.

So, complete rational homotopy information of $G/P$ can be derived from the cohomology algebra $H^*(G/P)$. In particular, the rational homotopy groups $\pi_n(G/P) \otimes \mathbb{Q}$, viewed as a graded Lie-algebra under Whitehead product, depends only on the cohomology ring $H^*(G/P)$. Moreover, all Massey products of any order are zero over $\mathfrak{g}$.

(2.8) Remarks.

(a) Compare the above theorem with formality of Kähler manifold proved in [DGMS].

*A more detailed proof can be found in Chapter 12 of "Lectures on Minimal models" by S. Halperin, Publications de l' U.B.R. Mathematicsiques pure et Appliquees".

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(b) Since $H^\bullet(G,\mathbb{Q})$ is a Borel algebra, the minimal model
\[ \mu_G \) of G (i.e. the minimal model of $DA(G)$ of \( \mathfrak o \)-polynomial forms $E_\mathfrak g^*$) in $H^\bullet(G,\mathbb{Q})$, so that $H^\bullet(G,\mathbb{Q}) \cong \Lambda(\pi_{2n+1}(G)) \otimes \mathbb{Q}$ as graded algebra, where $\pi_{even}(G)$ (resp. $\pi_{odd}(G)$) denotes $\sum_{n=0}^\infty \pi_{2n+1}(G)$ (resp. $\sum_{n=0}^\infty \pi_{2n+1}(G)$).

(c) In the next section, we would specifically determine the minimal model of $G/B$ and the Lie algebra $\pi_{\omega}(G/B) \otimes \mathbb{Q}$ under Whitehead product.

As an application of our theorem (2.2), we prove degeneracy of the Leray-Šerre fiber spectral sequence $\mathbb{Q}$ corresponding to the fibration $K \rightarrow E/T$.

Recently, Kac-Peterson have proved an important result that this spectral sequence degenerates at $E_3$ even over any finite field.

(2.9) Proposition. Let $K$ be the standard compact real form of a Kac-Moody algebraic group $G$ and let $B$ be the standard Borel subgroup of $G$. (See (6.10) for the notations.)

Leray-Šerre spectral sequence in cohomology $\mathbb{Q}$ corresponding to the fibration $K \rightarrow E/T$ where $T = B \cap K$ degenerates at $E_3$, i.e., $H^n(K; \mathbb{Q}) = 0$ for all $p$ and $n$.

Proof. Step 1. Let $A$ be a one connected DG algebra and let $\mathfrak{g}$ be a finite dimensional connected Lie algebra. Given a linear map $\phi : P \rightarrow Z(A)$ of degree $+1$ (where $P \subset H^0(G,E)$ is the linear subspace generated by primitive elements and $Z(A) = \{ z \in A : da = 0 \}$), we put a twisted differential $D = D_P$ on the tensor product of graded algebra $A \otimes H^\bullet_P$, to make it a $DA$, as follows.

\[ D_A \text{ a differential of } A \text{ and} \]
\[ D_A = \phi(x), \text{ for all } x \in P. \]

Denote the DG algebra thus obtained, by $A_{\phi}$.

There is a filtration $F = \{ F_i \}$, of the co-chain complex.
$A$, defined by $F_p = \sum A^i \otimes H^i(\mathcal{M}_p)$. Clearly, $F_p$ is $B$-stable. Further, it is easy to see that the corresponding spectral sequence has $E^3_{p,q} = H^q(A \otimes H^p(\mathcal{M}_p))$ and converges to $H^*(\mathcal{M}_p)$.

The above construction is motivated by Hochschild. Also the only property of $H^*(\mathcal{M}_p)$, which we are using is that it is free (in the graded sense) graded algebra on $P$.

**Step II.** In Step I, if we assume that the homotopy type of $A$ is formal, consequence of its cohomology and $C_0$ is a torus $T$ then the above spectral sequence degenerates at $E_4$.

To prove this, fix a minimal model $\mu : \mu \rightarrow A$ and a DGA morphism, inducing the identity at cohomology, $\psi : \mu \rightarrow H^*(\mu)$. There exist linear maps $\delta_i : H^i(T) \rightarrow \mu^2$ and $\gamma_i : H^i(T) \rightarrow \mu^3$, such that $\mu + \delta_i(\mu) = \psi_i(\mu)$ for all $i \in H^i(T)$. Further, there exists a DGA isomorphism $\epsilon : A \otimes \mu \rightarrow A$ defined by $\epsilon(A) = \mu + \delta_i(A)$ and $\epsilon(\mu) = 1_\mu + \gamma_i(\mu)$, for $x \in H^*(\mu)$. We have the following DGA morphisms

$$H^*(\mu) \otimes A \rightarrow H^*(\mu) \otimes A,$$

All of these morphisms preserve filtrations and induce isomorphisms at $E_4$ level. Hence degeneracy of the spectral sequence for $A_p$ at $E_3$ is equivalent to the degeneracy of the spectral sequence for $H^*(\mu) \otimes A_p$ at $E_3$.

We come to prove the degeneracy of the spectral sequence for $H^*(\mu) \otimes A_p$ at $E_4$. By definition, see, e.g., (Gh: page 438)

$$E_4^p = H^p(T)^{p-1} + D_{p-1}^{p-1}(\mu),$$

and the differential $d_p : E_4^p \rightarrow E_4^{p+1}$ is $d_s \circ s$ for $s \in E_4^p$. So, it suffices to show that $D_{\mu,1}^{p-1} \subset D_{p-1}^{p-1}$, for all $p > 3$. Let $A = \mu \otimes \mu$, where $A \in E_4^p$, with $A_{p,q} = H_{q}^*(\mu) \otimes \mu^3$. By definition of $D$, $D_\mu A \subset H_{p+1}^{p+1}(\mu)$. Since $s \in E_4^p$, $s \in F_{p+1}$; in particular $D_{p+1} \subset 0$ and hence $D_{p+1} = 0$. Then $s \in E_{p+1}^{p+1}, \eta_i \in D_{p+1}^{p+1}$. 

**Step III.** Consider the DGA $(C^1, \lambda)$ and a degree 1 (transgression)
Let \( \sigma \) denote the usual co-chain map of \( C(g^1) \). It is easy to see that \( \tilde{\theta}^\sigma \) is, in fact, an element of \( C(g^1, \mathfrak{h}^1) \). As in Step I, \( \sigma \) gives rise to a DGA \( C(g^1, \mathfrak{h}^1)_G = C(g^1, \mathfrak{h}^1) \otimes \Lambda(\mathfrak{h}^1^*) \). There is a DGA morphism \( \psi : C(g^1, \mathfrak{h}^1)_G \to C(g^1) \), defined by \( \psi |_{C(g^1, \mathfrak{h}^1)_G} = 1 \).

In §1(3), we have defined a co-chain map \( \tilde{\theta} : C(g^1) \to C^*(G) \). Composing with \( \psi \), we get a co-chain map \( \tilde{\theta} \circ \psi : C(g^1, \mathfrak{h}^1)_G \to C^*(G) \). We have described a filtration \( F = \mathcal{F}_p \) of \( C(g^1, \mathfrak{h}^1)_G \) in Step I. Also \( C^*(G) \) has a Leray-Serre filtration \( G = C(g^2) \), described in §1(6). It is fairly easy to see that \( \int \circ \mathcal{F}_p \subset C^p \) for all \( p \). Further, by Step I \( \mathcal{E}_q^p \cdot \mathcal{F}_p \) is \( \mathbb{H}(g^1, \mathfrak{h}^1) \otimes \Lambda^q(\mathfrak{h}^1^*) \). In view of theorem (1.6) (ii) (applied to the special case \( X = \emptyset \)), we get that \( \int \circ \mathcal{F} \) induces isomorphism \( \mathcal{E}_q^p \cdot \mathcal{F}_p \to \mathcal{E}_q^p \cdot \mathcal{F}_p \mathcal{F}_q \) for all \( p \) and \( q \) and hence degeneracy of the spectral sequence, corresponding to the filtration \( G \), at \( E_2 \) is equivalent to the degeneracy corresponding to the filtration \( \mathcal{F} \), which, in turn, follows from Step II and theorem (2.3). This establishes the proposition.

(2.10) Remark. The proof of proposition (2.9) can be modified to give the following generalization of (2.9).

Let \( Y \) be a simply connected space such that \( H^*(Y, G) \) is finite dimensional, for all \( I \). Assume further that \( Y \) is a formal space/\( G \) and let \( \pi : Y \to B \) be any principal \( T \) bundle \( T \) is a torus), then the corresponding Leray-Serre spectral sequence in cohomology/\( G \) degenerates at \( E_2 \).

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3. Determination of Minimal Model for $\mathcal{G}(B)$

(3.1) From the proof of proposition (2.9), we know that

$H^*(\mathcal{G}(C))$, as a graded algebra, is isomorphic with cohomology of the
DGA $H^*(\mathcal{G}(T)) = H^*(\mathcal{G}(T)/\mathcal{G}) \otimes \mathcal{A}$, where the notation $H^*(\mathcal{G}(T))$ is as in step 1 of the proof of proposition (2.9) and

$\phi: k^{\bullet} \rightarrow H^*(\mathcal{G}(T))$ is the map defined by $\phi(f) = \int_{[d]} f$, for all $f \in k^{\bullet}$. If $\mathcal{A}$ is, as in step III of the proof of proposition (2.9), an

element of $C^1(\mathcal{G})$ satisfying $\int_{[d]} f = f$ and $\int_{[d]} g = 0$, for root spaces

$\mathcal{G}$ corresponding to all the roots $a$, $f$ is the integration map,
defined in (1.3), from $C^1(\mathcal{G})$ to $C_c^{\infty}(\mathcal{G}(T))$ and $[\int_{[d]}$ denotes the cohomology class]. Extend $\mathcal{A}$ (again, denoted by $\mathcal{A}$ itself) to an

algebra homomorphism (called the Borel homomorphism) from

$\text{Sym}(k^{\bullet}) \rightarrow H^*(\mathcal{G}(T))$. $H^*(\mathcal{G}(T))$ becomes a $\text{Sym}(k^{\bullet})$-module under $\mathcal{A}$.

It is fairly easy to see that the DGA $H^*(\mathcal{G}(T))$ can be

identified with the standard chain complex $\mathcal{H}^*(\mathcal{G}(T))$,
corresponding to the abelian Lie-algebra $k^{\bullet}$ with coefficients in

$H^*(\mathcal{G}(T))$ (considered as $k^{\bullet}$-module under $\mathcal{A}$). So $H^*(\mathcal{G}(C))$, which is

isomorphic with the cohomology of the DGA $H^*(\mathcal{G}(T))$ is isomorphic

(as a graded algebra) with $H^*(\mathcal{G}(C))$.

By [EG, page 4, assertion $\mathcal{A}$] (in fact it is valid even over $\mathbb{Z}/(p)$), $H^*(\mathcal{G}(C))$ is free as $\text{Sym}(k^{\bullet}) = \text{Sym}(k^{\bullet})/\ker \mathcal{A}$-module (where $\ker \mathcal{A}$ denotes the kernel of $\mathcal{A}$). Hence

$H^*(\mathcal{G}(C)) = \mathcal{A}(\text{Sym}(k^{\bullet}), H^*(\mathcal{G}(T)))$

$= \mathcal{A}(\text{Sym}(k^{\bullet}), \text{Sym}(k^{\bullet})) \otimes (\mathcal{A})^* H^*(\mathcal{G}(T))$

$= \mathcal{A}(\text{Sym}(k^{\bullet}), \text{Sym}(k^{\bullet})) \otimes (\mathcal{A})^* H^*(\mathcal{G}(T))$

$(1, \ldots, H^*(\mathcal{G}(C)) = \mathcal{A}(\text{Sym}(k^{\bullet}), \text{Sym}(k^{\bullet})) \otimes (\mathcal{A})^* H^*(\mathcal{G}(T))/<\mathcal{A}>$

as graded algebra. (Since $\mathcal{A}(\text{Sym}(k^{\bullet}), \text{Sym}(k^{\bullet}))$ is trivial $k^{\bullet}$-module.)

$<\mathcal{A}>$ denotes the ideal in $H^*(\mathcal{G}(T))$, generated by

$\sum_{i \in \mathcal{A}} \mathcal{A}$.

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A graded algebra $A$ is said to be 
free if $A$ is isomorphic (as graded algebra) with $SW_{\mathbb{I}} \oplus A(W_{\mathbb{I}})$, 
where $W_0$ (resp. $W_{\mathbb{I}}$) is evenly $> 0$ (resp. oddly) graded vector space.

3.3 Lemma. Let $A$ be a free graded algebra and let $B$ and 
$C$ be two graded subalgebras of $A$ such that $A$, as a graded algebra, 
is isomorphic with $B \oplus C$ then $B$ and $C$ are free algebras.

Proof. Choose a graded algebra isomorphism $\varphi : B \oplus C \rightarrow A$. It is 
fairly easy to see that if $V \subseteq B^e$ (resp. $V \subseteq C^e$) is any graded vector space such that 
$V \subseteq B^e \cdot B^e = B^e$ (resp. $V \subseteq C^e \cdot C^e = C^e$) and $B^e$ denotes $\Sigma_{e \geq 0} B^e$. 

Furthermore, for a free graded algebra $D$ and any graded vector space 
$W \subseteq D^e$ such that $W \subseteq D^e \cdot D^e = D^e$, $FW$ is isomorphic as graded 
algebras, with $D$. (Where $FW$ denotes $SW_{\mathbb{I}} \oplus A(W_{\mathbb{I}})$). $W_0$ (resp. 
$W_{\mathbb{I}}$) is linear span of evenly (resp. oddly) graded elements in $W$.

In particular $FW \oplus B \oplus C \rightarrow A$ is an isomorphism, 
where $\theta$ is the graded algebra homomorphism with $\theta(V \oplus B^e \oplus C^e) = V \oplus B^e \oplus C^e$. Clearly 
$\theta(FW) \oplus B \oplus C \subseteq MW$ and $\theta(FW) \subseteq M(\mathbb{I})$. But since $\theta$ is an (additive) isomorphism, we get 
$\theta(FW) \oplus B \oplus C = MW$ and $\theta(FW) = M(\mathbb{I})$. This prove the 
lemma.

3.4 We return to the situation of 13.3. Let $E$ be a 
topological group. $H^0(E; E)$ is a free graded algebra. Write

$$H^0(E; E) = A(W_{\mathbb{I}}) \oplus SW_{\mathbb{I}}$$

where $W_0$ (resp. $W_{\mathbb{I}}$) is an evenly (resp. oddly) graded vector space.

Since, clearly, all the elements of $UH^0(E; E)$ of positive 
degree are nilpotent and $H^0(E; E)$ consists of evenly graded elements 
only, we get from (Ia) and lemma (3.3)

$Ia$...

$H^0(E; E) = A(W_{\mathbb{I}}) \oplus SW_{\mathbb{I}}$

and

$Ib$...

$H^0(E; E)/<\Sigma H^0(E; E)> = SW_{\mathbb{I}}$ as graded algebras.

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We prove the following.

Lemma. $H^*(\mathbb{C}/\mathbb{T}) = \tilde{S}(h^{1*}) \otimes SW_0$ as graded algebras.

Proof. Consider the graded algebra homomorphism $p: \tilde{S}(h^{1*}) \otimes SW_0 \rightarrow H^*(\mathbb{C}/\mathbb{T})$ defined by $p(a \otimes b) = a(a) \cdot \theta(b)$, for $a \in \tilde{S}(h^{1*})$ and $b \in SW_0$. (\$2; \tilde{S}(h^{1*}) \rightarrow H^*(\mathbb{C}/\mathbb{T})$ is the Borel homomorphism defined in (1.1); $\theta$ denotes a mod $\mathbb{F}_{9}$ char $\theta$ and $\theta$ is any graded algebra homomorphism: $SW_0 \rightarrow H^*(\mathbb{C}/\mathbb{T})$ such that $x \mapsto \theta(x)$ is an isomorphism as in (1.1), where $\pi: H^*(\mathbb{C}/\mathbb{T}) \rightarrow H^*(\mathbb{C}/\mathbb{T})/\mathcal{S}(h^{1*})$ is the canonical projection.) From (1.1) it is fairly easy to see that $p$ is surjective. We assert that $p$ is injective as well.

Let $J$ be the kernel of $p$, so there is an exact sequence of $\tilde{S}(h^{1*})$-modules $\tilde{S}(h^{1*})$ acts on $H^*(\mathbb{C}/\mathbb{T})$ via $\theta$ and it acts on $\tilde{S}(h^{1*}) \otimes SW_0$ by multiplication on the first factor.

$$0 \rightarrow J \rightarrow \tilde{S}(h^{1*}) \otimes SW_0 \rightarrow H^*(\mathbb{C}/\mathbb{T}) \rightarrow 0$$

considering $\mathcal{E} = \tilde{S}(h^{1*})/\mathcal{S}(h^{1*})$ as $\tilde{S}(h^{1*})$-module by multiplication, we get an exact sequence.

$$0 \rightarrow \mathcal{T} \rightarrow \mathcal{E} \rightarrow J \rightarrow 0$$

(Since $\mathcal{E} = \mathcal{M} / \mathcal{S}(h^{1*}), \mathcal{M}$ is an $\tilde{S}(h^{1*})$-module $M$.)

By (1.1), $\mathcal{E} = \mathcal{M} / \mathcal{S}(h^{1*}) \otimes SW_0 \cong SW_0$ and $H^*(\mathbb{C}/\mathbb{T})/\mathcal{S}(h^{1*})$ is an isomorphism. Also, $H^*(\mathbb{C}/\mathbb{T})$ is $\mathcal{E}$-free module and hence $\mathcal{T} \otimes \mathcal{E} \cong H^*(\mathbb{C}/\mathbb{T}) \otimes \mathcal{E} = 0$. Putting these in (1.1), we get $J/\mathcal{S}(h^{1*}) \cdot J = 0$, i.e.,

$$J = \mathcal{S}(h^{1*}) \cdot J$$
Assume, if possible, that $J \neq 0$. Pick a homogeneous element $a \neq 0 \in J$ of minimal degree. By (1.29), $a$ can be written as $a = \sum_{i} \lambda_i y_i$ for some homogeneous elements $\lambda_i \in S(\hat{h}, h)$ and $y_i \in N$. Since $S(\hat{h}, h)$ has no elements of degree 0, we have deg $y_i <$ deg $a$, contradicting the minimality of deg $a$. This proves the lemma.

(3.6) Determination of minimal model for $G/B$. Since by theorem (2.7) $G/B$ is a formal space over $\mathcal{O}$, in view of the Lemma (3.5), it suffices to determine the minimal model for the DG $\hat{S}(\hat{h}, h)$ (with $d \neq 0$).

Denote by $I$ the graded ideal Ker $\delta$. Choose a $\mathcal{O}$-linear graded splitting $s$ of the canonical projection $I \rightarrow I/\hat{h}^n(\hat{h}, h)$. Let $\{f_1, \ldots, f_{n_0}\}$ be a homogeneous $\mathcal{O}$-basis of $(S/(\hat{h}, h))_n$ with $f_1$ of degree 0 (assigning deg 1 to the elements of $\hat{h}, h$). By Lemma 2.1, $\{f_1, \ldots, f_{n_0}\}$ is a $\hat{S}(\hat{h}, h)$-regular sequence. Since $\hat{S}(\hat{h}, h)$ is a Noetherian, $n_0$ is finite. As $\mathcal{O}: \hat{h}^{1} \rightarrow \hat{h}^{n}(G/B)$ is an isomorphism, $\delta(1) \neq 0$ for all $1 \leq 1 \leq n_0$.

Define a minimal differential algebra $\nu_0 = \hat{S}(\hat{h}, h) \otimes_{\mathcal{O}} \bigoplus_{1 \leq 1 \leq n_0} \Lambda^{1/2} \hat{h}(1)$. Define a DGA homomorphism $\phi: \nu_0 \rightarrow \hat{S}(\hat{h}, h)/\text{Ker} \delta$ by $\phi(\nu_0) = (S/(\hat{h}, h))_n$. Since $\phi$ is a co-chain map, Lemma 2.2, $\phi$ induces isomorphism in cohomology.

Proof. $H^k(\nu_0)$ can be easily identified with $\mathcal{C}[\nu_0]/\mathcal{C}[\nu_0]$, where $\mathcal{C}$ is trivial $\mathcal{C}[\nu_0]$ module.
and $\mathbb{S}(\mathfrak{h})$ is $\mathbb{E}[[\gamma_1, \ldots, \gamma_{N_0}]]$ module under $\gamma_i f = f, i \neq i$ for all $1 \leq i \leq N_0$ and $f \in \mathbb{S}(\mathfrak{h})$.

$\mathbb{E}[[\gamma_1, \ldots, \gamma_{N_0}]]$ is a $\mathbb{S}(\mathfrak{h})$-regular sequence. Of course

$\text{Tor}_1^g(\mathbb{E}[[\gamma_1, \ldots, \gamma_{N_0}]], \mathbb{S}(\mathfrak{h})) = 0$ for all $i \geq 1$.

Further, $\text{Tor}_1^g(\mathbb{E}[[\gamma_1, \ldots, \gamma_{N_0}]], \mathbb{S}(\mathfrak{h})) = S(\mathfrak{h})/\text{Ker} \beta$. (Since the ideal $\langle \gamma_1, \ldots, \gamma_{N_0} \rangle$, generated by $\gamma_1, \ldots, \gamma_{N_0}$, is equal to $\text{Ker} \beta$.) This easily gives that $\beta$ induces isomorphism in cohomology.

We summarize all this in the following

**Lemma.** Let $G$ be a Kac-Moody algebraic group and $B$ the standard Borel subgroup of $G$. (See §10.10.) Then

(i) Let $\langle f_1, \ldots, f_{N_0} \rangle \subset \text{Ker} \beta$ be a homogeneous $\mathfrak{g}$-basis of $\text{Ker} \beta$

module $S(\mathfrak{h})$-mod with degree $\delta$. (Assigning degree 1 to the elements of $L$.) (i.e., the Borel map $\beta$ defined in §3.11).

Then the minimal model of the space $G/B$ (this is defined to be the minimal model of the LGA $G/B \otimes \mathbb{F}$, with respect to some triangulation of $G/B$.) See §2.9. of the form

$\mu_{G/B} = S(\mathfrak{g}) \otimes \mathbb{S}(\mathfrak{h})$.

where $\mathfrak{w}_0$ is an ordered graded vector space which is isomorphic (as graded vector space) to $\prod_{\mathfrak{n}} \mathfrak{g}$ and $A_{\mathfrak{t}}(\mathfrak{d})$ is the exterior algebra on a 1-dim. vector space in grade degree 1 with 1. Further the differential $\partial$ on $\mu_{G/B}$ is described as follows:

$\partial^1 |_{\mathfrak{W}_0} = 0$
\[ d_{1}(\text{Sh}_{1}^{1}) = 0 \]
\[ \text{dim}_{\mathbb{F}}(f_{1}^{i}) = 1 \]

In particular, \( \sum_{\sigma_{\mathbb{F}}} \text{dim}_{\mathbb{F}}(\sigma_{\mathbb{F}}) \otimes \mathbb{F} = \sum_{\tau_{\mathbb{F}}} \text{dim}_{\mathbb{F}}(\tau_{\mathbb{F}}) \otimes \mathbb{F} \) is finite dimensional and \( \text{dim}_{\mathbb{F}}(\sigma_{\mathbb{F}}) \otimes \mathbb{F} = 1 \text{dim}_{\mathbb{F}}(f_{1}^{i}) \text{ dim}_{\mathbb{F}}(f_{1}^{i}) = n \).

(2) The map \( H^{*}(G/B, \mathbb{F}) \rightarrow H^{*}(G, \mathbb{F}) \) induced by the canonical projection: \( G \rightarrow G/B \) has the kernel precisely equal to the ideal generated by \( H^{*}(G/B) \) and the image of \( H^{*}(G/B, \mathbb{F}) \) in \( H^{*}(G, \mathbb{F}) \) is isomorphic (as a graded algebra) with \( \text{Sym}(\mathbb{F}) \).

(3) Determination of Whitehead product in \( \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \). The Whitehead product map \( \{ . : \} : \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \otimes \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \rightarrow \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \) is given by

(4) \( [a, b] = 0 \) for \( a \in \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \) and \( b \in \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \) unless \( n = m = 2 \)

(5) \( \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \otimes \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \rightarrow \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \) is surjective.

Proof: (3) follows easily from theorem (2.7); (4) uses lemma (3.5); (5) coupled with [DGMS; Theorem 3.3(4)].

From (3), it is easy to see that the map: \( H^{*}(G/B, \mathbb{F}) \rightarrow H^{*}(G, \mathbb{F}) \) has kernel precisely equal to \( \sigma_{\mathbb{F}}^{2}(1) \). Hence, by (4), (2) follows.

To prove (3), observe that the Whitehead product

\[ \{ . : \} : \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \otimes \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \rightarrow \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \]

is zero for \( G \) being a group. From the homotopy exact sequence, corresponding to the fibration \( G \rightarrow G/B, \sigma_{\mathbb{F}}(G) = \sigma_{\mathbb{F}}(G/B) \) for \( n \geq 3 \). Hence \( \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \otimes \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \rightarrow \sigma_{\mathbb{F}}(G/B) \otimes \mathbb{F} \) is zero unless one of \( m \) and \( n \) is equal to 2. From first part of this theorem and [DGMS; Theorem 3.3(4)], it is fairly easy to see that.
\[ 1 \). \( \Sigma (\Sigma G/B) \otimes \mathcal{E} \otimes (\Sigma G/B) \otimes \mathcal{E} \rightarrow \Sigma^m (G/B) \otimes \mathcal{E} \) is also zero for \( m \geq 3 \). Finally the map \( d: \Sigma^2 \mathcal{E}/B \otimes \mathcal{F}/B \rightarrow \mathcal{E}/B \otimes \mathcal{H}/B \) can be easily seen (using its definition) to be injective.

This completes the proof of the theorem.

References


[L] Lepowsky, J.: Generalised Verma modules, loop


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