RATIONAL HOMOTOPY THEORY OF FLAG VARIETIES ASSOCIATED TO KAC-MOODY GROUPS

By

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Introduction

This paper is a sequel to my earlier paper "Geometry of Schubert cells and cohomology of Kac-Moody Lie-algebras". It uses many results from the paper, just mentioned, in an essential manner.

Let $\mathfrak{g}$ be a Kac-Moody Lie-algebra and let $\mathfrak{p}_X$ be a parabolic subalgebra of finite type. Let G be the algebraic group (in general infinite dimensional), in the sense of Šafarević, associated with $\mathfrak{g}$ (called a Kac-Moody algebraic group) and let P_X be the parabolic subgroup (of finite type) of G, associated with $\mathfrak{p}_X$. One of the principal aims of this paper is to study the rational homotopy theory of the flag varieties G/P_X. We prove that G/P_X is a "formal" space in the sense of rational homotopy theory. Further, we explicitly determine the minimal models of the flag varieties G/B. We also prove that the Lie-algebra cohomology, with trivial coefficients, $H^*(\mathfrak{g}^1)$ (resp. $H^*(\mathfrak{g},\mathfrak{r}_X)$) is isomorphic, as graded algebras, with singular cohomology $H^*(G,\mathbb{C})$ (resp. $H^*(G/P_X,\mathbb{C})$) and the isomorphism is explicitly given by an integration map. ($\mathfrak{g}^1$ denotes the commutator subalgebra of $\mathfrak{g}$ and $\mathfrak{r}_X$ is the reductive part of $\mathfrak{p}_X$.)

Now we describe the contents of this paper in more detail.

Chapter (0) is devoted to recalling various definitions and well known elementary facts from Kac-Moody theory. We fix notations to be used throughout the paper.

Chapter (1). Main result of this section is theorem (1.6). This states that $H^*(\mathfrak{g}, r_X)$ (resp. $H^*(\mathfrak{g}^1)$) is isomorphic with $H^*(G/P_X, \mathbb{C})$ (resp. $H^*(G,\mathbb{C})$), as graded algebras and moreover the isomorphism is explicitly given by an integration map. In particular, this gives a "complete" description of the cohomology algebra of the loop algebra $\mathfrak{g}_0 \otimes \mathbb{C}[t,t^{-1}]$ and its central extension (the affine algebra), for any finite dimensional semi-simple Lie-algebra $\mathfrak{g}_0$. Kac-Peterson also claim to have proved that $H^*(\mathfrak{g}^1)$ is isomorphic with $H^*(G,\mathbb{C})$. Their proofs have not yet appeared, but presumably, it is very different from ours. As more or less immediate corollaries (corollaries (1.9)) we deduce that $H^*(\mathfrak{g})$ and $H^*(\mathfrak{g}^1)$ are both Hopf algebras; for a finite dimensional simple Lie-algebra $\mathfrak{g}_0$, $H^2(\mathfrak{g}_0 \otimes \mathbb{C}[t,t^{-1}])$ is one dimensional; $H^2(\mathfrak{g}^1)$ is always 0 for any symmetrizable Kac-Moody Lie-algebra and hence, in particular, the standard map $\mathfrak{g}^1 \longrightarrow \mathfrak{g}_0 \otimes \mathbb{C}[t,t^{-1}]$ (where $\mathfrak{g}$ is the affine Lie-algebra associated with the finite dimensional simple Lie-algebra $\mathfrak{g}_0$) is a universal central extension. A similar result is true in the twisted affine case. Universality of this central extension is originally due to H. Garland, R. Wilson and V. Chari.

<u>Chapter</u> **2**. One of the main results of this section is theorem (2.2), which states that the DGA (differential graded algebra) $C(\mathfrak{g}, r_X)$ is formal (in the sense of rational homotopy theory). Our proof of this is similar to one of the proofs given by Deligne-Griffiths-Morgan and Sullivan for the formality of compact Kähler manifolds, but there is one essential difference in that the usual Hodge decomposition for Kähler manifolds is replaced by the "Hodge decomposition" with respect to the disjoint operators d and ∂ developed in [Ku_1]. This theorem, coupled with a technical lemma (lemma 2.6), gives rise to theorem (2.7) which states that G/P_X is a formal space (where P_X is any standard parabolic of G of finite type). So that, complete rational homotopy information of G/P_X can be derived from the cohomology algebra $H^*(G/P_X)$. Also, in particular, all the Massey products of any order are zero over $\mathbb{Q}$. As a second application of theorem (2.2), we prove that the Leray-Serre spectral sequence in cohomology corresponding to the fibration $G \longrightarrow G/B$ degenerates at E_3 over $\mathbb{Q}$. In fact, recently, Kac-Peterson have proved a far reaching result that this spectral sequence degenerates

at E_3 even over $\mathbb{Z}/p\,\mathbb{Z}$, for any prime p.

In Chapter 3, we explicitly determine the minimal models for the flag varieties G/B (for any symmetrizable Kac-Moody group G). We also determine the Lie-algebra structure (under Whitehead product) on $\pi_*(G/B) \otimes_{\mathbb{Z}} \mathbb{Q}$. See theorem (3.8) for the complete description.

After this work was done, I learnt from Victor Kac that theorem (2.7) was observed by P. Deligne (using the machinery of ℓ-adic cohomology) in a private communication to him. My very sincere thanks are due to Dale Peterson for many helpful conversations. I thank Heisuke Hironaka, Victor Kac, James R. Munkres, Leslie D. Saper and Pradeep Shukla for some helpful conversations.

0. Preliminaries and Notations

(0.1) Definitions.

(a) A *symmetrizable generalized Cartan matrix* $A = (a_{ij})_{1\leq i,j\leq \ell}$ is a matrix of integers satisfying $a_{ii} = 2$ for all i, $a_{ij} \leq 0$ if $i \neq j$, DA is symmetric for some diagonal matrix D = diag. $(q_1,\ldots,q_\ell)$ with $q_i > 0 \in \mathbb{Q}$.

(b) Choose a triple $(\mathfrak{h},\pi,\pi^{\vee})$, unique up to isomorphism, where $\mathfrak{h}$ is a vector space over $\mathbb{C}$ of dimℓ+co-rank A, $\pi = \{\alpha_i\}_{1\leq i\leq \ell} \subset \mathfrak{h}^*$ and $\pi^{\vee} = \{h_i\}_{1\leq i\leq \ell} \subset \mathfrak{h}$ are linearly independent indexed sets satisfying $\alpha_j(h_i) = a_{ij}$. The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie-algebra over $\mathbb{C}$, generated by $\mathfrak{h}$ and the symbols e_i and f_i $(1\leq i\leq \ell)$ with the defining relations $[\mathfrak{h},\mathfrak{h}] = 0$; $[h,e_i] = \alpha_i(h)e_i$, $[h,f_i] = -\alpha_i(h)f_i$ for $h\in\mathfrak{h}$ and all $1\leq i\leq \ell$; $[e_i,f_j] = \delta_{ij}h_j$ for all $1\leq i,j\leq \ell$; $(\mathrm{ad}\, e_i)^{1-a_{ij}}(e_j) = 0 = (\mathrm{ad}\, f_i)^{1-a_{ij}}(f_j)$ for all $1\leq i\neq j\leq \ell$.

$\mathfrak{h}$ is canonically embedded in $\mathfrak{g}$.

(0.2) Root space decomposition [K1]. There is available the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta\subset\mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}: [h,x] = \alpha(h)x$, for all $h \in \mathfrak{h}\}$ and $\Delta = \{\alpha \in \mathfrak{h}^* - (0)$ such that $\mathfrak{g}_\alpha \neq 0\}$. Moreover $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_+ \subset \{\sum_{i=1}^{\ell} n_i\alpha_i: n_i \in \mathbb{Z}_+$ (= the non-negative integers) for all $i\}$ and $\Delta_- = -\Delta_+$. Elements of Δ_+ (resp. Δ_-) are called positive (resp. negative) roots.

(0.3) Parabolics. We fix a subset X (including $X = \emptyset$) of $\{1,\ldots,\ell\}$ of finite type, i.e., the submatrix $A_X = (a_{ij})_{i,j\in X}$ is a classical Cartan matrix of finite type. There is a natural injection $\mathfrak{g}_X = \mathfrak{g}(A_X) \hookrightarrow \mathfrak{g}(A)$. Define Δ_+^X (resp. Δ_-^X) $= \Delta_+ \cap \{\sum_{i\in X} \mathbb{Z}\, \alpha_i\}$ (resp. $\Delta_- \cap \{\sum_{i\in X} \mathbb{Z}\, \alpha_i\}$), then $\mathfrak{g}_X = \mathfrak{h}_X \oplus \sum_{\alpha\in\Delta_+^X} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\Delta_-^X} \mathfrak{g}_\alpha$, where $\mathfrak{h}_X$ = linear span of $\{h_i\}_{i\in X}$.

Define the following Lie-subalgebras. $\mathfrak{n} = \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$; $\mathfrak{u} = \mathfrak{u}_X = \sum_{\alpha\in\Delta_+\setminus\Delta_+^X} \mathfrak{g}_\alpha$; $\mathfrak{r} = \mathfrak{r}_X = \mathfrak{g}_X + \mathfrak{h}$ and $\mathfrak{p} = \mathfrak{p}_X = \mathfrak{r} + \mathfrak{u}$. Of course $\mathfrak{r}$ is a reductive algebra. $\mathfrak{p}$ is called the *F-parabolic* *subalgebra* (F for finite dimensionality of g_X) defined by X. If $X = \emptyset$, the associated parabolic $\mathfrak{p}$ (= $\mathfrak{h} + \mathfrak{n}$) is the *Borel subalgebra*. If A itself is of finite type (i.e. A is a classical Cartan matrix), then the F-parabolic subalgebras are precisely the parabolic subalgebras of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}$.

(0.4) __**Weyl group**__ $[K_1]$. There is a *Weyl group* $W \subset \mathrm{Aut}(\mathfrak{h}^*)$ generated by the reflections $\{r_i\}_{1\leq i\leq \ell}(r_i(\beta) = \beta-\beta(h_i)\alpha_i)$, associated to the Lie-algebra $\mathfrak{g}$. $(W, \{r_i\}_{1\leq i\leq \ell})$ is a Coxeter system, hence we can talk of the lengths of elements of W.

W preserves Δ. Δ^{re} is defined to be $W\cdot\pi$ and $\Delta^{im} = \Delta\setminus\Delta^{re}$. For $\alpha \in \Delta^{re}$, $\dim \mathfrak{g}_\alpha = 1$ and $\Delta \cap \mathbb{Z}\alpha = \{\alpha,-\alpha\}$.

Given a subset X of finite type, as in §(0.3), there is defined a subset W_X^1, of the Weyl group W, by

$$W_X^1 = \{w \in W: \Delta_+ \cap w\Delta_- \subset \Delta_+\setminus\Delta_+^X\}.$$

(0.5) __**Cartan involution**__. There is a ($\mathbb{C}$-linear) unique involution ω of $\mathfrak{g}$ defined by $\omega(f_i) = -e_i$ for all $1\leq i\leq \ell$ and $\omega(h) = -h$, for all $h \in \mathfrak{h}$. It is easy to see that ω leaves $\mathfrak{g}(\mathbb{R})$ (= "real points" of $\mathfrak{g}$) stable.

Further, there is a unique *conjugate linear* involution ω_0 of $\mathfrak{g}$ which coincides with ω on $\mathfrak{g}(\mathbb{R})$.

(0.6) __**Algebraic group associated to a Kac-Moody Lie-algebra**__ $\mathfrak{g}$ $[KP_1]$, $[KP_2]$ and [T]. A $\mathfrak{g}^1$ (= $[\mathfrak{g},\mathfrak{g}]$) module (V,θ) (θ: $\mathfrak{g}^1 \longrightarrow \mathrm{End}\, V$) is called integrable, if $\theta(e)$ is locally nilpotent whenever $e \in \mathfrak{g}_\alpha$, for $\alpha \in \Delta^{re}$. Let G^* be the free product of the additive groups $\{\mathfrak{g}_\alpha\}_{\alpha\in\Delta^{re}}$, with canonical inclusions i_α: $\mathfrak{g}_\alpha \longrightarrow G^*$. For any integrable $\mathfrak{g}^1$-module (V,θ), define a homomorphism θ^*: $G^* \longrightarrow \mathrm{Aut}_{\mathbb{C}}V$ by $\theta^*(i_\alpha(e)) = \exp(\theta(e))$ for $e \in \mathfrak{g}_\alpha$. Let N^* be the intersection of all Ker θ^*. Put $G = G^*/N^*$. Let q be the canonical homomorphism: $G^* \longrightarrow G$. For $e \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta^{re}$), put $\exp e = q(i_\alpha e)$, so that $U_\alpha = \exp\mathfrak{g}_\alpha$ is an additive one parameter subgroup of G. Denote by U the subgroup of G generated by the U_α's with $\alpha \in \Delta_+^{re}$.

Choose $\Lambda_i \in \mathfrak{h}^*(1 \leq i \leq \ell)$, satisfying $\Lambda_i(h_j) = \delta_{ij}$ for all $1 \leq j \leq \ell$. There is an embedding [KP_2; page 162–163]

$$i: G \longrightarrow \mathbf{A} = [\bigoplus_{i=1}^{\ell} L(\Lambda_i)] \oplus [\bigoplus_{i=1}^{\ell} L^*(\Lambda_i)]$$

defined by $i(g) = g(\sum_{i=1}^{\ell} v_{\Lambda_i}) + g(\sum_{i=1}^{\ell} v^*_{\Lambda_i})$.

Here $(L(\Lambda_i), \pi(\Lambda_i))$ is the integrable highest weight module with highest weight Λ_i. $L^*(\Lambda_1)$ is the vector space $L(\Lambda_i)$ regarded as a $\mathfrak{g}$-module under $\pi^*(\Lambda_i) = \pi(\Lambda_i) \circ \omega$; v_{Λ_i} is a highest weight vector in $L(\Lambda_i)$ and v_{Λ_i} is denoted $v^*_{\Lambda_i}$ regarded as an element in $L^*(\Lambda_i)$.

By "differentiating" i, we get an embedding $\bar{i}: \mathfrak{g}^1 \longrightarrow \mathbf{A}$. More explicitly $\bar{i}(x) = x(\sum_{i=1}^{\ell} v_{\Lambda_i}) + x(\sum_{i=1}^{\ell} v^*_{\Lambda_i})$, for $x \in \mathfrak{g}^1$.

$\mathbf{A}$ is endowed with a Hausdorff topology defined as follows. A set $V \subset \mathbf{A}$ is open if and only if $V \cap F$ is open in F, for all the finite dimensional vector sub-spaces F of $\mathbf{A}$. Now, put the subspace (through i) topology on G. G may be viewed as a, possibly infinite dimensional, affine algebraic group in the sense of Šafarevič [Sa] with Lie-algebra $\mathfrak{g}^1$. For a proof, see [KP_2; §4]. In [KP_2; §4(G)], (a priori) a different topology is put on G but it can be seen that these two topologies, on G, actually coincide.

(0.7) Recall, from §(0.5), the conjugate linear involution ω_0 of $\mathfrak{g}$. On "integration" this gives rise to an involution $\tilde{\omega}_0$ of G. Let K denote the fixed point set of this involution.

(0.8) The subgroup of $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{h})$ generated by the reflections $\{\bar{r}_i\}_{1 \leq i \leq \ell}$ (resp. $\{\bar{r}_i\}_{i \in X}$) is denoted by $\bar{W}$ (resp. $\bar{W}_X$), where $\bar{r}_i(h) = h - \alpha_i(h)h_i$, for all $h \in \mathfrak{h}$. It is easy to see that, under the canonical identification χ: Aut $\mathfrak{h} \longrightarrow$ Aut $(\mathfrak{h}^*)$ (given by $(\chi f)\beta(h) = \beta(f^{-1}h)$, for $f \in$ Aut $\mathfrak{h}$; $\beta \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$), $\bar{W}$ corresponds with W, in fact $\chi(\bar{r}_i) = r_i$ for all $1 \leq i \leq \ell$. From now on, we would identify $\bar{W}$ with W (under χ) and use the same symbol W for both.

For each $1 \leq i \leq \ell$, there exists a unique homomorphism β_i: $SL_2(\mathbb{C}) \longrightarrow G$ satisfying $\beta_i\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp(te_i)$ and $\beta_i\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp(tf_i)$ (for all $t \in \mathbb{C}$). Define $H_i = \beta_i\left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}: t \in \mathbb{C}^*\right\}$; $G_i =$

$\beta_i(SL_2(\mathbb{C}))$; N_i = Normalizer of H_i in G_i; H = the subgroup (of G) generated by all H_i; N = the subgroup (of G) generated by all N_i. There is an isomorphism $\gamma: W \longrightarrow N/H$, such that $\gamma(r_i)$ is the coset $N_iH\setminus H$ mod H. See [KP_1; §2]. We would, sometimes, identify W with N/H under γ.

Put $B = HU$ (U is defined in §(0.6)) and $P = P_X = BW_XB$. Denote by K_X the subgroup $K \cap P_X$. It is easy to see that the canonical inclusion $K/K_X \longrightarrow G/P_X$ is a (surjective) homeomorphism. Use [KP_2; Theorem 4(d)]. ($K \subset G$ is given the subspace topology and topology on G is described in §(0.6)).

(0.9) **Bruhat decomposition** **[KP_1]; [KP_2] and [T].** Recall the definition of W_X^1 from §(0.4). W_X^1 can be characterized as the set of elements of minimal length in the cosets W_Xw ($w \in W$) (each such coset contains a unique element of minimal length).

G can be written as disjoint union $G = \bigcup_{w \in W_X^1} (U\, a(w)^{-1}\, P_X)$, so

that $G/P_X = \bigcup_{w \in W_X^1} (U\, a(w)^{-1}\, P_X/P_X)$.

($a(w)$ is an element of N satisfying $a(w) \bmod H = \gamma(w)$. In fact, we will choose $a(w) \in N \cap K$, which is possible because $KH \supset N$.)

G/P_X is a C–W complex with cells $\{V_w = U\, a(w)^{-1}\, P_X/P_X\}_{w \in W_X^1}$ and $\dim_{\mathbb{R}} V_w = 2$ length w. (To interchange

right and left cosets we have, in the expression of V_w, $a(w)^{-1}$ instead of $a(w)$ as in [KP_2].)

(0.10) **Notations.** Throughout the paper, unless otherwise specifically stated, all the vector spaces will be over $\mathbb{C}$ and linear maps would be $\mathbb{C}$-linear maps. For a vector space V, $\Lambda(V)$ denotes the exterior algebra and $S(V)$ denotes the symmetric algebra.

For a Lie-algebra pair $(\mathfrak{g}, \mathfrak{r})$, $C(\mathfrak{g}, \mathfrak{r})$ denotes the standard co-chain complex associated to the pair $(\mathfrak{g}, \mathfrak{r})$. See, e.g., [HS; §1]. For a topological space X, $C(X, \mathbb{C})$ will denote the (usual) singular co-chain complex of X with coefficients in $\mathbb{C}$. Unless otherwise stated, *cohomologies would be with complex coefficients.*

The symmetrizability assumption on the Kac-Moody Lie-algebras $\mathfrak{g}(A)$ *(i.e. A is symmetrizable) would be implicitly assumed throughout the paper.* By a *Kac-Moody algebraic group,* we mean a group G (as defined in §(0.6)), associated to some Kac-Moody Lie algebra $\mathfrak{g}$. The subgroup K (defined in §(0.7)) would be called the *standard compact real* form of G (though it is non-compact, in general!). By a *standard parabolic of G,* we would mean P_X (defined in §(0.8)) for some $X \subset \{1,\ldots,\ell\}$. If, in addition, X is of finite type P_X would be called a *standard parabolic of finite type.* When $X = \emptyset$, so that $P_X = B$, it is called the *standard Borel subgroup of G.*

1. An Analogue of Cartan-deRham Theorem for Kac-Moody Groups

(1.1) Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac-Moody Lie-algebra associated to a generalized Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq \ell}$ and let $X \subset \{1,\ldots,\ell\}$ be a subset of finite type. There is associated a group G, its standard compact real form K and a standard parabolic subgroup P $= P_X$ as described in §(0.10).

(1.2) **Definitions.**

(a) We recall the definition of a smooth map from a finite dimensional smooth manifold M to K or K/K_X from [Ku$_1$; §(4.3)] ($K_X = K \cap P_X$).

Let $f: M \longrightarrow K$ be a continuous map. Consider the composite of the maps

$$M \xrightarrow{f} K \hookrightarrow G \xrightarrow{i} \mathcal{A} \qquad \text{(i is defined in §(0.6))}.$$

Since $i \circ f: M \longrightarrow \mathcal{A}$ is continuous, given any $x_0 \in M$, there exists an open neighborhood $N(x_0)$ of x_0 in M such that $i \circ f(N(x_0)) \subset F$, for some finite dimensional vector subspace F of $\mathcal{A}$. We say that f *is smooth at* x_0 if the restricted map $i \circ f|_{N(x_0)}: N(x_0) \longrightarrow F$ is smooth ($= C^\infty$) in the usual sense. *The map* f *itself is said to be smooth* if f is smooth at all $x_0 \in M$.

A map $f: M \longrightarrow K/K_X$ *is said to be smooth* if for any $x_0 \in M$, there exists an open neighborhood $N(x_0)$ (of x_0 in M) and a smooth lift $\tilde{f}: N(x_0) \longrightarrow K$ (i.e. $\tilde{f}$ is smooth and $\pi \circ \tilde{f} = f|_{N(x_0)}$, where π is the canonical projection: $K \longrightarrow K/K_X$).

(b) By *a smooth singular* n-*simplex in* K (*resp.* K/K_X), we mean a continuous map $f: \Delta^n = \{(t_1,\ldots,t_n) \in \mathbb{R}^n: t_i \geq 0 \text{ and } \sum t_i \leq 1\} \longrightarrow K$ (resp. $f: \Delta^n \longrightarrow K/K_X$) such that there exists an open neighborhood N of Δ^n in $\mathbb{R}^n$ and a smooth map $f_{ext}: N \longrightarrow K$ (resp. $f_{ext}: N \longrightarrow K/K_X$) extending f.

Let us denote by $\Delta^n_{C^\infty}(K)$ (resp. $\Delta^n_{C^\infty}(K/K_X)$), the free abelian group on the set of all the smooth singular n-simplexes f in K (resp. in K/K_X).

Finally, denote $\sum_{n \geq 0} \mathrm{Hom}_{\mathbb{Z}}(\Delta^n_{C^\infty}(K), \mathbb{C})$ (resp.

$\sum_{n\geq 0} Hom_{\mathbb{Z}}(\Delta^n_{C^\infty}(K/K_X),\mathbb{C}))$ by $C_{C^\infty}(K,\mathbb{C})$ (resp. $C_{C^\infty}(K/K_X,\mathbb{C})$).

(c) Let M be a finite dimensional smooth manifold with a smooth map $f: M \longrightarrow K$ (resp. $f: M \longrightarrow K/K_X$). Given a $u \in C^n(\mathfrak{g}^1)$ (resp. $u \in C^n(\mathfrak{g}^1,\mathfrak{r}^1)$, $\mathfrak{r} = \mathfrak{r}_X$ is defined in §(0.3) and $\mathfrak{r}^1 = \mathfrak{r} \cap \mathfrak{g}^1$), we construct a smooth n-form $f^*(u)$ on M as follows.

Fix a $x_0 \in M$. Choose a local smooth lift $\tilde{f}: N(x_0) \longrightarrow K$. (When $f: M \longrightarrow K$, $\tilde{f}$ is, of course, f itself.) Consider the map

$i \circ L_{\tilde{f}(x_0)^{-1}} \circ \tilde{f}: N(x_0) \longrightarrow \mathcal{A}$, where $L_{\tilde{f}(x_0)^{-1}}$ is the left translation (by $\tilde{f}(x_0)^{-1}$): $K \longrightarrow K$. Define $(f^*u)_{x_0} = (i \circ L_{\tilde{f}(x_0)^{-1}} \circ \tilde{f})^* \tilde{u}$, where $\tilde{u}$ is any translation invariant n-form on $\mathcal{A}$ (so that $\tilde{u}$ is given by $\tilde{u}_0 \in Hom_{\mathbb{C}}(\Lambda^n(\mathcal{A}),\mathbb{C}))$ satisfying $\tilde{u}_0\big|_{\Lambda^n(\mathfrak{g}^1)} = u$. ($\mathfrak{g}^1$ is identified as a subspace of $\mathcal{A}$ via $\bar{i}$, see §(0.6).)

It is a routine checking, using the following facts, that $f^*(u)$ is well defined, i.e., $(f^*u)_{x_0}$ does not depend upon the particular choices of $\tilde{f}$; $\tilde{u}$ and further (f^*u) is a smooth n-form on M.

Let M be a (finite dim.) smooth manifold and $m_0 \in M$. Given two smooth maps $f,f': (M,m_0) \longrightarrow (G,e)$ (i.e. $\bar{f} = i \circ f: M \longrightarrow \mathcal{A}$ is smooth and so is $\bar{f}'$), then the following are true.

(1) The map $ff'^{-1}: (M,m_0) \longrightarrow (G,e)$, defined by $ff'^{-1}(m) = f(m)\cdot(f'(m))^{-1}$ for all $m \in M$, is smooth and $d(\overline{ff'^{-1}})_{m_0} = (d\bar{f})_{m_0} - (d\bar{f}')_{m_0}$.

(2) Fix any $a \in \mathcal{A}$, then the map $f_a: M \longrightarrow \mathcal{A}$, defined by $f_a(m) = f(m)\cdot a$ is smooth.

(3) $(d\bar{f})_{m_0}(T_{m_0}(M)) \subset \bar{i}(\mathfrak{g}^1)$.

(4) Fix a $g \in G$, then the map $gfg^{-1}: (M,m_0) \longrightarrow (G,e)$, defined by

$(gfg^{-1})m = gf(m)g^{-1}$, is smooth and for any $v \in T_{m_0}(M)$,

$\overline{d(gfg^{-1})}_{m_0} v = \bar{i}((\mathrm{Ad}\ g)x(v))$, where $x(v) \in \mathfrak{g}^1$ is the element satisfying $(d\bar{f})_{m_0} v = \bar{i}(x(v))$. (Ad: $G \longrightarrow \mathrm{Aut}(\mathfrak{g}^1)$ is defined in [KP_1; §2].)

(1) and (2) are easy in view of [KP_2; §4]. Dale Peterson showed me proofs of (3) and (4).

(1.3) __**Integration map**__. We describe an "integration" map $\int: C(\mathfrak{g}^1) \longrightarrow C_{C^\infty}(K,\mathbb{C})$ as follows.

$$(\int u)f = \int_{\Delta^n} (f^* u), \text{ for } u \in C^n(\mathfrak{g}^1) \text{ and } f: \Delta^n \longrightarrow K \text{ a}$$

simplex $\in \Delta^n_{C^\infty}(K)$.

Exactly similarly, we can define an integration map $\int: C(\mathfrak{g}^1, \mathfrak{r}^1) \longrightarrow C_{C^\infty}(K/K_X, \mathbb{C})$.

We have the following two technical lemmas.

(1.4) __**Lemma**__. The integration maps $\int: C(\mathfrak{g}^1) \longrightarrow C_{C^\infty}(K,\mathbb{C})$ and $\int: C(\mathfrak{g}^1, \mathfrak{r}^1) \longrightarrow C_{C^\infty}(K/K_X, \mathbb{C})$ are both co-chain maps. Further they induce algebra homomorphisms in cohomology.

__**Proof**__. We would prove that $\int: C(\mathfrak{g}^1) \longrightarrow C_{C^\infty}(K,\mathbb{C})$ is a co-chain map, which induces algebra homomorphism in cohomology. The proof of the analogous statement for K/K_X is similar.

To prove that $\int$ is a co-chain map, in view of Stokes' theorem, it suffices to show that for any (finite dimensional) smooth manifold M and a smooth map $f: M \longrightarrow K$, we have, for any $u \in C^n(\mathfrak{g}^1)$, $d(f^* u) = f^*(du)$.

Extend u arbitrarily to an element u_0 of $\mathrm{Hom}_{\mathbb{C}}(\Lambda^n_{\mathbb{C}}(\mathbf{A}), \mathbb{C})$. ($\mathfrak{g}^1$ is canonically embedded in $\mathbf{A}$ via $\bar{i}$. See §(0.6).) The embedding $i|_K: K \longrightarrow \mathbf{A}$ is K-equivariant (K acting on K by left multiplication and of course $\mathbf{A}$ is a representation space for K). Extend u_0 to a K-invariant form $\hat{u}_0$ on $\mathbf{A}$, though defined only on i(K). Since the representation map: $G \times \mathbf{A} \to \mathbf{A}$ is regular (see [KP_2; §4]), $\hat{u}_0$ can further be extended to a smooth (in the obvious sense) n-form $\bar{u}_0$ defined on whole of $\mathbf{A}$. Of course, $(i \circ f)^*(d\bar{u}_0) = d((i \circ f)^* \bar{u}_0)$. Further, $(i \circ f)^* \bar{u}_0$ can be easily seen to be the form $f^*(u)$. So, in view of K-invariance of $\hat{u}_0$ on i(K), it is enough to show that

$$(d\bar{u}_0)_{i(e)}(\bar{i}x_0,\ldots,\bar{i}x_n) = du(x_0,\ldots,x_n), \text{ for all } x_0,\ldots,x_n \in \mathfrak{g}^1.$$

Fix any ad locally-finite elements $x_0,\ldots,x_n \in \mathfrak{g}^1$. Consider the 1-parameter group of diffeomorphisms $\phi(x_i)$: $\mathbb{R} \times \mathbf{A} \to \mathbf{A}$, defined by $\phi(x_i)(t,a) = \exp(tx_i)a$. It can be easily seen that the corresponding vector field $\bar{x}_i$ on $\mathbf{A}$ is given by $\bar{x}_i(a) = x_i a$. Now (we would write e for i(e)),

$$(d\bar{u}_0)_e(\bar{x}_0,\ldots,\bar{x}_n) = \sum_{i=0}^{n} (-1)^i (\bar{x}_i)_e(\bar{u}_0(\bar{x}_0,\ldots,\hat{\bar{x}}_i,\ldots,\bar{x}_n)) +$$

$$\sum_{i<j} (-1)^{i+j} (\bar{u}_0)_e([\bar{x}_i,\bar{x}_j],\bar{x}_0,\ldots,\hat{\bar{x}}_i,\ldots,\hat{\bar{x}}_j,\ldots,\bar{x}_n)$$

$$(*) = \sum_{i=0}^{n} (-1)^i \underset{t\to 0}{\mathrm{Lt.}} \frac{1}{t}[(\bar{u}_0)_e(\exp(-tx_i)x_0 \exp(tx_i)e,\ldots,\underset{i\text{-th place}}{\wedge},\ldots,\exp(-tx_i)x_n \exp(tx_i)e) - (\bar{u}_0)_e(x_0e,\ldots,\hat{x_i}e,\ldots,x_ne)]$$

$$+ \sum_{i<j} (-1)^{i+j} (\bar{u}_0)_e([\bar{x}_i,\bar{x}_j],\bar{x}_0,\ldots,\hat{\bar{x}}_i,\ldots,\hat{\bar{x}}_j,\ldots,\bar{x}_n)$$

But since, for all $0 \leqslant j \leqslant n$,

$$\underset{t\to 0}{\mathrm{Lt.}} \frac{\exp(-tx_i)x_j \exp(tx_i)e - x_je}{t}$$

$$= \underset{t\to 0}{\mathrm{Lt.}} \frac{(\mathrm{Ad}(\exp(-tx_i))x_j)e - x_je}{t}$$

$$= \underset{t\to 0}{\mathrm{Lt.}} \frac{((\exp(\mathrm{ad}(-tx_i)))x_j)e - x_je}{t}$$

$$= -[x_i,x_j]e$$

Also, $[\bar{x}_i,\bar{x}_j]_e = \underset{t\to 0}{\mathrm{Lt.}} \dfrac{\exp(-tx_i)x_j \exp(tx_i)e - x_je}{t}$

$$= -[x_i, x_j]e.$$

Putting these in (*) we get

$$(d\bar{u}_0)_e(\bar{x}_0,\dots,\bar{x}_n) = \sum_{i=0}^{n} (-1)^{i+1} \sum_{j\neq i} (\bar{u}_0)_e(x_0e,\dots,[x_i,x_j]e,\dots, \hat{x_i}e,\dots,x_ne)$$

$$+ \sum_{i<j} (-1)^{i+j+1}(\bar{u}_0)_e([x_i,x_j]e, x_0e,\dots,\hat{x_i}e,\dots, \hat{x_j}e,\dots,x_ne)$$

$$= \sum_{j<i} (-1)^{i+j+1}(\bar{u}_0)_e([x_i,x_j]e, x_0e,\dots,\hat{x_j}e,\dots, \hat{x_i}e,\dots,x_ne)$$

$$+ \sum_{j>i} (-1)^{i+j}(\bar{u}_0)_e([x_i,x_j]e, x_0e,\dots,\hat{x_i}e,\dots, \hat{x_j}e,\dots,x_ne)$$

$$+ \sum_{i<j} (-1)^{i+j+1}(\bar{u}_0)_e([x_i,x_j]e, x_0e,\dots,\hat{x_i}e,\dots, \hat{x_j}e,\dots,x_ne)$$

$$= \sum_{j<i} (-1)^{i+j+1}(\bar{u}_0)_e([x_i,x_j]e, x_0e,\dots,\hat{x_j}e,\dots, \hat{x_i}e,\dots,x_ne)$$

(since the last two expressions cancel each other)

$$= \sum_{i<j} (-1)^{i+j}(\bar{u}_0)_e([x_i,x_j]e, x_0e,\dots,\hat{x_i}e,\dots, \hat{x_j}e,\dots,x_ne)$$

(interchanging i and j)

$$= du\ (x_0,\dots,x_n).$$

Since ad locally-finite elements in g^1 span g^1, this proves that $\int$ is a co-chain map.

Now we prove that $\int$ induces algebra homomorphism in cohomology.

Let $\mathrm{Sing}_{C^\infty}(K)$ (resp. $\mathrm{Sing}(K)$) denote the simplicial set $n \to \mathrm{Sing}^n_{C^\infty}(K)$ (resp. $\mathrm{Sing}^n(K)$), where $\mathrm{Sing}^n_{C^\infty}(K)$ (resp. $\mathrm{Sing}^n(K)$) denotes the set of all the smooth (resp. continuous) singular n-simplexes in K with the standard face and degeneracy maps. Let $\Omega_{p\cdot dR}(\mathrm{Sing}_{C^\infty}(K)) = \sum_{p\geq 0} \Omega^p_{p\cdot dR}(\mathrm{Sing}_{C^\infty}(K))$ denote the piece-wise smooth de-Rham complex associated to the simplicial set $\mathrm{Sing}_{C^\infty}(K)$, where an element of $\Omega^p_{p\cdot dR}(\mathrm{Sing}_{C^\infty}(K))$ is, by definition, a function θ which assigns to each element of $\mathrm{Sing}^n_{C^\infty}(K)$ $(n = 0,1,2,\ldots)$ a complex valued smooth p-form on Δ^n (i.e. a p-form on $\Delta^n \subset \mathbb{R}^n$, which extends to a smooth p-form on an open neighborhood U of Δ^n), such that θ commutes with the face and degeneracy operators. $\Omega_{p\cdot dR}$ is made into a DGA (DGA is defined in §(2.1)(a)) under pointwise addition, multiplication and the usual differential of forms. Define a DGA morphism η: $C(g^1) \longrightarrow \Omega_{p\cdot dR}(\mathrm{Sing}_{C^\infty}(K))$, by $(\eta w)s = s^* w$, for $w \in C^p(g^1)$ and for any smooth singular n-simplex s: $\Delta^n \longrightarrow K$.

There is a canonical integration map $\tilde{\int}$: $\Omega_{p\cdot dR}(\mathrm{Sing}_{C^\infty}(K)) \longrightarrow C_{C^\infty}(K)$, defined by

$$(\tilde{\int}\theta)s = \int_{\Delta^p} \theta(s), \text{ for } \theta \in \Omega^p_{p\cdot dR}(\mathrm{Sing}_{C^\infty}(K))$$

and for any smooth singular simplex s: $\Delta^p \longrightarrow K$. (We denote the integration map here by $\tilde{\int}$ to distinguish it from our earlier integration map $\int$.)

By Stokes' theorem $\tilde{\int}$ is a co-chain map. Further, it is known (see $[S_2;$ §7] and our next lemma (1.5)) that $\tilde{\int}: \Omega_{p\cdot dR}(\mathrm{Sing}_{C^\infty}(K)) \longrightarrow C_{C^\infty}(K)$ induces algebra isomorphism in cohomology. Of course, by definition, $\tilde{\int} \circ \eta = \int$ and hence the assertion, that $\int$ induces algebra homomorphism in cohomology, follows.

(1.5) Lemma. The restriction map $\gamma: C(K,\mathbb{C}) \longrightarrow C_{C^\infty}(K,\mathbb{C})$ induces isomorphism in cohomology, where $C(K,\mathbb{C})$ is the usual (continuous) singular co-chain complex with complex coefficients. A similar statement holds good with K replaced by K/K_X throughout.

Proof. For any $n \geqslant 0$, let $\mathcal{C}^n_\infty$ be the sheaf on K associated with the presheaf (for any open set U in K) $U \longmapsto \mathrm{Hom}_{\mathbb{Z}}(\Delta^n_{C^\infty}(U),\mathbb{C})$, where $\Delta^n_{C^\infty}(U)$ denotes the free abelian group on the set of all the smooth singular simplexes $\phi: \Delta^n \longrightarrow U$. There is clearly a sheaf sequence

$$(S) \ldots \; 0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{C}^0_\infty \xrightarrow{d} \mathcal{C}^1_\infty \xrightarrow{d} \mathcal{C}^2_\infty \xrightarrow{d} \ldots$$

($\mathbb{C}$ denotes the constant sheaf on K).

To prove the lemma, it suffices (see [Wa; Chapter 5]) to show that the above sequence (S) is exact and all the sheaves $\mathcal{C}^n_\infty$ are fine sheaves.

(a) $\mathcal{C}^n_\infty$ are fine sheaves. Choose a locally finite open cover $\{U_\alpha\}$ of K (K being a paracompact space, this is possible). Choose a (discontinuous) partition of unity $\{\phi_\alpha\}$ subordinate to the cover $\{U_\alpha\}$ in which the functions ϕ_α take values 0 or 1 only. For each α and n, define an endomorphism e_α of $\mathcal{C}^n_\infty$ by setting $(e_\alpha f)\sigma = \phi_\alpha(\sigma(0))f(\sigma)$, for σ a smooth singular n-simplex in U and $f \in \mathrm{Hom}_{\mathbb{Z}}(\Delta^n_{C^\infty}(U),\mathbb{C})$. This provides a partition of unity for all the sheaves $\mathcal{C}^n_\infty$, concluding that they are fine.

(b) The sequence (S) is exact. We need to prove a Poincaré type lemma. Write any element $a \in \mathbf{A}$ as

$$a = \sum_{i=1}^{l} \sum_{\lambda_i \in \text{Weights of } L(\Lambda_i)} v^i_{\lambda_i}(a) + \sum_{i=1}^{l} \sum_{\mu_i \in \text{Weights of } L^*(\Lambda_i)} w^i_{\mu_i}(a),$$

where $v^i_{\lambda_i}$ (resp. $w^i_{\mu_i}$) denotes λ_i (resp. μ_i) weight vector $\in L(\Lambda_i)$ (resp. $L^*(\Lambda_i)$). Let $N = \{a \in \mathcal{A}: v^i_{\Lambda_i}(a) \notin \mathbb{R}^+(-v_{\Lambda_i})$ and $w^i_{-\Lambda_i}(a) \notin \mathbb{R}^+(-v^*_{\Lambda_i})$ for any $i\}$. Fix a smooth function φ: $\mathbb{R} \longrightarrow [0,1]$ satisfying $\varphi(t) = 0$ for all $t \leqslant 3/4$ and $\varphi(t) = 1$ for all $t \geqslant 1$. Define a contraction $H: \mathbb{R} \times N \longrightarrow N$ by

$$H(t,a) = \sum_{i=1}^{l} \frac{\sum_{\lambda_i} \varphi(t)^{ht \cdot (\Lambda_i - \lambda_i)} v^i_{\lambda_i}(a)}{\|\sum_{\lambda_i} \varphi(t)^{ht \cdot (\Lambda_i - \lambda_i)} v^i_{\lambda_i}(a)\|}$$

$$+ \sum_{i=1}^{l} \frac{\sum_{\mu_i} \varphi(t)^{ht \cdot (\Lambda_i + \mu_i)} w^i_{\mu_i}(a)}{\|\sum_{\mu_i} \varphi(t)^{ht \cdot (\Lambda_i + \mu_i)} w^i_{\mu_i}(a)\|}, \text{ for } t \geqslant 1/2 \text{ and}$$

$$= \sum_{i=1}^{l} \frac{\varphi(2t)\alpha_i(a) + 1 - \varphi(2t)}{|\varphi(2t)\alpha_i(a) + 1 - \varphi(2t)|} v_{\Lambda_i} +$$

$$+ \sum_{i=1}^{l} \frac{\varphi(2t)\beta_i(a) + 1 - \varphi(2t)}{|\varphi(2t)\beta_i(a) + 1 - \varphi(2t)|} v^*_{\Lambda_i}, \text{ for } t \leqslant 3/4,$$

where $\dfrac{v^i_{\Lambda_i}(a)}{\|v^i_{\Lambda_i}(a)\|} = \alpha_i(a) v_{\Lambda_i}$ and $\dfrac{w^i_{-\Lambda_i}(a)}{\|w^i_{-\Lambda_i}(a)\|} = \beta_i(a) v^*_{\Lambda_i}$ (If $\lambda_i = \Lambda_i - \sum_j n_j \alpha_j$ (resp. $\mu_i = -\Lambda_i + \sum_j n_j \alpha_j$), then $ht \cdot \Lambda_i - \lambda_i$ (resp.

$\mu_i+\Lambda_i)$ is defined to be $\sum n_j \cdot \varphi(t)^0$ is defined to be $\equiv 1$.)

H has the following properties.

(1) H is smooth (in the obvious sense).

(2) $H(\mathbb{R} \times (N \cap i(K))) \subset N \cap i(K)$.

(3) $H(t,a) = a$, for all $t \geqslant 1$ and $a \in N \cap i(K)$.

(4) $H(t,a) = i(e)$, for all $t \leqslant 1/4$ and all $a \in N$.

Now we are ready to show that

$$0 \longrightarrow \mathbb{C} \longrightarrow C^0_{C^\infty}(K \cap i^{-1}(N)) \xrightarrow{d} C^1_{C^\infty}(K \cap i^{-1}(N)) \xrightarrow{d} \cdots$$

is exact. It suffices to find a homotopy operator, i.e., a linear map h_p: $C^p_{C^\infty}(K \cap i^{-1}(N)) \longrightarrow C^{p-1}_{C^\infty}(K \cap i^{-1}(N))$, for all $p \geqslant 1$, satisfying

(*) $$d \circ h_p + h_{p+1} \circ d = \mathrm{Id}.$$

For a smooth singular simplex σ: $\Delta^{p-1} \longrightarrow K \cap i^{-1}(N)$, define a smooth singular simplex $\tilde{h}_p\sigma$: $\Delta^p \longrightarrow K \cap i^{-1}(N)$ by,

$$(\tilde{h}_p\sigma)(t_1,\ldots,t_p) = i^{-1}H(t_1+\ldots+t_p,\ i\sigma(\frac{t_2}{t_1+\ldots+t_p},\ldots,\frac{t_p}{t_1+\ldots+t_p}))$$

for $t_1+\ldots+t_p>0$

$= e$ for $t_1+\ldots+t_p<1/4$.

Now, define $(h_pf)\sigma = f(\tilde{h}_p\sigma)$, for $f \in C^p_{C^\infty}(K \cap i^{-1}(N))$.

It is easy to see that (*) is satisfied.

Since K is homogeneous under left multiplication and also that, we can choose a co-final system $\{N^\epsilon\}$ of open neighborhoods of $i(e)$ in N such that $H(\mathbb{R} \times (N^\epsilon \cap i(K))) \subset N^\epsilon \cap i(K)$, we get that the sheaf sequence (S) is exact.

The case of K/K_X is similar. (We can define a similar smooth contraction of the open set $\bigcap_{w \in W_X} a(w)\ U^- \ a(w)^{-1} \subset G/P_X$.)

Remark. A contraction, similar to H above, has earlier been used by Kac-Peterson to prove contractibility of U.

We come to the main theorem of this section.

(1.6) **Theorem**. Let $g = g(A)$ be the Kac-Moody Lie-algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq \ell}$ and let $X \subset \{1,\ldots,\ell\}$ be a subset of finite type. Then the integration maps (defined in §(1.3))

(a) $$\int: C(g^1, r_X^1) \longrightarrow C_{C^\infty}(K/K_X, \mathbb{C}) \text{ and}$$

(b) $$\int: C(g^1) \longrightarrow C_{C^\infty}(K, \mathbb{C})$$

both induce algebra isomorphisms in cohomology.

Recall that $g^1 = [g,g]$; $r_X^1 = g^1 \cap r_X$ (r_X is defined in §(0.3)) K is the standard compact real form of the Kac-Moody algebraic group G associated to g and $K_X = K \cap P_X$ (P_X is the standard parabolic subgroup of G). See §(0.10).

In particular, in view of lemma (1.5), the Lie-algebra cohomology $H^*(g^1, r_X^1)$ (resp. $H^*(g^1)$) is algebra isomorphic with the singular cohomology $H^*(K/K_X, \mathbb{C})$ (resp. $H^*(K, \mathbb{C})$). Also, by [L; §6], the canonical inclusion $C(g, r_X) \hookrightarrow C(g^1, r_X^1)$ induces isomorphism in cohomology.

Proof.

(a) By lemma (1.5) and the Bruhat decomposition §(0.9), $\dim H^n(C_{C^\infty}(K/K_X, \mathbb{C})) = \dim H^n(K/K_X, \mathbb{C}) = \#\{\text{elements of length } n/2 \text{ in } W_X^1\}$. ($W_X^1$ is defined in §(0.4).) Also, by [L; §6] (see also [Ku_1; §(3.3)]), $\dim H^n(g, r_X) = \#\{\text{elements of length } n/2 \text{ in } W_X^1\}$. Hence $\dim H^n(K/K_X, \mathbb{C}) = \dim H^n(g, r_X)$. Since, by lemma (1.4), $\int$ induces algebra homomorphism, it suffices to show (for dimensional considerations) that the induced map $H(\int)$: $H^{2n}(g, r_X) \longrightarrow$

$H^{2n}(C_{C^\infty}(K/K_X,\mathbb{C}))$ is injective for all $n \geq 0$.

For any $\omega \in W_X^1$ of length n, U $a(\omega)^{-1} P_X/P_X$ is an open cell (of real dim 2n) in G/P_X (i.e. homeomorphic with $\mathbb{C}^n$). See §(0.9). Further, this extends to a smooth singular simplex $\sigma_\omega: \Delta^{2n} \longrightarrow G/P_X \in \Delta^{2n}_{C^\infty}(G/P_X)$, so that $\partial(\sigma_\omega)$ is a 2n-1 dim cycle in $(G/P_X)^{2n-2}$. But since $H_{2n-1}((G/P_X)^{2n-2}) = 0$, there exists a 2n-dim chain b_ω in $(G/P_X)^{2n-2}$ such that $\partial(b_\omega) = \partial(\sigma_\omega)$. In fact, we can further choose $b_\omega \in \Delta_{C^\infty}(G/P_X)$.

By [Ku₁; Theorem 4.5], there are "d;∂ harmonic" forms $\{s_o^\omega\}_{\omega\in W_X^1}$ such that $\int_{\sigma_\eta} s_o^\omega = \int_{B\, a(\eta)^{-1}P_X/P_X} s_o^\omega = \delta_{\omega,\eta}$, for $\omega;\eta \in W_X^1$ with $\ell(\omega) = \ell(\eta) = n$. So $\int_{\sigma_\eta - b_\eta} s_o^\omega = \int_{\sigma_\eta} s_o^\omega - \int_{b_\eta} s_o^\omega = \delta_{\omega,\eta}$, since $\int_{b_\eta} s_o^\omega = 0$. (Actually the integrand itself is 0, as s_o^ω is a 2n-form and b_η is a chain in $(G/P_X)^{2n-2}$.) Since $\{s_o^\omega\}_{w\in W_X^1 \text{ with } l(w) = n}$ is a $\mathbb{C}$-basis of $H^{2n}(g,r_X)$, by [Ku₁; §3], this immediately gives injectivity of $H(\int)$. Hence (a) follows.

(b) There is a Hochschild-Serre filtration $\tilde{F} = \{\tilde{F}_p\}_{p\geq 0}$ of $C(g^1)$ with respect to the subalgebra $\mathfrak{h}^1$, defined as follows. $\tilde{F}_p^n = \{u \in C^n(g^1): u(r_1,\ldots,r_n) = 0$, whenever n-p+1 of the arguments r_j belong to $\mathfrak{h}^1\}$ and $\tilde{F}_p = \sum_{n\geq 0} \tilde{F}_p^n$.

Also there is a Leray-Serre filtration $G = \{G_p\}$ of $C_{C^\infty}(K)$, associated with the fibration $\pi: K \longrightarrow K/T$ (where $T = B \cap K$), defined by $G_p = \{c \in C_{C^\infty}(K): c|_{\Delta(K^{p-1})\cap\Delta_{C^\infty}(K)} = 0$, where K^{p-1} denotes $\pi^{-1}((K/T)^{p-1})$ and $\Delta(K^{p-1})$ denotes the usual (continuous) singular chain complex of K^{p-1}. ($(K/T)^{p-1}$, as earlier, denotes (p-1)-th skeleton of $K/T \approx G/B$ under the Bruhat decomposition.)

It is fairly easy to see that $\int(\tilde{F}_p) \subset G_p$ for all p.

Let $E_r(\tilde{F})$ and $E_r(G)$ be the spectral sequences associated with the filtrations $\tilde{F}$ and G respectively. Since $\int$ preserves filtrations, it induces a map $E_r(\int)$: $E_r(\tilde{F}) \longrightarrow E_r(G)$ for all r.

By [HS; §6], $E_2^{p,q}(\tilde{F}) \cong H^p(\mathfrak{g}^1,\mathfrak{h}^1) \otimes H^q(\mathfrak{h}^1)$ and converging to the cohomology $H^*(\mathfrak{g}^1)$. (Although, in [HS], this is proved under the assumption that $\mathfrak{g}^1$ is finite dimensional, it can easily be adopted to our situation since $\mathfrak{h}^1$ acts reductively on $\mathfrak{g}^1$.)

Further, we can suitably modify the proof of lemma (1.5) to give the following generalization (of lemma (1.5)).

For any $p \geq 0$, the restriction map: $\mathrm{Hom}_{\mathbb{Z}}(\Delta(K^{p-1}),\mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Delta_{C^\infty}(K) \cap \Delta(K^{p-1}),\mathbb{C})$ induces isomorphism in cohomology.

Using the five lemma, this gives that the filtration $G = \{G_p\}$ is regular (and hence strongly convergent) in the sense of [CE; page 324] and also (by Leray-Serre) $E_2^{p,q}(G) \cong H^p(K/T) \otimes H^q(T)$.

By part (a) of this theorem $H(\int)$: $H^p(\mathfrak{g}^1,\mathfrak{h}^1) \xrightarrow{\sim} H^p(K/T)$, for all $p \geq 0$. From this it is fairly easy to see that $E_2^{p,q}(\tilde{F}) \xrightarrow{\sim} E_2^{p,q}(G)$ for all p and q. Hence $E_r^{p,q}(\tilde{F}) \xrightarrow{\sim} E_r^{p,q}(G)$ for all $r \geq 2$ and all $p,q \geq 0$.

This completes the proof of part (b) as well.

(1.7) __Remark.__ Kac-Peterson also claim to have proved that $H^*(\mathfrak{g}^1)$ is isomorphic with $H^*(K,\mathbb{C})$, although their proofs have not yet appeared.

The following lemma is trivial to verify.

(1.8) __Lemma.__ For any Lie-algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{s}$, $H^*(\mathfrak{g},\mathfrak{s}) \approx H^*(\mathfrak{g}/\mathfrak{z},\mathfrak{s}/\mathfrak{z})$, for a central subalgebra $\mathfrak{z}$ of $\mathfrak{g}$ such that $\mathfrak{z} \subset \mathfrak{s}$.

__Proof.__ In fact the co-chain complex $C(\mathfrak{g},\mathfrak{s})$ itself is isomorphic with $C(\mathfrak{g}/\mathfrak{z},\mathfrak{s}/\mathfrak{z})$,

(1.9) __Corollaries.__

(a) For any Kac-Moody Lie-algebra $\mathfrak{g}$, $H^*(\mathfrak{g}^1)$ and $H^*(\mathfrak{g})$ are both Hopf algebras.

(b) Let $\mathfrak{g}_0$ be a finite dimensional simple Lie-algebra and let θ be an automorphism of $\mathfrak{g}_0$, of order k, induced by an automorphism of the Dynkin diagram (so that k = 1, 2 or 3). Then

$H^2(\tilde{g}^{(k)})$ is one dimensional, where $\tilde{g}^{(k)} = \sum_{m=-\infty}^{\infty} g_m \otimes t^m$ ($g_m = \{x \in g: \theta(x) = e^{2\pi(-1)^{1/2}m/k} \cdot x\}$).

(c) Let g be any Kac-Moody Lie-algebra then $H^2(g^1) = 0$. In particular, let g be the affine Lie-algebra associated to a finite dim. simple Lie-algebra g_0 and an automorphism θ (of g_0), of order k, as in (b). Then the one dimensional central extension $0 \longrightarrow \mathfrak{z} \longrightarrow g^1 \longrightarrow \tilde{g}^{(k)} \longrightarrow 0$ (see [W_i; page 210], g^1 is nothing but $\hat{g}^{(k)}$ in the notation of [W_i]) is universal.

(d) $H^2(g) \approx \Lambda^2(g/g^1)$ and $H^3(g) \approx H^3(K) \oplus \Lambda^3(g/g^1)$, for any g.

Proof.

(a) Since $H^*(g^1) \approx H^*(K,\mathbb{C})$; K is a topological group and $H^*(g) \approx H^*(g^1) \otimes \Lambda(g/g^1)$ by [Ku_2; Proposition 1.9], (a) follows.

(b) We prove (b) in the special case $\theta = 1$. The general case is exactly similar. Specializing theorem (1.6) (a) to the affine Lie-algebra g associated with g_0 and choosing an appropriate maximal parabolic p_X, we get that $H^2(g^1, h^1+g_0)$ is one dimensional, since, from Bruhat decomposition, K/K_X can be easily seen to have only one cell in dim 2.

By lemma (1.8), taking $\mathfrak{s} = h^1 + g_0$ and $\mathfrak{z}$ = centre of g^1, we get $H^*(g^1, h^1+g_0) \approx H^*(g_0 \otimes \mathbb{C}[t,t^{-1}], g_0)$. Now using [HS; Corollary in §6] (since g_0 is acting reductively on $g_0 \otimes \mathbb{C}[t,t^{-1}]$, this is available) and the fact that $H^1(g_0) = H^2(g_0) = 0$, we get (b).

(c) By theorem (1.6) (b), we have $H^2(g^1) \approx H^2(K)$. But, by [KP_2; Theorem 4], K is simply connected. Further, using the long exact homotopy sequence for the fibration $K \longrightarrow K/T$, we get

$$0 \longrightarrow \pi_2(K) \longrightarrow \pi_2(K/T) \longrightarrow \pi_1(T) \longrightarrow 0 .$$

Since $\pi_2(K/T)$ ($\approx H_2(K/T)$) and $\pi_1(T)$ are free abelian groups with equal ranks, we get $\pi_2(K) = 0$. This gives $H^2(g^1) = 0$.

Now universality of the central extension follows immediately from (1.9) (b) together with standard facts on central extensions. See

[G; §1].

(d) This follows easily from (c) and [Ku_2; Proposition 1.9].

(1.10) **Remarks**. (1.9) (b) is due to the referee of [G]. See [G; §2]. (1.9) (c) in the affine case is, independently, due to Garland [G, Theorem (3.14)] and Vyjayanthi Chari (unpublished) and the twisted affine case is due to Wilson [W_i]. (1.9) (d) is strengthening of some results due to Berman [B].

(1.11) **Remark**. Using mixture of topological and geometric arguments, we show that, in general, the inclusion of the space of bi-invariant forms $C(\mathfrak{g}^1)^{\mathfrak{g}^1} \longrightarrow C(\mathfrak{g}^1)$ *does not* induce isomorphism in cohomology. The counterexample exists in any irreducible Kac-Moody Lie-algebra except in the case when it is a finite dimensional Lie-algebra or $\widehat{s\ell}(2)$.

2. <u>Formality of Flag Varieties Associated to Kac-Moody Groups</u>

We recall some, fairly known, definitions from rational homotopy theory. See, e.g., [DGMS]; [GM]; [Q]; [S_1]; [S_2].

(2.1) <u>Definitions</u>.

(a) A *differential graded algebra/$\mathbb{C}$* (*abbreviated to DGA*) is a graded algebra (over $\mathbb{C}$) $A = \bigoplus_{p\geq 0} A^p$ with a differential (i.e. $d^2 = 0$) $d: A \longrightarrow A$ of degree +1, such that

(1) A is graded commutative, i.e.,

$$a\cdot b = (-1)^{pq} b\cdot a \text{ for } a \in A^p \text{ and } b \in A^q.$$

(2) d is a derivation, i.e.,

$$d(a\cdot b) = (da)\cdot b + (-1)^p a\cdot db \text{ for } a \in A^p.$$

A is said to be *connected* if $H^0(A)$ is the ground field $\mathbb{C}$ and A is *one-connected* if, in addition, $H^1(A) = 0$.

(b) A DGA μ is a *minimal differential* algebra, if

(1) d is decomposable, i.e., $d(\mu^+) \subset \mu^+ \cdot \mu^+$ (μ^+ denotes the augmentation ideal $\sum_{p>0} \mu^p$).

(2) μ may be written as an increasing union of sub DGA's $\mu_0 = \mathbb{C} \subset \mu_1 \subset \mu_2 \subset \ldots$, $\bigcup_{i\geq 0} \mu_i = \mu$ with $\mu_i \subset \mu_{i+1}$ an elementary extension for all $i \geq 0$, i.e., μ_{i+1} is a graded algebra of the form $\mu_i \otimes F(V_{d_i})$, for some $d_i > 0$ ($F(V_{d_i})$ denotes the symmetric (resp. exterior) algebra on V_{d_i} if d_i is even (resp. odd). We assign grade degree d_i to elements of V_{d_i}) and such that $d_{\mu_{i+1}}|_{\mu_i} = d_{\mu_i}$ and $d_{\mu_{i+1}}(V_{d_i}) \subset \mu_i$.

(c) A *minimal model* for a DGA A is a minimal differential algebra μ_A together with a DGA homomorphism $\rho: \mu_A \longrightarrow A$ such that ρ induces isomorphism in cohomology.

An important fact is that every one connected DGA A, such that $H^i(A)$ is finite dimensional for all i, has a minimal model unique up to isomorphism. See [DGMS; Theorem 1.1 (a)].

In this paper, we would only consider one-connected DGA's A with the additional assumption that $H^i(A)$ is finite dimensional for all i. *From now on, this would be our implicit assumption on DGA's.*

(d) A minimal differential algebra μ is said to be *formal* if there is a DGA homomorphism $\psi: \mu \longrightarrow H^*(\mu)$ inducing the identity on cohomology. ($H^*(\mu)$ is equipped with identically zero differential.)

(e) *The homotopy type of a DGA* A *is a formal consequence of its cohomology* if its minimal model is formal.

Now we can state one of the main theorems of this section.

(2.2) **Theorem.** Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac–Moody Lie-algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq \ell}$ and let $X \subset \{1,\ldots,\ell\}$ be a subset of finite type.

Then, the homotopy type of the DGA $C(\mathfrak{g},\mathfrak{r})$ is formal consequence of its cohomology, where $\mathfrak{r} = \mathfrak{r}_X$ and $C(\mathfrak{g},\mathfrak{r})$ are defined in §(0.3) and §(0.10) respectively.

Proof. Our proof of this theorem is similar to the first proof of formality of Kähler manifolds, given by Deligne-Griffiths-Morgan-Sullivan [DGMS; §6]. One essential difference, however, is that the Hodge decomposition with respect to the operators $d;d^*$ (d^* is the adjoint of d) is replaced by the 'Hodge decomposition' proved in [Ku_1], for the "disjoint" operators d and ∂.

We need the following dd^c lemma.

(2.3) **Lemma.** Recall the definition of the operators $d;d';d''$: $C(\mathfrak{g},\mathfrak{r}) \longrightarrow C(\mathfrak{g},\mathfrak{r})$ from [Ku_1; §3]. As in [DGMS], define the operator $d^c = i(d'' - d')$ acting on $C(\mathfrak{g},\mathfrak{r})$. Then, we have

(1) $\operatorname{Im} d \cap \operatorname{Ker} d^c \subset \operatorname{Im} (dd^c)$ and

$$(2)\ \mathrm{Im}\ d^c \cap \mathrm{Ker}\ d \subset \mathrm{Im}\ (dd^c).$$

Proof. Let $\omega \in \mathrm{Im}\ d \cap \mathrm{Ker}\ d^c$. Since $d = d' + d''$

$$(I_1)\ldots \qquad d'\omega = 0 = d''\omega$$

From the 'Hodge type decomposition' [Ku_1; Theorem 3.13 and Remark 3.14] and disjointness of (d',∂') [Ku_1; Proposition 3.7], we get $\omega \in \mathrm{Im}\ d' \oplus \mathrm{Ker}\ S'$. Further, again by using [Ku_1; Lemma (3.8), Theorem (3.13), Remark (3.14) and Lemma (3.5)], we get $\mathrm{Im}\ d' \subset \mathrm{Im}\ S = \mathrm{Im}\ d \oplus \mathrm{Im}\ \partial$. Since, by assumption, $\omega \in \mathrm{Im}\ d$, we get $\omega \in \mathrm{Im}\ d'$. Write $\omega = d'\eta$, for some $\eta \in C(g,r)$. Express $\eta = d''\eta_1 + \partial''\eta_2 + \eta_3$, for some $\eta_1,\eta_2 \in C(g,r)$ and $\eta_3 \in \mathrm{Ker}\ S'' = \mathrm{Ker}\ S'$. This gives, on taking d',

$$(I_2)\ldots \qquad \omega = d'\eta = d'd''\eta_1 + d'\partial''\eta_2 \quad (\text{since } \eta_3 \in \mathrm{Ker}\ S').$$

Using $d'\partial'' + \partial''d' = 0$ and $d'd'' + d''d' = 0$ (see [Ku_1; Lemma (3.1) and identity (I_{18})]), we get $d'\eta = -d''d'\eta_1 - \partial''d'\eta_2$. So $d''d'\eta = -d''\partial''\ d'\eta_2 = 0$ (since $d''d'\eta = d''\omega = 0$, by (I_1)). By disjointness of the pair (d'',∂'') [Ku_1; Proposition 3.7], $\partial''d'\eta_2 = 0$. Putting this in (I_2), we get the first part of this lemma. The second part follows exactly similarly.

(2.4) **Proof of Theorem (2.2).** Denote by $H_{d^c}(g,r)$ the cohomology of the complex $C(g,r)$ under d^c and by $Z_{d^c}(g,r)$ the d^c closed forms in $C(g,r)$. Consider the diagram

$$C(g,r) \xleftarrow{i} Z_{d^c}(g,r) \xrightarrow{\alpha} H_{d^c}(g,r),$$

where i is the canonical inclusion and α the canonical projection.

Since $dd^c = -d^cd$, $Z_{d^c}(g,r)$ is stable under d. Moreover, by the previous lemma, the differential induced by d on $H_{d^c}(g,r)$ is zero.

We prove that i and α both induce isomorphism in cohomology, if we consider $C(g,r)$; $Z_{d^c}(g,r)$ and $H_{d^c}(g,r)$ as co-chain complexes under d.

(1) α^* *is surjective*: Given $\omega \in Z_{d^c}(g,r)$, we need to

show that there exists a $\eta \in C(g,r)$ such that $\omega + d^c\eta$ is d closed. By dd^c-lemma, $d\omega = -dd^c\eta$, so $d(\omega + d^c\eta) = 0$.

(2) α^* *is injective*: We need to show that $\text{Im } d^c \cap \text{Ker } d \subset d(Z_{d^c}(g,r))$, which is immediate from dd^c-lemma.

(3) i^* *is injective*: We need to prove that $\text{Im } d \cap \text{Ker } d^c \subset d(Z_{d^c}(g,r))$. Use dd^c-lemma.

(4) i^* *is surjective*: We need to show that

$$\text{Im } d + (\text{Ker } d^c \cap \text{Ker } d) = \text{Ker } d.$$

By [Ku_1; Theorem 3.13, Remark 3.14 and Lemma (3.5)], $\text{Ker } S \subset \text{Ker } d^c \cap \text{Ker } d$ and $\text{Ker } S + \text{Im } d = \text{Ker } d$. This gives surjectivity of i^*.

Theorem follows, now, by choosing a minimal model μ for the DGA $Z_{d^c}(g,r)$. (Observe that $C(g,r)$, hence $Z_{d^c}(g,r)$, is one-connected and $H^i(C(g,r))$ is finite dimensional for all i, by [L; §6] or [Ku_1; Theorem 3.15].) □

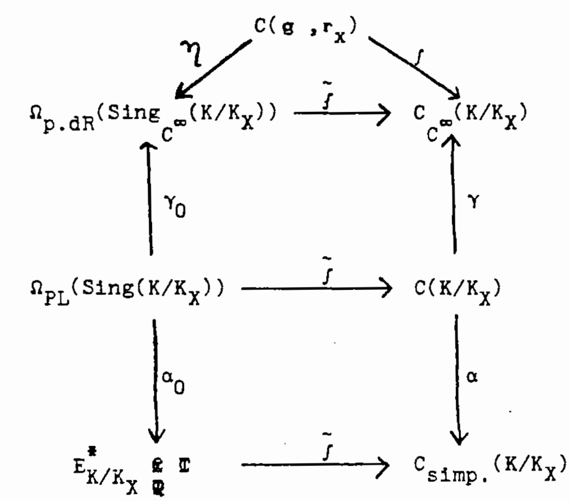
We recall the following

(2.5) __**Definition**__ **[S_2]; [DGMS].** A *polyhedron* Y (we assume, for simplicity, that Y is simply connected and $H^i(Y,\mathbb{Q})$ is finite dimensional for all i) *is said to be a formal consequence of its cohomology over* $\mathbb{Q}$ (*or a formal space over* $\mathbb{Q}$), if the homotopy type of the DGA of $\mathbb{Q}$-polynomial forms E_Y^* (see [DGMS; §2] for the definition of E_Y^*) is a formal consequence of its cohomology.

The formality of Y does not depend upon particular choice of simplicial structure on Y, in fact let $\tilde{E}_Y^*$ denote the DGA of $\mathbb{Q}$-polynomial forms on Y with respect to some other triangulation of Y then the minimal models of $E_Y \otimes_{\mathbb{Q}} \mathbb{C}$ and $\tilde{E}_Y \otimes_{\mathbb{Q}} \mathbb{C}$ are isomorphic. This can be easily seen by taking a common subdivision. Now using [HaS; Theorem 6.8] or [S_2; §12], we see that the minimal models of E_Y^* and $\tilde{E}_Y^*$ are themselves isomorphic.

(2.6) **Lemma.** Minimal models of the DGA's $C(g,r)$ and $E^*_{G/P_X} \otimes_{\mathbb{Q}} \mathbb{C}$ are isomorphic.

Proof. We have described the DGA $\Omega_{p \cdot dR}(\mathrm{Sing}_{C^\infty}(K/K_X))$ of piece-wise smooth forms associated to the simplicial set $\mathrm{Sing}_{C^\infty}(K/K_X)$ during the proof of lemma (1.4). Further, we described an integration map $\tilde{\int}: \Omega_{p \cdot dR}(\mathrm{Sing}_{C^\infty}(K/K_X)) \longrightarrow C_{C^\infty}(K/K_X)$. Exactly similarly, we can define $\Omega_{p \cdot dR}(\mathrm{Sing}\ K/K_X)$ associated to the simplicial set $\mathrm{Sing}(K/K_X)$ and also $\Omega_{PL}(\mathrm{Sing}(K/K_X)) \hookrightarrow \Omega_{p \cdot dR}(\mathrm{Sing}(K/K_X))$, where Ω_{PL} consists of only polynomial forms $/\mathbb{C}$ (with respect to the Barycentric co-ordinates on Δ^n). We have the following commutative diagram



, where E^*_{K/K_X} is the space of $\mathbb{Q}$-polynomial forms on K/K_X with respect to some (fixed) triangulation of K/K_X, $C_{simp.}(K/K_X)$ is the simplicial co-chain complex of K/K_X, the maps α, α_0, γ, γ_0 are the canonical restrictions and the map η is defined during the proof of lemma (1.4).

All the three horizontal maps induce algebra isomorphisms in cohomology.* (See [S_2; §7] and our lemma (1.5).) By lemma (1.5) (resp. theorem (1.6)) γ (resp. η) induces algebra isomorphism in cohomology. α, of course, induces algebra isomorphism in cohomology. Hence α_0 and γ_0 both induce isomorphisms in cohomology, which proves the lemma.

As an immediate corollary of theorem (2.2), lemma (2.6) and [HaS; Corollary 6.9], we get the following.

(2.7) **Theorem**. Let G be a Kac-Moody algebraic group and let $P = P_X$ be a standard parabolic (of G) of finite type (see §(0.10) for terminologies).

Then the space G/P is a formal space over $\mathbb{Q}$.

So, complete rational homotopy information of G/P can be derived from the cohomology algebra $H^*(G/P,\mathbb{Q})$. In particular, the rational homotopy groups $\pi_*(G/P) \otimes_{\mathbb{Z}} \mathbb{Q}$, viewed as a graded Lie-algebra under Whitehead product, depends only on the cohomology ring $H^*(G/P,\mathbb{Q})$. Moreover, all Massey products of any order are zero over $\mathbb{Q}$.

(2.8) **Remarks**.

(a) Compare the above theorem with formality of Kähler manifolds proved in [DGMS].

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*A more detailed proof can be found in Chapter 12 of "Lectures on Minimal models by S. Halperin, Publications de L' U.E.R. Mathematiques pures et Appliquees".

(b) Since $H^*(G,\mathbb{Q})$ is a Hopf algebra, the minimal model μ_G of G (i.e. the minimal model of DGA of $\mathbb{Q}$-polynomial forms E_G^*) is $H^*(G,\mathbb{Q})$, so that $H^*(G,\mathbb{Q}) \approx S(\pi_{\text{even}}(G) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \otimes \Lambda(\pi_{\text{odd}}(G) \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ as graded algebras, where $\pi_{\text{even}}(G)$ (resp. $\pi_{\text{odd}}(G)$) denotes $\sum_{n=0}^{\infty} \pi_{2n}(G)$ (resp. $\sum_{n=0}^{\infty} \pi_{2n+1}(G)$).

(c) In the next section, we would specifically determine the minimal model of G/B and the Lie-algebra $\pi_*(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ under Whitehead product.

As an application of our theorem (2.2), we prove degeneracy of the Leray-Serre fiber spectral sequence/$\mathbb{Q}$ corresponding to the fibration $K \longrightarrow K/T$.

Recently, Kac-Peterson have proved an important result that this spectral sequence degenerates at E_3 even over any finite field.

(2.9) __Proposition__. Let K be the standard compact real form of a Kac-Moody algebraic group G and let B be the standard Borel subgroup of G. (See §(0.10) for the notations.)

Leray-Serre spectral sequence in cohomology/$\mathbb{Q}$ corresponding to the fibration $K \longrightarrow K/T$ where $T = B \cap K$ degenerates at E_3, i.e., $E_3^{p,q} \approx E_\infty^{p,q}$ for all p and q.

__Proof__. __Step I__. Let A be a one connected DGA and let G_o be a finite dimensional connected Lie-group. Given a linear map θ: P $\longrightarrow$ Z(A) of degree +1 (where $P \subset H^*(G,\mathbb{C})$ is the linear subspace generated by primitive elements and $Z(A) = \{a \in A: da = 0\}$), we put a *twisted differential* $D = D_\theta$ on the tensor product of graded algebras $A \underset{\mathbb{C}}{\otimes} H^*(G_o)$, to make it a DGA, as follows.

$$D|_A = \text{differential of A and}$$

$$Dx = \theta(x), \text{ for all } x \in P.$$

Denote the DGA, thus obtained, by A_θ.

There is a filtration $F = \{F_p\}$, of the co-chain complex

A_θ, defined by $F_p = \sum_{\ell \geqslant p} A^\ell \otimes H^*(G_0)$. Clearly, F_p is D-stable. Further, it is easy to see that the corresponding spectral sequence has $E_2^{p,q} \approx H^p(A) \otimes H^q(G_0)$ and converges to $H^*(A_\theta)$.

The above construction is motivated by Hirsch lemma. Also the only property of $H^*(G_0)$, which we are using is that it is free (in the graded sense) graded algebra on P.

__Step II__. In Step I, if we assume that the homotopy type of A is formal consequence of its cohomology and G_0 is a torus T then the above spectral sequence degenerates at E_3.

To prove this, fix a minimal model $\rho: \mu \longrightarrow A$ and a DGA morphism, inducing the identity at cohomology, $\psi: \mu \longrightarrow H^*(\mu)$. There exist linear maps $\tilde{\theta}: H^1(T) \longrightarrow \mu^2$ and $y: H^1(T) \longrightarrow A^1$, such that $\rho \circ \tilde{\theta}(x) - \theta(x) = dy(x)$ for all $x \in H^1(T)$. Further, there exists a DGA isomorphism $\xi: A_{\rho\circ\tilde{\theta}} \longrightarrow A_\theta$ defined by $\xi|_A = \mathrm{Id}\cdot$ and $\xi(x)$

$= 1 \otimes x + y(x) \otimes 1$, for $x \in H^1(T)$. We have the following DGA morphisms

$$H^*(\mu)_{\psi\circ\tilde{\theta}} \xleftarrow{\psi\otimes \mathrm{Id}\cdot} \mu_{\tilde{\theta}} \xrightarrow{\rho\otimes \mathrm{Id}\cdot} A_{\rho\circ\tilde{\theta}} \xrightarrow{\xi} A_\theta$$

All of these morphisms preserve filtrations and induce isomorphisms at E_2 level. Hence degeneracy of the spectral sequence for A_θ at E_3 is equivalent to the degeneracy of the spectral sequence for $H^*(\mu)_{\psi\circ\tilde{\theta}}$ at E_3.

We come to prove the degeneracy of the spectral sequence for $H^*(\mu)_{\psi\circ\tilde{\theta}}$ at E_3. By definition (see, e.g., [GH; page 441]) $E_s^p = Z_s^p/(Z_{s-1}^{p+1} + D(Z_{s-1}^{p-s+1}))$, where $Z_s^p = \{a \in F_p\colon Da \in F_{p+s}\}$ and the differential $d_s\colon E_s^p \longrightarrow E_s^{p+s}$ is $\bar{a} \longrightarrow \overline{Da}$ for $a \in Z_s^p$. So, it suffices to show that $D(Z_s^p) \subset DZ_{s-1}^{p+1}$, for all $s \geqslant 3$. Let $a = \sum_{t \geqslant p} a_t \in Z_s^p$, where $a_t \in H^t(\mu) \otimes H^*(T)$. By definition of D, $Da_t \in H^{t+2}(\mu) \otimes H^*(T)$. Since $a \in Z_s^p$, $Da \in F_{p+s}$; in particular $Da_p = 0$ and hence $Da = D(\sum_{t \geqslant p+1} a_t) \in DZ_{s-1}^{p+1}$.

__Step III__. Consider the DGA $C(\mathfrak{g}^1, \mathfrak{h}^1)$ and a degree +1 (transgression) map $\theta(f) = d_{g^1}(\tilde{f})$, for all $f \in (h^1)^*$, where $\tilde{f}|_{h^1} = f$ and $\tilde{f}|_{g_\alpha} = 0$ for root spaces g_α corresponding to all the (nonzero) roots α. (d_{g^1} denotes the usual co-chain map of $C(g^1)$. It is easy to see that $d_{g^1}(\tilde{f})$ is, in fact, an element of $C(g^1,h^1)$.) As in Step I, θ gives rise to a DGA $C(g^1,h^1)_\theta = C(g^1,h^1) \otimes_{\mathbb{C}} \Lambda(h^{1*})$. There is a DGA morphism ψ: $C(g^1,h^1)_\theta \longrightarrow C(g^1)$, defined by $\psi|_{C(g^1,h^1)} = i$ (i is the canonical inclusion: $C(g^1,h^1) \longrightarrow C(g^1)$) and $\psi(f) = \tilde{f}$ for $f \in h^{1*}$.

In §(1.3), we have defined a co-chain map $\int: C(g^1) \longrightarrow C_{C^\infty}(K)$. Composing with ψ, we get a co-chain map $\int \circ\psi: C(g^1,h^1)_\theta \longrightarrow C_{C^\infty}(K)$. We have described a filtration $F = \{F_p\}$ of $C(g^1,h^1)_\theta$ in Step I. Also $C_{C^\infty}(K)$ has a Leray-Serre filtration $G = \{G_p\}$, described in §(1.6). It is fairly easy to see that $\int\circ\psi(F_p) \subset G_p$ for all p. Further, by Step I, $E_2^{p,q}(F) \simeq H^p(g^1,h^1) \otimes \Lambda^q(h^{1*})$. In view of theorem (1.6) (a) (applied in the special case $X = \emptyset$), we get that $\int\circ\psi$ induces isomorphism: $E_2^{p,q}(F) \longrightarrow E_2^{p,q}(G)$ for all p and q and hence degeneracy of the spectral sequence, corresponding to the filtration G, at E_3 is equivalent to the degeneracy corresponding to the filtration F, which, in turn, follows from Step II and theorem (2.2). This establishes the proposition.

(2.10) Remark. The proof of proposition (2.9) can be modified to give the following generalization of (2.9).

Let Y be a simply connected space such that $H^i(Y,Q)$ is finite dimensional, for all i. Assume further that Y is a formal space/$\mathbb{Q}$ and let $E \longrightarrow Y$ be any principal T bundle (T is a torus), then the corresponding Leray-Serre spectral sequence in cohomology/$\mathbb{Q}$ degenerates at E_3.

3. Determination of Minimal Model for G/B.

(3.1) From the proof of proposition (2.9), we know that $H^*(K,\mathbb{C})$, as a graded algebra, is isomorphic with the cohomology of the DGA $H^*(K/T)_\beta = H^*(K/T,\mathbb{C}) \otimes_{\mathbb{C}} \Lambda(\mathfrak{h}^{1*})$, where the notation $H^*(K/T)_\beta$ is as in Step I of the proof of proposition (2.9) and $\beta: \mathfrak{h}^{1*} \longrightarrow H^2(K/T)$ is the map defined by $\beta(f) = [\int(d_{\mathfrak{g}^1}\tilde{f})]$, for all $f \in \mathfrak{h}^{1*}$. ($\tilde{f}$ is, as in Step III of the proof of proposition (2.9), an element of $C^1(\mathfrak{g}^1)$ satisfying $\tilde{f}|_{\mathfrak{h}^1} = f$ and $\tilde{f}|_{\mathfrak{g}^\alpha} = 0$, for root spaces $\mathfrak{g}^\alpha$ corresponding to all the roots α. $\int$ is the integration map, defined in §(1.3), from $C(\mathfrak{g}^1,\mathfrak{h}^1)$ to $C_{C^\infty}(K/T)$ and [ ] denotes the cohomology class.) Extend β (again denoted by β itself) to an algebra homomorphism (called the Borel homomorphism) from $S(\mathfrak{h}^{1*}) \longrightarrow H^*(K/T)$. $H^*(K/T)$ becomes a $S(\mathfrak{h}^{1*})$-module under β.

It is fairly easy to see that the DGA $H^*(K/T)_\beta$ can be identified with the standard chain complex $\Lambda(\mathfrak{h}^{1*},H^*(K/T))$, corresponding to the abelian Lie-algebra $\mathfrak{h}^{1*}$ with coefficients in $H^*(K/T)$ (considered as $\mathfrak{h}^{1*}$-module under β). So $H^*(K,\mathbb{C})$, which is isomorphic with the cohomology of the DGA $H^*(K/T)_\beta$, is isomorphic (as a graded algebra) with $H_*(\mathfrak{h}^{1*},H^*(K/T))$.

By [K_2; page 4, assertion 4] (in fact it is valid even over $\mathbb{Z}/p\mathbb{Z}$), $H^*(K/T,\mathbb{C})$ is free as $\bar{S}(\mathfrak{h}^{1*}) = S(\mathfrak{h}^{1*})/\mathrm{Ker}\ \beta$-module (Ker β denotes the kernel of $\beta: S(\mathfrak{h}^{1*}) \longrightarrow H^*(K/T,\mathbb{C})$). Hence

$$H^*(K,\mathbb{C}) \approx H_*(\mathfrak{h}^{1*},H^*(K/T))$$

$$\approx H_*(\mathfrak{h}^{1*},\bar{S}(\mathfrak{h}^{1*}) \otimes_{\bar{S}(\mathfrak{h}^{1*})} H^*(K/T))$$

$$\approx H_*(\mathfrak{h}^{1*},\bar{S}(\mathfrak{h}^{1*})) \otimes_{\bar{S}(\mathfrak{h}^{1*})} H^*(K/T)$$

$$(I_3)\ldots\quad H^*(K,\mathbb{C}) \approx H_*(\mathfrak{h}^{1*},\bar{S}(\mathfrak{h}^{1*})) \otimes_{\mathbb{C}} [H^*(K/T)/\langle\bar{S}^+(\mathfrak{h}^{1*})\rangle]$$

as graded algebras. (Since $H_*(\mathfrak{h}^{1*},\bar{S}(\mathfrak{h}^{1*}))$ is trivial $\mathfrak{h}^{1*}$-module.)

$\langle\bar{S}^+(\mathfrak{h}^{1*})\rangle$ denotes the ideal, in $H^*(K/T)$, generated by $\sum_{i\geq 1} \bar{S}^i(\mathfrak{h}^{1*})$.

(3.2) <u>Definition</u>. *A graded algebra A is said to be free* if A is isomorphic (as graded algebra) with $S(W_0) \otimes \Lambda(W_1)$, where W_0 (resp. W_1) is evenly > 0 (resp. oddly) graded vector space.

(3.3) <u>Lemma</u>. Let A be a free graded algebra and let B and C be two graded subalgebras of A such that A, as a graded algebra, is isomorphic with $B \otimes C$ then B and C are free algebras.

<u>Proof</u>. Choose a graded algebra isomorphism $\phi: B \otimes C \longrightarrow A$. It is fairly easy to see that $\phi(V' \otimes 1 \oplus 1 \otimes V'') \oplus A^+ \cdot A^+ = A^+$, where $V' \subset B^+$ (resp. $V'' \subset C^+$) is any graded vector space such that $V' \oplus B^+ \cdot B^+ = B^+$ (resp. $V'' \oplus C^+ \cdot C^+ = C^+$) and A^+ denotes $\sum_{i>0} A^i$. Further, for a free graded algebra D and any graded vector space $W \subset D^+$ such that $W \oplus D^+ \cdot D^+ = D^+$, F(W) is isomorphic as graded algebras, with D. (Where F(W) denotes $S(W_0) \otimes \Lambda(W_1)$; W_0 (resp. W_1) is linear span of evenly (resp. oddly) graded elements in W.)

In particular $F(V' \otimes 1 \oplus 1 \otimes V'') \xrightarrow{\theta} A$ is an isomorphism, where θ is the graded algebra homomorphism with $\theta|_{V'\otimes 1 \oplus 1 \otimes V''} = \phi|_{V'\otimes 1\oplus 1\otimes V''}$. Clearly $\theta(F(V' \otimes 1)) \subset \phi(B)$ and $\theta(F(1 \otimes V'')) \subset \phi(C)$. But since θ is an (surjective) isomorphism, we get $\theta(F(V' \otimes 1)) = \phi(B)$ and $\theta(F(1 \otimes V'')) = \phi(C)$. This proves the lemma.

(3.4) We return to the situation of §(3.1). K being a topological group, $H^*(K,\mathbb{C})$ is a free graded algebra. Write

$$(I_4)\ldots \qquad H^*(K,\mathbb{C}) \approx \Lambda(W_1) \otimes S(W_0),$$

where W_0 (resp. W_1) is an evenly (resp. oddly) graded vector space.

Since, clearly, all the elements of $H_*(\mathfrak{h}^{1*},\bar{S}(\mathfrak{h}^{1*}))$ of positive degree are nilpotent and $H^*(K/T)$ consists of evenly graded elements only, we get from (I_3) and lemma (3.3)

$$(I_5)\ldots \qquad H_*(\mathfrak{h}^{1*},\bar{S}(\mathfrak{h}^{1*})) \approx \Lambda(W_1) \text{ and}$$

$$(I_6)\ldots \qquad H^*(K/T)/\langle\bar{S}^+(\mathfrak{h}^{1*})\rangle \approx S(W_0) \text{ as graded algebras.}$$

We prove the following.

(3.5) **Lemma**. $H^*(K/T) \approx \bar{S}(\mathfrak{h}^{1*}) \otimes S(W_0)$ as graded algebras.

Proof. Consider the graded algebra homomorphism $p: \bar{S}(\mathfrak{h}^{1*}) \underset{\mathbb{C}}{\otimes} S(W_0)$ $\longrightarrow H^*(K/T)$ defined by $p(\bar{a} \otimes b) = \beta(a) \cdot \theta(b)$, for $a \in S(\mathfrak{h}^{1*})$ and $b \in S(W_0)$. (β: $S(\mathfrak{h}^{1*}) \longrightarrow H^*(K/T)$ is the Borel homomorphism defined in §(3.1); $\bar{a}$ denotes a mod Ker β and θ is any graded algebra homomorphism: $S(W_0) \longrightarrow H^*(K/T)$ such that $\pi \circ \theta$ is an isomorphism as in (I_6), where π: $H^*(K/T) \longrightarrow H^*(K/T)/\langle\bar{S}^+(\mathfrak{h}^{1*})\rangle$ is the canonical projection.) From (I_6) it is fairly easy to see that p is surjective. We assert that p is injective as well.

Let J be the kernel of p, so there is an exact sequence of $\bar{S}(\mathfrak{h}^{1*})$-modules ($\bar{S}(\mathfrak{h}^{1*})$ acts on $H^*(K/T)$ via β and it acts on $\bar{S}(\mathfrak{h}^{1*}) \otimes S(W_0)$ by multiplication on the first factor.)

$$0 \longrightarrow J \longrightarrow \bar{S}(\mathfrak{h}^{1*}) \underset{\mathbb{C}}{\otimes} S(W_0) \longrightarrow H^*(K/T) \longrightarrow 0$$

considering $\mathbb{C} = \bar{S}(\mathfrak{h}^{1*})/\bar{S}^+(\mathfrak{h}^{1*})$ as $\bar{S}(\mathfrak{h}^{1*})$-module by multiplication, we get an exact sequence.

$$(I_7)\ldots \quad \mathrm{Tor}_1^{\bar{S}(\mathfrak{h}^{1*})}(\mathbb{C},H^*(K/T)) \longrightarrow J/\bar{S}^+(\mathfrak{h}^{1*})\cdot J \longrightarrow$$

$$(\bar{S}(\mathfrak{h}^{1*})\underset{\mathbb{C}}{\otimes}S(W_0))/(\bar{S}^+(\mathfrak{h}^{1*})\underset{\mathbb{C}}{\otimes}S(W_0)) \longrightarrow H^*(K/T)/\langle\bar{S}^+(\mathfrak{h}^{1*})\rangle \longrightarrow 0$$

(Since $\mathbb{C} \underset{\bar{S}(\mathfrak{h}^{1*})}{\otimes} M \cong M/\bar{S}^+(\mathfrak{h}^{1*})\cdot M$, for any $\bar{S}(\mathfrak{h}^{1*})$-module M.)

By (I_6), $(\bar{S}(\mathfrak{h}^{1*}) \underset{\mathbb{C}}{\otimes} S(W_0))/(\bar{S}^+(\mathfrak{h}^{1*}) \underset{\mathbb{C}}{\otimes} S(W_0)) \approx S(W_0) \longrightarrow$ $H^*(K/T)/\langle\bar{S}^+(\mathfrak{h}^{1*})\rangle$ is an isomorphism. Also, $H^*(K/T)$ is $\bar{S}(\mathfrak{h}^{1*})$-free module and hence $\mathrm{Tor}_1^{\bar{S}(\mathfrak{h}^{1*})}(\mathbb{C},H^*(K/T)) = 0$. Putting these in (I_7), we get $J/\bar{S}^+(\mathfrak{h}^{1*})\cdot J = 0$, i.e.,

$$(I_8)\ldots \qquad J = \bar{S}^+(\mathfrak{h}^{1*})\cdot J$$

Assume, if possible, that $J \neq 0$. Pick a homogeneous element $a \neq 0 \in J$ of minimal degree. By (I_8), a can be written as $a = \sum_i \lambda_i a_i$, for some homogeneous elements $\lambda_i \in \bar{S}^+(h^{1*})$ and $a_i \in J$. Since $\bar{S}^+(h^{1*})$ has no elements of degree 0, we have deg $a_i <$ deg a, contradicting the minimality of deg a. This proves the lemma.

(3.6) __Determination of minimal model for G/B.__ Since by theorem (2.7) G/B is a formal space over $\mathbb{Q}$, in view of the Lemma (3.5), it suffices to determine the minimal model for the DGA $\bar{S}(h^{1*})$ (with $d \equiv 0$).

Denote by I the graded ideal Ker β. Choose a $\mathbb{C}$-linear graded splitting s of the canonical projection: $I \longrightarrow I/S^+(h^{1*})\cdot I$ and let $\{f_1,\ldots,f_{m_0}\}$ be a homogeneous $\mathbb{C}$-basis of $s(I/S^+(h^{1*})\cdot I)$ with f_i of degree $\ell(i)$ (assigning deg 1 to the elements of h^{1*}). By $[K_2]$, $\{f_1,\ldots,f_{m_0}\}$ is a $S(h^{1*})$-regular sequence. (Since $S(h^{1*})$ is Noetherian, m_0 is finite.) As β: $h^{1*} \longrightarrow H^2(G/B)$ is an isomorphism, $\ell(i) \geq 2$ for all $1 \leq i \leq m_0$.

Define a minimal differential algebra $\mu_0 = S(h^{1*}) \otimes [\bigotimes_{i=1}^{m_0} \Lambda(x^i_{2\ell(i)-1})]$ ($\Lambda(x^i_{2\ell(i)-1})$ denotes the exterior algebra on 1 dim vector space in grade degree $2\ell(i)-1$ and the elements of h^{1*} are assigned grade degree 2) with $d|_{S(h^{1*})} \equiv 0$ and $d(x^i_{2\ell(i)-1}) = f_i$.

Define a DGA homomorphism θ: $\mu_0 \longrightarrow S(h^{1*})/\mathrm{Ker}\ \beta$ by $\theta|_{S(h^{1*})}$ is the canonical projection and $\theta(x^i_{2\ell(i)-1}) = 0$. (Since $f_i \in$ Ker β, θ is a co-chain map.)

(3.7) __Lemma.__ θ induces isomorphism in cohomology.

__Proof.__ $H^*(\mu_0)$ can be, easily, identified with $\mathrm{Tor}^{\mathbb{C}[y_1,\ldots,y_{m_0}]}(\mathbb{C},S(h^{1*}))$, where $\mathbb{C}$ is trivial $\mathbb{C}[y_1,\ldots,y_{m_0}]$ module and $S(\mathfrak{h}^{1*})$ is $\mathbb{C}[y_1,\ldots,y_{m_0}]$ module under $y_i \cdot f = f_i \cdot f$ for all $1 \leqslant i \leqslant m_0$ and $f \in S(\mathfrak{h}^{1*})$.

Further $\mathrm{Tor}_i^{\mathbb{C}[y_1,\ldots,y_{m_0}]}(\mathbb{C}, S(\mathfrak{h}^{1*})) = 0$ for all $i \geqslant 1$. This follows from [Se; Proposition 2 on page IV-4], $\{f_1,\ldots,f_{m_0}\}$ is a $S(\mathfrak{h}^{1*})$-regular sequence. Of course $\mathrm{Tor}_o^{\mathbb{C}[y_1,\ldots,y_{m_0}]}(\mathbb{C}, S(\mathfrak{h}^{1*})) = S(\mathfrak{h}^{1*})/\langle f_1,\ldots,f_{m_0}\rangle = S(\mathfrak{h}^{1*})/\mathrm{Ker}\ \beta$. (Since the ideal $\langle f_1,\ldots,f_{m_0}\rangle$, generated by $f_1,\ldots,f_{m_0}$, is equal to Ker β.) This easily gives that θ induces isomorphism in cohomology.

We summarize all this in the following

(3.8) __**Theorem**__. Let G be a Kac-Moody algebraic group and B the standard Borel subgroup of G. (See §(0.10).) Then

(1) Let $\{f_1,\ldots,f_{m_0}\} \subset \mathrm{Ker}\ \beta$ be a homogeneous $\mathbb{C}$-basis of Ker β modulo $S^+(\mathfrak{h}^{1*})\cdot\mathrm{Ker}\ \beta$, with f_i of degree $\ell(i)$ (assigning degree 1 to the elements of $\mathfrak{h}^{1*}$) (β is the Borel map defined in §(3.1)).

Then the minimal model of the space G/B (this is defined to be the minimal model of the DGA $E^*_{G/B} \underset{\mathbb{Q}}{\otimes} \mathbb{C}$, with respect to some triangulation of G/B. See §(2.5)) is of the form

$$\mu_{G/B} = S(W_0) \otimes S(\mathfrak{h}^{1*}) \otimes [\bigotimes_{i=1}^{m_0} \Lambda(x^i_{2\ell(i)-1})],$$

where W_0 is an evenly graded vector space which is isomorphic (as graded vector spaces/$\mathbb{C}$) with $\sum_{n\geqslant 1} \pi_{2n}(G) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ and $\Lambda(x^i_{2\ell(i)-1})$ is the exterior algebra on a 1 dim. vector space in grade degree $2\ell(i)-1$. Further the differential d on $\mu_{G/B}$ is described as follows

$$d\big|_{S(W_0)} \equiv 0$$

$$d\big|_{S(\mathfrak{h}^{1*})} \equiv 0$$

$$d(x^{i}_{2\ell(i)-1}) = f_i$$

In particular, $\sum_{n\geq 1} \pi_{2n-1}(G) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \approx \sum_{n\geq 1} \pi_{2n-1}(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ is finite dimensional and $\dim_{\mathbb{C}}(\pi_{2n-1}(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) = \#\{f_j: \deg f_j = n\}$.

(2) The map: $H^*(G/B,\mathbb{C}) \longrightarrow H^*(G,\mathbb{C})$ (induced by the canonical projection: $G \longrightarrow G/B$) has the kernel precisely equal to the ideal generated by $H^2(G/B)$ and the image of $H^*(G/B,\mathbb{C})$ in $H^*(G,\mathbb{C})$ is isomorphic (as a graded algebra) with $S(W_0)$.

(3) Determination of Whitehead product in $\pi_*(G/B) \otimes \mathbb{C}$. The Whitehead product map $[\ ,\ ]: (\pi_n(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) \otimes (\pi_m(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) \longrightarrow \pi_{n+m-1}(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ is given by

(a) $[\alpha,\beta] = 0$ for $\alpha \in \pi_n(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ and $\beta \in \pi_m(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ unless $n = m = 2$

(b) $(\pi_2(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) \otimes (\pi_2(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) \longrightarrow \pi_3(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ is surjective.

Proof. (1) follows easily from theorem (2.7); (I_4); lemma (3.5); §(3.6) and lemma (3.7) coupled with [DGMS; Theorem (3.3)(a)].

From §(3.1), it is easy to see that the map: $H^*(G/B,\mathbb{C}) \longrightarrow H^*(G,\mathbb{C})$ has kernel precisely equal to $\langle\bar{S}^+(\mathfrak{h}^{1*})\rangle$. Hence, by (I_6), (2) follows.

To prove (3), observe that the Whitehead product $[\ ,\ ]: (\pi_n(G) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) \otimes (\pi_m(G) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) \longrightarrow \pi_{n+m-1}(G) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ is zero (G being a group). From the homotopy exact sequence, corresponding to the fibration $G \longrightarrow G/B$, $\pi_n(G) \approx \pi_n(G/B)$ for $n \geq 3$. Hence $[\ ,\ ]: (\pi_n(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) \otimes (\pi_m(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) \longrightarrow \pi_{n+m-1}(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ is zero unless one of m and n is equal to 2. From first part of this theorem and [DGMS; Theorem (3.3)(a)], it is fairly easy to see that

$[\ ,\ ]\colon (\pi_2(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) \otimes (\pi_m(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}) \longrightarrow \pi_{m+1}(G/B) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ is also zero for $m \geqslant 3$. Finally the map $d\colon \pi^3_{G/B} = \sum_{\substack{\text{those } i \\ \text{s.t. } \ell(i)=2}} \mathbb{C}x^i_{2\ell(i)-1} \longrightarrow \mu^2_{G/B} \otimes \mu^2_{G/B} = \mathfrak{h}^{1*} \otimes \mathfrak{h}^{1*}$ can be easily seen (using its definition) to be injective.

This completes the proof of the theorem.

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