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A Remark on Universal Connections

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Introduction

For a connected compact Lie group G, the Weil algebra $W(\mathfrak{G})$ of the Lie algebra $\mathfrak{G}$ (of G) is an algebraic analogue of a universal G-bundle. Moreover, trivially, there is a "universal" connection on $W(\mathfrak{G})$ given by the identity map. On the other hand, Narasimhan and Ramanan have given in [6] a universal connection (in the geometric sense) on a certain universal G-bundle. Moreover, in the case of $G=SO(k)$, $U(k)$ or $\mathrm{Sp}(k)$, they take for universal G-bundle the Stiefel manifold i.e. the bundle of all orthonormal k-frames in appropriate spaces and show that the homogeneous connection on this bundle serves the purpose of a universal connection. In this paper, we show that this algebraic and geometric "universal" connections correspond with each other in a nice manner [see Theorem (1.1)]. Theorem (1.1) is seen to be equivalent to Theorem (1.3) (see Sect. 2). We isolate it as a theorem, because this seems to be interesting on its own.

Section 1 contains the precise statements of the main theorems [Theorems (1.1) and (1.3)] and Sect. 2 contains their proofs.

Notations

For an integer $k \geqq 1$, $G=G_F(k)$ will denote the group $SO(k)$, $U(k)$ or $\mathrm{Sp}(k)$ according as F is the field $\mathbb{R}$, $\mathbb{C}$ or Q (the skew field of quaternions) in that order and $\mathfrak{G}=\mathfrak{G}_F(k)$ its real Lie algebra. All the G-bundles will be principal and in the smooth category. For a vector space V (over $\mathbb{R}$, $\mathbb{C}$ or Q) $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ is abbreviated as V^*. Isomorphism will mean surjective isomorphism. $W(\mathfrak{G})$ denotes the Weil algebra of $\mathfrak{G}$ i.e., $W(\mathfrak{G})=\Lambda(\mathfrak{G}^*)\otimes S(\mathfrak{G}^*)$, where Λ denotes the exterior algebra and S denotes the symmetric algebra. For a manifold M, $\Omega(M)$ denotes its De Rham algebra. For $0 \leqq p < \infty$ and $1 \leqq N$, $k \leqq \infty$, we denote the set of all the partitions $\lambda=(\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_l > 0)$ of p (i.e. $\sum \lambda_i = p$) with $l \leqq N$ and $\lambda_1 \leqq k$ by $\mathscr{P}(p, N, k)$.

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1. Statements of the Main Theorems

A connection in a G-bundle E gives rise to a DGA morphism from $W(\mathfrak{G})$ to $\Omega(E)$ commuting with the i and θ actions and conversely. See [2a, Sects. 5, 6] or [3, Sect. 1]. Narasimhan and Ramanan showed in [6] that, for a fixed $n_0>0$, the $G(k+N)$ invariant $G(k)$ connection in the Stiefel bundle $G(k+N)/I_k\times G(N)$, for large $N=N(n_0)$, is "universal" for connections in any principal $G(k)$ bundle with base space a manifold of $\dim\leqq n_0$. Here I_k denotes the unit $k\times k$ matrix. This universal connection gives rise to a morphism

$$\Phi(N): W(\mathfrak{G}(k))\to\Omega(G(k+N)/I_k\times G(N))$$

and in fact, since the connection is $G(k+N)$ invariant, we get that the image of $\Phi(N)$ lands in the space of $G(k+N)$ invariant forms on $G(k+N)/I_k\times G(N)$, i.e.,

$$\Phi_0(N): W(\mathfrak{G}(k))\to\Omega_{\mathrm{Inv}}(G(k+N)/I_k\times G(N)).$$

Since $$G(k+N)\hookrightarrow G(k+N+1)$$

under the map $$\gamma\mapsto\left(\begin{array}{c|c}\gamma & 0\\ \hline 0 & 1\end{array}\right),$$

we get a commutative diagram

$$\begin{array}{ccc} & \overset{\Phi_0(N)}{\nearrow} & \Omega_{\mathrm{Inv}}(G(k+N)/I_k\times G(N)) \\ W(\mathfrak{G}(k)) & & \uparrow \\ & \underset{\Phi_0(N+1)}{\searrow} & \Omega_{\mathrm{Inv}}(G(k+N+1)/I_k\times G(N+1)). \end{array}$$

Let $\Omega_{\mathrm{Inv}}(E(G(k)))$ denote the inverse limit of the sequence

$$\Omega_{\mathrm{Inv}}(G(k+N)/I_k\times G(N))\leftarrow\Omega_{\mathrm{Inv}}(G(k+N+1)/I_k\times G(N+1))\leftarrow\dots.$$

We get a DGA morphism (in fact commuting with canonical i and θ actions also) $\Phi_0: W(\mathfrak{G}(k))\to\Omega_{\mathrm{Inv}}(E(G(k)))$.

Now we state the following main theorem of the paper.

(1.1) **Theorem.** *Φ_0 is an isomorphism for any $k\geqq 1$ and any $F=R$, $\mathbb{C}$ or Q.*

Before coming to the proof of the theorem, we state the following immediate corollary. This result (corollary) is well known and in fact is true for all compact connected Lie groups. See, e.g., [2b, Sect. 7].

Let $B(G(k))$ denote the direct limit of the spaces

$$G(k+N)/G(k)\cdot G(N)\hookrightarrow G(k+N+1)/G(k)\cdot G(N+1)\hookrightarrow\dots.$$

The map $\Phi_0(N): W(\mathfrak{G}(k))\to\Omega_{\mathrm{Inv}}(G(k+N)/I_k\times G(N))$ on restriction to basic elements [i.e., the set of all those elements β for which $i_X\beta=0=\theta_X\beta$ for all $X\in\mathfrak{G}(k)$] gives rise to a DGA morphism

$$\Phi_0(N)_{\mathrm{res}}: S(\mathfrak{G}(k)^*)^{\mathfrak{G}(k)}\to\Omega_{\mathrm{Inv}}(G(k+N)/G(k)\cdot G(N)),$$

where $S(\mathfrak{G}(k)^*)^{\mathfrak{G}(k)}$ denotes the algebra of $\mathfrak{G}(k)$ invariants in the $\mathfrak{G}(k)$ module $S(\mathfrak{G}(k)^*)$ and d on $S(\mathfrak{G}(k)^*)^{\mathfrak{G}(k)}$, of course, is 0. This map $\Phi_0(N)_{\mathrm{res}}$ is nothing but the evaluation of the invariant polynomial after substituting the curvature. By taking the cohomology classes, we get a homomorphism

$$\tilde{\Phi}_0(N): S(\mathfrak{G}(k)^*)^{\mathfrak{G}(k)} \to H^*(G(k+N)/G(k)\cdot G(N), \mathbb{R})$$

and hence taking inverse limit, we get a homomorphism

$$\tilde{\tilde{\Phi}}_0 : S(\mathfrak{G}(k)^*)^{\mathfrak{G}(k)} \to H^*(B(G(k)), \mathbb{R}).$$

(1.2) **Corollary.** $\tilde{\tilde{\Phi}}_0 : S(\mathfrak{G}(k)^*)^{\mathfrak{G}(k)} \to H^*(B(G(k)), \mathbb{R})$ *is an isomorphism.*

Proof (of the corollary). Let $\Omega_{\mathrm{Inv}}(B(G(k)))$ denote the inverse limit of

$$\Omega_{\mathrm{Inv}}(G(k+N)/G(k)\cdot G(N)) \leftarrow \Omega_{\mathrm{Inv}}(G(k+N+1)/G(k)\cdot G(N+1)) \leftarrow \dots .$$

Since

$$\Omega_{\mathrm{Inv}}(G(k+N)/G(k)\cdot G(N)) \hookrightarrow \Omega(G(k+N)/G(k)\cdot G(N))$$

induces isomorphism in cohomology (a result of E. Cartan), we have $H^*(\Omega_{\mathrm{Inv}}(B(G(k)))) \xrightarrow{\sim} H^*(B(G(k)))$.

As can be easily seen, $\Omega_{\mathrm{Inv}}(B(G(k)))$ consists exactly of the basic elements in $\Omega_{\mathrm{Inv}}(E(G(k)))$. Hence, from Theorem (1.1), $S(\mathfrak{G}(k)^*)^{\mathfrak{G}(k)} \xrightarrow[\Phi_{0\,\mathrm{res}}]{\sim} \Omega_{\mathrm{Inv}}(B(G(k)))$. This, in particular, gives that $\Omega_{\mathrm{Inv}}(B(G(k)))$ consists of evenly graded elements only. So the corollary follows.

We state another theorem which will be seen to be equivalent to Theorem (1.1).

Let $M(N,k,F)$ denote the space of $N\times k$ matrices over the field F. This is canonically a $G_F(N)\times G_F(k)$ module, $G_F(N)$ acting on $M(N,k,F)$ from the left and $G_F(k)$ acting from the right [more specifically, the action is $(A,B)\cdot X = AXB^{-1}$ for $A\in G_F(N)$, $B\in G_F(k)$, and $X\in M(N,k,F)$] and hence the exterior algebra $\Lambda M(N,k,F)^*$ also inherits a $G_F(N)\times G_F(k)$ module structure. Moreover $M(N,k,F)\hookrightarrow M(N+1,k,F)$ as the subspace with bottom row being 0 and hence there is an inverse system of $G_F(k)$ modules

$$\overset{G_F(N)}{\Lambda}(M(N,k,F)^*) \leftarrow \overset{G_F(N+1)}{\Lambda}(M(N+1,k,F)^*) \leftarrow \dots,$$

where $\overset{G_F(N)}{\Lambda}(M(N,k,F)^*)$ denotes the set of $G_F(N)$ invariants in the exterior algebra $\Lambda(M(N,k,F)^*)$.

(1.3) **Theorem.** *As graded algebras with $G_F(k)$ module structure,*

$$S(\mathfrak{G}_F(k)^*) \xrightarrow{\sim} {}_{\mathrm{Lt}.N\to\infty} \overset{G_F(N)}{\Lambda}(M(N,k,F)^*),$$

where we assign grade degree 2 to the elements of $\mathfrak{G}_F(k)^*$.

2. Proofs of the Theorems

Before coming to the proofs, we state the following proposition.

(2.1) **Proposition.** *As $U(N)\times U(k)$ module, $\Lambda^p_{\mathbb{C}}(M(N,k,\mathbb{C}))$ is isomorphic with $\sum_{\lambda\in\mathscr{P}(p,N,k)} \square^{\lambda}(\mathbb{C}^N)\otimes_{\mathbb{C}}\square^{\tilde{\lambda}}(\mathbb{C}^k)$ where $\tilde{\lambda}$ denotes the conjugate partition. For the notation $\mathscr{P}(p,N,k)$ see "Notations".*

$\square^{\lambda}(\mathbb{C}^N)$ is an irreducible representation space over $\mathbb{C}$ of $U(N)$. We describe its character ch_λ restricted to a maximal torus

$$D=\left\{t=\begin{pmatrix} t_1 & & 0\\ & \ddots & \\ 0 & & t_N\end{pmatrix} \text{ with } |t_i|=1,\, t_i\in\mathbb{C}\right\} \text{ of } U(N).$$

$$\mathrm{ch}_\lambda(t)=\frac{|t^{l_1},\dots,t^{l_N}|}{|t^{N-1},\dots,t^0|}$$

where $l_i=\lambda_i+N-i$. (We define $\lambda_i=0$ for $i>l$) and $|t^{l_1},\dots,t^{l_N}|$ denotes the determinant of the matrix

$$\begin{pmatrix} t_1^{l_1}, & \dots, & t_1^{l_N}\\ \vdots & & \vdots\\ t_N^{l_1}, & \dots, & t_N^{l_N}\end{pmatrix}.$$

The character theoretic equivalent of this proposition can be found, for example, in [5, p. 35, identity (4.3)′].

We turn to the proof of Theorem (1.3). First, we prove that, for any $p\geqq 1$ and $N\geqq 2p+1$, $S^p_{\mathbb{C}}(\mathfrak{G}_F(k)^*\otimes\mathbb{C})$ is $\mathbb{C}$-isomorphic with $\left[\overset{G_F(N)}{\Lambda^{2p}}(M(N,k,F)^*)\right]\otimes\mathbb{C}$ as a $G_F(k)$ module and $\overset{G_F(N)}{\Lambda^r}(M(N,k,F)^*)=0$ for odd r and any $N\geqq r+1$. Unfortunately, we did not succeed in giving a unified proof of this assertion, so we have to deal with the three cases separately, although the proofs in the case of $SO(k)$ and $\mathrm{Sp}(k)$ are very similar.

(a) *Case 1* [Unitary group $U(k)$]. Since $M(N,k,\mathbb{C})^*\otimes\mathbb{C}$, as $U(N)\times U(k)$ module, is $\mathbb{C}$-isomorphic with $M(N,k,\mathbb{C})\oplus\mathrm{Hom}_{\mathbb{C}}(M(N,k,\mathbb{C}),\mathbb{C})$, we get

$$[\Lambda(M(N,k,\mathbb{C})^*)]\otimes\mathbb{C}\simeq\Lambda_{\mathbb{C}}(M(N,k,\mathbb{C}))\otimes_{\mathbb{C}}\Lambda_{\mathbb{C}}(\mathrm{Hom}(M(N,k,\mathbb{C}),\mathbb{C})).$$

From the Proposition (2.1), we get that

$$\Lambda^p_{\mathbb{C}}(M(N,k,\mathbb{C}))\otimes\Lambda^q_{\mathbb{C}}(\mathrm{Hom}(M(N,k,\mathbb{C}),\mathbb{C}))$$
$$\simeq\bigoplus_{\substack{\lambda\in\mathscr{P}(p,N,k)\\ \mu\in\mathscr{P}(q,N,k)}}\square^{\lambda}(\mathbb{C}^N)\otimes\square^{\tilde\lambda}(\mathbb{C}^k)\otimes\mathrm{Hom}_{\mathbb{C}}(\square^{\tilde\mu}(\mathbb{C}^N),\mathbb{C})\otimes\mathrm{Hom}_{\mathbb{C}}(\square^{\mu}(\mathbb{C}^k),\mathbb{C}).$$

Hence $\overset{U(N)}{\Lambda^r}(M(N,k,\mathbb{C})^*)=0$ for odd r and

$$\left[\overset{U(N)}{\Lambda^{2p}}(M(N,k,\mathbb{C})^*)\right]\otimes\mathbb{C}\simeq\sum_{\lambda\in\mathscr{P}(p,N,k)}\square^{\tilde\lambda}(\mathbb{C}^k)\otimes\mathrm{Hom}_{\mathbb{C}}(\square^{\tilde\lambda}(\mathbb{C}^k),\mathbb{C}).$$

Now, for $N\geqq p$, we get that

$$\left[\overset{U(N)}{\Lambda^{2p}}(M(N,k,\mathbb{C})^*)\right]\otimes\mathbb{C}\simeq\sum_{\eta\in\mathscr{P}(p,k,\infty)}\square^{\eta}(\mathbb{C}^k)\otimes_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}}(\square^{\eta}(\mathbb{C}^k),\mathbb{C})$$

as $U(k)$ modules. But

$$\sum_{\eta\in\mathscr{P}(p,k,\infty)}\square^{\eta}\otimes\mathrm{Hom}_{\mathbb{C}}(\square^{\eta},\mathbb{C})$$

is known to be isomorphic with $S^p_{\mathbb{C}}(\mathfrak{G}_{\mathbb{C}}(k)^* \otimes \mathbb{C})$ as a $U(k)$ module. We give a brief sketch of its proof.

Let $\mathbb{C}[\mathrm{GL}(k,\mathbb{C})]$ denote the affine algebra of the algebraic group $\mathrm{GL}(k,\mathbb{C})$. In view of Tannaka duality $\mathbb{C}[\mathrm{GL}(k,\mathbb{C})]$ can be identified, as a $\mathrm{GL}(k,\mathbb{C})$ module over $\mathbb{C}$, with $\sum_{\eta \in \widehat{\mathrm{GL}(k,\mathbb{C})}} V_\eta \otimes \mathrm{Hom}_{\mathbb{C}}(V_\eta, \mathbb{C})$, where $\widehat{\mathrm{GL}(k,\mathbb{C})}$ denotes the set of all the finite dimensional inequivalent irreducible representations (over $\mathbb{C}$) of $\mathrm{GL}(k,\mathbb{C})$. Of course $\widehat{\mathrm{GL}(k,\mathbb{C})}$ can be parametrized (in a bijective way) by the set

$$S_k = \{\eta = (\eta_1, \ldots, \eta_k) \in \mathbb{Z}^k \text{ such that } \eta_1 \geqq \ldots \geqq \eta_k\}.$$

The algebra $S_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(M(k,\mathbb{C}),\mathbb{C}))$ sits inside $\mathbb{C}[\mathrm{GL}(k,\mathbb{C})]$ as the set of all those polynomial functions on $\mathrm{GL}(k,\mathbb{C})$ which extend to the affine space $M(k,\mathbb{C})$. It can be verified that the $\mathrm{GL}(k,\mathbb{C})$ module $S^p_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(M(k,\mathbb{C}),\mathbb{C}))$, under the identification, corresponds to $\sum_{\substack{\eta \in S_k \text{ with} \\ \eta_k \geqq 0 \text{ and } \sum \eta_i = p}} V_\eta \otimes \mathrm{Hom}_{\mathbb{C}}(V_\eta, \mathbb{C})$ that is the same as $\sum_{\eta \in \mathscr{P}(p,k,\infty)} \square^\eta(\mathbb{C}^k) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(\square^\eta(\mathbb{C}^k), \mathbb{C})$.

Observe that V_η is nothing but the $\mathrm{GL}(k,\mathbb{C})$ module $\square^\eta(\mathbb{C}^k)$ introduced earlier.

(b) *Case II* [Special orthogonal group $SO(k)$].

Since the real representations are always self dual, $[\Lambda^p(M(N,k,\mathbb{R})^*)] \otimes \mathbb{C}$ is isomorphic with $\Lambda^p_{\mathbb{C}}(M(N,k,\mathbb{C}))$ as $SO(N) \times SO(k)$ module. The problem, in this case, reduces to the study of $[\square^\lambda(\mathbb{C}^N)]^{SO(N)}$, i.e., the $SO(N)$ invariants in the space $\square^\lambda(\mathbb{C}^N)$.

(2.2) **Lemma.** *For a partition $\lambda = (\lambda_1 \geqq \ldots \geqq \lambda_l > 0)$ of p, if we take $N \geqq p+1$ then $[\square^\lambda(\mathbb{C}^N)]^{SO(N)} \neq 0$ if and only if all the λ_i's are even and in that case $[\square^\lambda(\mathbb{C}^N)]^{SO(N)}$ is one dimensional (over $\mathbb{C}$).*

The lemma is clearly false for $N = p$. Consider the partition $(1 \geqq 1 \geqq \ldots \geqq 1)$ of p.

Remark. This lemma was proved by a method similar to the proof of Lemma (2.3), but we give a slightly different proof suggested by Ramanan.

Proof. Let $O(N,\mathbb{C})$ denote the algebraic group of all the $N \times N$ matrices A over $\mathbb{C}$ satisfying $A.\ A^t = I_N$ (A^t denotes the transpose of the matrix A). From Frobenius reciprocity, taking the induced representation of $\mathrm{GL}(N,\mathbb{C})$ corresponding to the 1-dimensional trivial representation of $O(N,\mathbb{C})$, we get that

$$\mathbb{C}[\mathrm{GL}(N,\mathbb{C})/O(N,\mathbb{C})] \simeq \sum_{\eta \in \widehat{\mathrm{GL}(N,\mathbb{C})}} m_\eta V_\eta \qquad (*)$$

as $\mathrm{GL}(N,\mathbb{C})$ modules, where $m_\eta = \dim[\mathrm{Hom}_{O(N,\mathbb{C})}(V_\eta, \mathbb{C})]$. Clearly, the variety $\mathrm{GL}(N,\mathbb{C})/O(N,\mathbb{C})$ can be identified with the variety of all the nondegenerate symmetric $N \times N$ matrices over $\mathbb{C}$. Now, $S(S^2(\mathbb{C}^N))$ is embedded in $\mathbb{C}[\mathrm{GL}(N,\mathbb{C})/O(N,\mathbb{C})]$ as the set of all those polynomial functions on $\mathrm{GL}(N,\mathbb{C})/O(N,\mathbb{C})$ which extend to the affine space of all the $N \times N$ symmetric matrices over $\mathbb{C}$. On the other hand $S^p(S^2(\mathbb{C}^N))$, as a $\mathrm{GL}(N,\mathbb{C})$ module over $\mathbb{C}$, is identified with $\sum_{\substack{\eta \in \mathscr{P}(2p,N,\infty) \\ \text{with all the entries } \eta_i \\ \text{of } \eta \text{ being even}}} V_\eta$. For a character theoretic analogue of this, see [4,

p. 238, identity 11.9.4]. Moreover, it can be checked that $S(S^2(\mathbb{C}^N))$ corresponds to $\sum_{\substack{\eta\in\widehat{GL(N,\mathbb{C})}\\ \text{with } \eta_N\geqq 0}} m_\eta V_\eta$ under the identification (∗).

Hence for any partition $\eta\in\mathscr{P}(\cdot, N, \infty)$

$$\dim[\square^\eta(\mathbb{C}^N)]^{O(N)}\begin{cases}=1 & \text{if all the entries of } \eta \text{ are even}\\ =0 & \text{otherwise}.\end{cases} \tag{1}$$

Let det denote the 1-dimensional representation of $GL(N,\mathbb{C})$ given by the determinant. Applying (1) to the irreducible representation $\square^\eta(\mathbb{C}^N)\otimes\det$ [which corresponds to the partition $(\eta_1+1\geqq\ldots\geqq\eta_N+1)$], we get

$$\dim[\square^\eta\otimes\det]^{O(N)}\begin{cases}=0 & \text{if some entry of } \eta \text{ is even}\\ =1 & \text{otherwise}.\end{cases} \tag{2}$$

Now since $[\square^\eta(\mathbb{C}^N)]^{SO(N)}\simeq[\square^\eta]^{O(N)}\oplus[\square^\eta\otimes\det]^{O(N)}$ and, by assumption, $N\geqq p+1$ (hence $\lambda_N=0$), we get the required result.

Now, from Proposition (2.1), $[\Lambda^p_{\mathbb{C}}(M(N,k,\mathbb{C}))]^{SO(N)}$ is isomorphic with

$$\sum_{\lambda\in\mathscr{P}(p,N,k)}[\square^\lambda(\mathbb{C}^N)]^{SO(N)}\otimes_{\mathbb{C}}\square^{\tilde\lambda}(\mathbb{C}^k)$$

as a $U(k)$ module. This, from Lemma (2.2), is isomorphic to

$$\sum_{\substack{\lambda\in\mathscr{P}(p,\infty,k)\\ \text{with all the entries}\\ \text{of } \lambda \text{ being even}}}\square^{\tilde\lambda}(\mathbb{C}^k)\quad\text{for}\quad N\geqq p+1.$$

This is O if p is odd and is equal to

$$\sum_{\varrho=(\varrho_1\geqq\ldots\geqq\varrho_l>0)\in\mathscr{P}(p/2,[k/2],\infty)}\square^{\underset{\sim}{\varrho}}(\mathbb{C}^k)\quad\text{for even } p,$$

where $\underset{\sim}{\varrho}$ denotes the partition $(\varrho_1\geqq\varrho_1\geqq\varrho_2\geqq\varrho_2\geqq\ldots\varrho_l\geqq\varrho_l>0)$ of p.

From [4, p. 238, identity 11.9.2], we get that the above space is isomorphic with $S^{p/2}_{\mathbb{C}}(\Lambda^2_{\mathbb{C}}(\mathbb{C}^k))$ as a $U(k)$ module. But $\Lambda^2_{\mathbb{C}}(\mathbb{C}^k)$, as an $SO(k)$ module, is isomorphic with $\mathfrak{G}_{\mathbb{R}}(k)^*\otimes\mathbb{C}$, proving the assertion in this case.

(c) *Case III* [Symplectic group $\mathrm{Sp}(k)$]. It can be seen that $M(N,k,Q)^*\otimes\mathbb{C}$ is isomorphic over $\mathbb{C}$ with $M(2N,2k,\mathbb{C})$ as a $\mathrm{Sp}(N)\times\mathrm{Sp}(k)$ *module*. [Observe that $\mathrm{Sp}(N)$ sits canonically inside $U(2N)$.] Hence

$$\begin{aligned}[\Lambda^p(M(N,k,Q)^*)]^{\mathrm{Sp}(N)}\otimes\mathbb{C} &\simeq \overset{\mathrm{Sp}(N)}{\Lambda^p_{\mathbb{C}}}(M(2N,2k,\mathbb{C}))\\ &\simeq \sum_{\lambda\in\mathscr{P}(p,2N,2k)}[\square^\lambda(\mathbb{C}^{2N})]^{\mathrm{Sp}(N)}\otimes\square^{\tilde\lambda}(\mathbb{C}^{2k}).\end{aligned}$$

So, as in the case of $SO(k)$, we study the space $[\square^\lambda(\mathbb{C}^{2N})]^{\mathrm{Sp}(N)}$.

(2.3) **Lemma.** $[\square^\lambda(\mathbb{C}^{2N})]^{\mathrm{Sp}(N)}=0$ *unless* $\lambda=(\lambda_1\geqq\lambda_2\geqq\ldots)$ *satisfies* $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4,\ldots$, *and in this case it is one dimensional.*

Proof. Let (V,ϱ) be a finite dimensional representation, over $\mathbb{C}$, of a compact Lie group G. Let ch_ϱ denote its character. Then $\dim_{\mathbb{C}} V^G=\int_G \mathrm{ch}_\varrho(g)\,dg$, the integration being taken with respect to the normalized Haar measure on G. If we fix a maximal torus T in G, then, in view of the Weyl integration formula, $\dim_{\mathbb{C}} V^G$ can be expressed as the integral $\frac{1}{|W|}\int_T \mathrm{ch}_\varrho(t)\,\delta_G(t)\,\bar{\delta}_G(t)\,dt$, where $|W|$ denotes the order of the Weyl group of G and δ_G denotes its Weyl denominator.

If we take

$$D=\left\{t=(t_1,\ldots,t_N)=(e^{2\pi i\theta_1},\ldots,e^{2\pi i\theta_N})=\begin{pmatrix} e^{2\pi i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i\theta_N}\end{pmatrix}\right\}$$

as a maximal torus in $\mathrm{Sp}(N)$, we get

$$\delta_{\mathrm{Sp}(N)}(t)=\prod_{r=1}^{N}[e^{2\pi i\theta_r}-e^{-2\pi i\theta_r}]\prod_{1\leqq r<s\leqq N}[e^{\pi i(\theta_r+\theta_s)}-e^{-\pi i(\theta_r+\theta_s)}]$$
$$\prod_{1\leqq r<s\leqq N}[e^{\pi i(\theta_r-\theta_s)}-e^{-\pi i(\theta_r-\theta_s)}]\,.$$

Since $t\in D$ sits as $\mathbf{t}$ in $U(2N)$, where

$$\mathbf{t}=\begin{pmatrix} t_1 & & & & & & \\ & \bar{t}_1 & & & & & \\ & & t_2 & & 0 & & \\ & & & \bar{t}_2 & & & \\ & & 0 & & \ddots & & \\ & & & & & t_N & \\ & & & & & & \bar{t}_N \end{pmatrix}$$

and

$$\delta_{U(2N)}(\mathbf{t})=(-1)^{\frac{N(N-1)}{2}}\prod_{1\leqq r<s\leqq N}[e^{\pi i(\theta_r-\theta_s)}-e^{-\pi i(\theta_r-\theta_s)}][e^{-\pi i(\theta_r-\theta_s)}-e^{\pi i(\theta_r-\theta_s)}]$$
$$\cdot\prod_{r,s}[e^{\pi i(\theta_r+\theta_s)}-e^{-\pi i(\theta_r+\theta_s)}]\,,$$

we have

$$\frac{\delta_{\mathrm{Sp}(N)}\bar{\delta}_{\mathrm{Sp}(N)}(t)}{\delta_{U(2N)}(\mathbf{t})}=(-1)^N\prod_{r=1}^{N}(t_r-\bar{t}_r)\,,$$

so

$$\begin{aligned}\dim_{\mathbb{C}}[\square^\lambda(\mathbb{C}^{2N})]^{\mathrm{Sp}(N)}&=\frac{(-1)^N}{N!\,2^N}\int_D|\mathbf{t}^{l_1},\ldots,\mathbf{t}^{l_{2N}}|\prod_{r=1}^{N}(t_r-\bar{t}_r)\,dt\\&=\frac{(-1)^{\frac{N(N+1)}{2}}}{N!\,2^N}\int_D\left[\sum_{\varepsilon\in S_{2N}}s(\varepsilon)t_1^{l_{\varepsilon(1)}-l_{\varepsilon(N+1)}}\ldots t_N^{l_{\varepsilon(N)}-l_{\varepsilon(2N)}}\right]\prod_{r=1}^{N}(t_r-\bar{t}_r)\,dt\\&=\frac{(-1)^{\frac{N(N+1)}{2}}}{N!\,2^N}\sum_{\varepsilon\in R'}s(\varepsilon)(-1)^{m_\varepsilon}\,,\end{aligned}$$

where

$$R'=\{\varepsilon\in S_{2N}:|l_{\varepsilon(r)}-l_{\varepsilon(N+r)}|=1\quad\text{for all}\quad 1\leqq r\leqq N\}$$

and

$$m_\varepsilon=\#\{1\leqq r\leqq N:\varepsilon(r)<\varepsilon(N+r)\}\,.$$

Let H be the subgroup [of order $2^N(N!)$] of S_{2N} generated by the permutations $\{(i, N+i): 1 \leqq i \leqq N\}\ U\{\tilde{\sigma}: \sigma \in S_N\}$ where $\tilde{\sigma}(j) = \sigma(j)$ and $\tilde{\sigma}(N+j) = N + \sigma(j)$ for $1 \leqq j \leqq N$. [H corresponds to the Weyl group of $\mathrm{Sp}(N)$]. Clearly H acts from the right on R' by multiplication. Up to the H action, if R' is nonempty, any $\varepsilon \in R'$ is of the form $\varepsilon(1) = 1$, $\varepsilon(N+1) = 2$, $\varepsilon(2) = 3$, $\varepsilon(N+2) = 4, \ldots, \varepsilon(N) = 2N-1$, $\varepsilon(2N) = 2N$. This ε belongs to R' if and only if $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4, \ldots$ and in this case, the desired integral is 1. [Notice that $s(\varepsilon)(-1)^{m_\varepsilon} = s(\varepsilon h)(-1)^{m_{\varepsilon h}}$.] This proves Lemma (2.3).

Remark. This lemma can also be proved by a method similar to the proof of Lemma (2.2).

So, in view of the above lemma, for $N \geqq p/2$, we get that $[\Lambda_{\mathbb{C}}^p(M(2N, 2k, \mathbb{C}))]^{\mathrm{Sp}(N)}$ is isomorphic with

$$\sum_{\substack{\lambda \in \mathscr{P}(p, 2k, \infty) \\ \text{with all the } \lambda_i \\ \text{being even}}} \square^\lambda(\mathbb{C}^{2k})$$

as a $U(2k)$ module. This is clearly 0 for odd p and for even p, it is isomorphic with $S_{\mathbb{C}}^{p/2}[\mathfrak{G}_Q(k)^* \otimes \mathbb{C}]$. See [4; p. 238, identity 11.9.4].

This finishes the proof of the assertion that, $S_{\mathbb{C}}^p(\mathfrak{G}_F(k)^* \otimes \mathbb{C})$ is $\mathbb{C}$-isomorphic with $\left[\overset{G_F(N)}{\Lambda^{2p}}(M(N, k, F)^*)\right] \otimes \mathbb{C}$ as $G_F(k)$ module for any $N \geqq 2p+1$ and $\overset{G_F(N)}{\Lambda^r}(M(N, k, F)^*) = 0$ for odd values of r and $N \geqq r+1$, in all the three cases.

Now, we come to the proof of Theorem (1.3). We fix any $G_F(k)$ module isomorphism φ from $\mathfrak{G}_F(k)^*$ to $\overset{G_F(N)}{\Lambda^2}(M(N, k, F)^*)$ (it exists, for large N, as we have seen above). This extends to a unique algebra homomorphism [which is, of course, also a $G_F(k)$ module map] $\tilde{\varphi}: S(\mathfrak{G}_F(k)^*) \to \overset{G_F(N)}{\Lambda}(M(N, k, F)^*)$. The following lemma will finish the proof of Theorem (1.3) from dimensional considerations.

(2.4) **Lemma.** *$\tilde{\varphi}|_{S^p(\mathfrak{G}_F(k)^*)}$ is injective provided we take N to be sufficiently large compared to p.*

Proof. We take the case of $F = \mathbb{R}$, i.e., the special orthogonal group. Without loss of generality, we can assume that the map $\varphi: \mathfrak{G}_{\mathbb{R}}(k)^* \to \overset{SO(N)}{\Lambda^2}(M(N, k, \mathbb{R})^*)$ [after identifying $\mathfrak{G}_{\mathbb{R}}(k)^*$ with $\Lambda^2(\mathbb{R}^k)$ and $M(N, k, \mathbb{R})^*$ with $\mathbb{R}^N \otimes \mathbb{R}^k$] looks like $\varphi(v_i \wedge v_j) = \sum_{l=1}^N e_l \otimes v_i \wedge e_l \otimes v_j$ for $1 \leqq i < j \leqq k$, where $\{v_i\}_{1 \leqq i \leqq k}$ is a basis of $\mathbb{R}^k$ and $\{e_l\}_{1 \leqq l \leqq N}$ is the usual orthonormal basis of $\mathbb{R}^N$. Let us denote $v_i \wedge v_j$ by A_{ij}. Since

$$\begin{aligned} \tilde{\varphi}(A_{ij})^{\alpha_{ij}} &= \left[\sum_{l=1}^N e_l \otimes v_i \wedge e_l \otimes v_j\right]^{\alpha_{ij}} \\ &= \sum_{l^{ij} = (l_1^{ij} < \ldots < l_{\alpha_{ij}}^{ij})} c_{l^{ij}} e_{l^{ij}} \otimes v_i \wedge \ldots \wedge e_{l_{\alpha_{ij}}^{ij}} \otimes v_i \wedge e_{l^{ij}} \otimes v_j \wedge \ldots \wedge e_{l_{\alpha_{ij}}^{ij}} \otimes v_j, \end{aligned}$$

where $c_{l^{ij}}$ are nonzero constants, we have for

$$\begin{aligned} A^\alpha &= \prod_{i<j} [A_{ij}^{\alpha_{ij}}], \tilde{\varphi}(A^\alpha) = \prod_{i<j} [\tilde{\varphi}(A_{ij})]^{\alpha_{ij}} \\ &= \sum_{\substack{l \\ l = \{l^{ij} = (l_1^{ij} < \ldots < l_{\alpha_{ij}}^{ij})\}}} \prod_{i<j} [c_{l^{ij}} e_{l^{ij}} \otimes v_i \wedge \ldots \wedge e_{l_{\alpha_{ij}}^{ij}} \otimes v_i \wedge e_{l^{ij}} \otimes v_j \wedge \ldots \wedge e_{l_{\alpha_{ij}}^{ij}} \otimes v_j] \\ &= \sum_\mu d_\mu e_{\mu_1(1)} \otimes v_1 \wedge \ldots \wedge e_{\mu_1(p_1)} \otimes v_1 \wedge \ldots \wedge e_{\mu_k(1)} \otimes v_k \wedge \ldots \wedge e_{\mu_k(p_k)} \otimes v_k \end{aligned}$$

$$\mu = \{\mu_i = (\mu_i(l) < \ldots < \mu_i(p_i))\}_{\widehat{1 \leqq i \leqq k}}$$

for some constants d_μ, where

$$p_i = \alpha_{1,i} + \dots + \alpha_{i-1,i} + \alpha_{i,i+1} + \dots + \alpha_{i,k}.$$

Clearly, for any μ with $d_\mu \neq 0$, the intersection

$$S^\mu_{i,j} = \{\mu_i(l), \dots, \mu_i(p_i)\} \cap \{\mu_j(l), \dots, \mu_j(p_j)\}$$

has at least $\alpha_{i,j}$ elements. We choose a μ such that $d_\mu \neq 0$ and the set $S^\mu_{i,j}$ consists exactly of $\alpha_{i,j}$ elements for all $1 \leqq i < j \leqq k$. Such a choice is possible provided N is taken to be sufficiently large compared to $\sum_{i<j} \alpha_{i,j} = p$. Now, $\mathbb{R}$-linear independence of the set $\{\tilde{\varphi}(A^\alpha)\}_{\alpha = \{\alpha_{i,j}\} \text{ with } \sum \alpha_{i,j} = p}$ can be easily seen.

This proves that the map $\tilde{\varphi}|_{SP(\mathfrak{G}_F(k)^*)}$ is injective in this case.

The injectivity of $\tilde{\varphi}$, in the case of unitary and symplectic groups, can be similarly verified.

Now we return to the proof of Theorem (1.1).

(2.5) *Proof of Theorem (1.1).* The DGA $\Omega_{\mathrm{Inv}}(G_F(k+N)/I_k \times G_F(N))$ can be identified with the cochain complex of the Lie-Algebra pair $(\mathfrak{G}_F(k+N), \mathfrak{G}_F(N))$, i.e., $\Omega_{\mathrm{Inv}}(G_F(k+N)/I_k \times G_F(N)) \simeq C^*(\mathfrak{G}_F(k+N), \mathfrak{G}_F(N))$. But, by definition, $C^*(\mathfrak{G}_F(k+N), \mathfrak{G}_F(N)) = \overset{\mathfrak{G}_F(N)}{\Lambda}((\mathfrak{G}_F(k+N)/\mathfrak{G}_F(N))^*)$. Now the map $i: \mathfrak{G}_F(k) \oplus \mathfrak{G}_F(N) \oplus M(N,k,F) \to \mathfrak{G}_F(k+N)$ defined by

$$i(X+Y+Z) = \begin{pmatrix} X & -Z^t \\ Z & Y \end{pmatrix} \quad [\text{for } X \in \mathfrak{G}_F(k), Y \in \mathfrak{G}_F(N), Z \in M(N,k,F)]$$

is a vector space isomorphism. Moreover the space $M(N,k,F)$, under this identification, is stable under the action of $\mathfrak{G}_F(k)$ and $\mathfrak{G}_F(N)$. In fact $[X,Z] = -ZX$ and $[Y,Z] = YZ$. Of course $[\mathfrak{G}_F(k), \mathfrak{G}_F(N)] = 0$, and hence

$$\overset{\mathfrak{G}_F(N)}{\Lambda}((\mathfrak{G}_F(k+N)/\mathfrak{G}_F(N))^*) \simeq \overset{\mathfrak{G}_F(N)}{\Lambda}((\mathfrak{G}_F(k) \oplus M(N,k,F))^*)$$
$$\simeq \Lambda(\mathfrak{G}_F(k)^*) \otimes \overset{\mathfrak{G}_F(N)}{\Lambda}(M(N,k,F)^*).$$

So we get the isomorphism

$$\Omega_{\mathrm{Inv}}(E(G_F(k))) \simeq \Lambda(\mathfrak{G}_F(k)^*) \bigotimes \mathrm{Lt.}_{N\to\infty} \overset{\mathfrak{G}_F(N)}{\Lambda}(M(N,k,F)^*). \qquad (*)$$

Since the map $\Phi_0: W(\mathfrak{G}_F(k)) \to \Omega_{\mathrm{Ipv}}(E(G_F(k)))$ commutes with the i and θ actions, Φ_0 restricted to the set of the horizontal elements (i.e. the set of all elements ω, satisfying $i_X\omega = 0$ for all $X \in \mathfrak{G}$) gives a $G_F(k)$ module map $\Phi_0^h: S(\mathfrak{G}_F(k)^*) \longrightarrow \mathrm{Lt.}_{N\to\infty} \overset{\mathfrak{G}_F(N)}{\Lambda}(M(N,k,F)^*)$. It can be easily seen that this map Φ_0^h restricted to $\mathfrak{G}_F(k)^*$ is injective (e.g., one can use universality of connection). So, from Lemma (2.4), Φ_0^h itself is injective. Theorem (1.3), by a dimensional argument, ensures that Φ_0^h is actually an isomorphism. Moreover $\Phi_0|_{\Lambda(\mathfrak{G}_F(k)^*)}$ is the identity map under the identification $(*)$, since any map from $\mathfrak{G}_F(k)^* (\hookrightarrow \Lambda(\mathfrak{G}_F(k)^*)) \to \mathfrak{G}_F(k)^*$ commuting with the i action is, trivially, the identity map. This finishes the proof of Theorem (1.1). One could work back-ward to see that Theorem (1.1) implies Theorem (1.3).

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