EQUIVARIANT K-THEORY OF FLAG VARIETIES [1]

by Shrawan Kumar

(Notes by Valdemar Tsanov)

Contents

Introduction

Let G be a connected, simply connected, semisimple linear complex algebraic group. The flag varieties of G are the coset spaces of the form G/P, where $P \subset G$ is a parabolic subgroup. We present some results on the K-theory of flag varieties of G. By definition, the flag varieties carry a transitive G-action. The G-equivariant K-theory is relatively easy to understand. One gets more challenging questions and interesting results by restricting the action to subgroups of G, in particular, to a maximal torus T. We shall focus mainly on the T-equivariant K-theory of the complete flag variety G/B, where B is a Borel subgroup. Results about the K-groups of partial flag varieties can be deduced from those about complete flag varieties or by analogy; we only give an idea in section 2.4.

In the first chapter we introduce the fundamental notions of equivariant K-theory. In the second chapter we study the case of flag varieties. Proofs are only sketched or fully omitted. The results formulated in chapter 2, without a specific reference, can be found in [KK90]. In the last section we present some recent developments.

[1] Lectures presented at the Workshop on Symplectic Geometry with Applications to Representation Theory, July 29 - August 3, 2012, Jacobs University Bremen, Germany.

1 Basics of equivariant K-theory

1.1 K-groups of topological vector bundles

A general reference for the material in this section is [S68].

Let G_0 be a compact group. Let X be a compact topological space on which G_0 acts. We say that X is a G_0-space. A topological complex vector bundle on X consists of a topological space E, together with a continuous map $\pi : E \to X$, such that for every $x \in X$ the fibre $E_x = \pi^{-1}(x)$ has the structure of a complex vector space. In addition, we require a 'local triviality' condition. Unless otherwise specified, all vector bundles will be complex and we will occasionally suppress this adjective for brevity. A vector bundle $\pi : E \to X$ is called G_0-equivariant, if E caries a G_0-action, the map π is G_0-equivariant and, for every $x \in X$ and $g \in G_0$, the map $L_g : E_x \to E_{gx}$ is linear.

Definition 1.1. *The topological K-group $K^{\text{top}}_{G_0}(X)$ is defined as the Grothendieck group associated to the semigroup of G_0-equivariant topological complex vector bundles on X. The class of a vector bundles E is denoted by $[E]$.*

The K-group $K^{\text{top}}_{G_0}(X)$ admits a ring structure under the operations induced by direct sum and tensor product of vector bundles, i.e.,

$$[E] + [F] = [E \oplus F] \quad , \quad [E] \cdot [F] = [E \otimes F] \ .$$

1.1.1 Functoriality

Let $f : X \to Y$ be an equivariant map between compact G_0-spaces. A G_0-equivariant vector bundle E on Y can be pulled back to a G_0-equivariant vector bundle f^*E on X, with $(f^*E)_x = E_{f(x)}$. This induces a ring homomorphism

$$\begin{array}{rccc} f^* : & K^{\text{top}}_{G_0}(Y) & \longrightarrow & K^{\text{top}}_{G_0}(X) \\ & [E] & \longmapsto & [f^*E] \ . \end{array}$$

Thus, $K^{\text{top}}_{G_0}$ is a contravariant functor.

1.1.2 Algebra structure

We already know that $K^{\text{top}}_{G_0}(X)$ is a ring. We shall now put a $R(G_0)$-algebra structure on it, using the functoriality property. Here $R(G_0)$ denotes the representation ring of G_0. Note that, if $\{*\}$ is the one point space, then $K^{\text{top}}_{G_0}(*) = R(G_0)$. Now let $f : X \to \{*\}$ be the constant map. By functoriality, we have a ring homomorphism

$$R(G_0) = K^{\text{top}}_{G_0}(*) \xrightarrow{f^*} K^{\text{top}}_{G_0}(X) \ ,$$

which defines an $R(G_0)$-algebra structure on $K^{\text{top}}_{G_0}(X)$.

1.2 K-groups of algebraic vector bundles

A general reference for the sections 1.2 - 1.4 is [CG97], Ch. 5.

Let G be a reductive complex linear algebraic group. Let X be an algebraic variety with a G-action, i.e., a G-variety. We consider algebraic G-equivariant vector bundles on X.

Definition 1.2. *The algebraic K-group of $K^0_G(X)$ is defined as the Grothendieck group associated to the semigroup of algebraic G-equivariant vector bundles on X, under the equivalence relation induced from the short exact sequences of algebraic G-equivariant vector bundles (see the next remark). The class of a vector bundle E is denoted by $[E]$.*

Analogously to the topological case, $K^0_G(X)$ is a ring, with operations induced from direct sum and tensor product of vector bundles. Also, K^0_G is a contravariant functor, from the category of G-varieties to the category of rings. Thus, if $f : X \to Y$ is a G-equivariant morphism between G-varieties, we have a ring homomorphism

$$\begin{array}{rccc} f^* : & K^0_G(Y) & \longrightarrow & K^0_G(X) \\ & [E] & \longmapsto & [f^*E] \ . \end{array}$$

Taking the one point variety $\{*\}$ and the constant map $f : X \to \{*\}$, we obtain a homomorphism

$$f^* : R(G) = K^0_G(*) \longrightarrow K^0_G(X) \ ,$$

which defines an $R(G)$-algebra structure on $K^0_G(X)$.

Remark 1.1. *Notice the following difference between the topological and the algebraic case. All topological G_0-equivariant complex vector bundles are completely reducible in a G_0-equivariant way. Hence, in the construction of the Grothendieck group $K^{\mathrm{top}}_{G_0}(X)$, the splitting of short exact sequences is a redundant requirement. This is not the case with $K^0_G(X)$. There exist reducible but not completely reducible vector bundles. A simple example is obtained by restricting the tangent bundle on $\mathbb{P}^2$ to a nonsingular conic. We get an $SL_2(\mathbb{C})$-equivariant vector bundle of rank 2 on $\mathbb{P}^1$, with the tangent bundle on $\mathbb{P}^1$ as a subbundle and the normal bundle as a quotient. This bundle does not split in an $SL_2(\mathbb{C})$-equivariant way.*

1.3 K-groups of coherent sheaves

As in the previous section, G is a reductive linear algebraic group and X is a G-variety. We consider coherent sheaves on X.

Definition 1.3. *The group K^G_0 is defined as the Grothendieck group associated with the semigroup of algebraic G-equivariant coherent sheaves on X, under the equivalence relation induced from the short exact sequences of G-equivariant coherent sheaves on X. The class of a sheaf S is denoted by $[S]$.*

The group $K^G_0(X)$ does not have a natural ring structure for the following reason. Suppose

$$0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow S_3 \longrightarrow 0$$

is an exact sequence of G-equivariant coherent sheaves of X. For $K_0^G(X)$ to be a ring under tensor product, we would need the sequence

$$0 \longrightarrow S \otimes S_1 \longrightarrow S \otimes S_2 \longrightarrow S \otimes S_3 \longrightarrow 0$$

to be exact, for any G-equivariant coherent sheaf S on X. In general, it is not exact, but it is exact if S is a vector bundle. Thus, $K_0^G(X)$ is not a ring, but there is a morphism

$$K_G^0(X) \otimes K_0^G(X) \longrightarrow K_0^G(X) .$$

It follows that $K_0^G(X)$ has the structure of a module over the ring $K_G^0(X)$. Consequently, $K_0^G(X)$ is also an $R(G)$-module.

1.3.1 Functoriality

Let $f : X \to Y$ be a *proper* morphism of G-varieties. Then, if S is a coherent sheaf on X, the higher direct images $R^i f_* S$ are coherent sheaves on Y. Thus we can define a map

$$\begin{array}{rccc} f_* : & K_0^G(X) & \longrightarrow & K_0^G(Y) \\ & [S] & \longmapsto & \sum_{i=0}^{\infty} (-1)^i [R^i f_* S] . \end{array}$$

Observe that $R^i f_* S = 0$, for $i > \dim X$ and hence the above sum is a finite sum.

1.4 Relation between $K_0^G(X)$ and $K_G^0(X)$

Every algebraic vector bundle on X is a coherent sheaf. Thus there is a group homomorphism

$$\theta : K_G^0(X) \longrightarrow K_0^G(X) . \tag{1}$$

Also, since we work over the complex numbers, we can consider the analytic topology on G, on X and on any algebraic vector bundle on X. Let G_0 be a maximal compact subgroup of the complex Lie group G. Then, since G is reductive, we have $G = G_0^{\mathbb{C}}$. Thus, forgetting the algebraic structure, we get a ring homomorphism

$$\psi : K_G^0(X) \longrightarrow K_{G_0}^{\text{top}}(X) .$$

The following result can be found as Proposition 5.1.28 in [CG97].

Theorem 1.1. *If X is a smooth G-variety, then the map $\theta : K_G^0(X) \to K_0^G(X)$ is an isomorphism of $R(G)$-modules.*

Idea of proof: The surjectivity of θ is implied by the following argument. Let S be a G-equivariant coherent sheaf on X. We need to show that its class $[S] \in K_0^G(X)$ can be written as $[S] = [V_1] - [V_2]$, where V_1, V_2 are G-equivariant vector bundles on X. Since X is smooth, S admits a resolution of finite length by G-equivariant vector bundles (cf. [CG97], Proposition 5.1.28), i.e., there is an exact sequence

$$0 \longrightarrow V_n \longrightarrow \ldots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow S \longrightarrow 0 ,$$

where each V_i is a G-equivariant vector bundle. Thus,

$$[S] = \sum_{j=0}^{n} (-1)^j [V_j] = [\bigoplus_{j \text{ even}} V_j] - [\bigoplus_{j \text{ odd}} V_j] \ .$$

Hence θ is surjective. The proof of injectivity of θ is similar.

2 K-theory of flag varieties

We now turn to our main object of interest - the K-theory of flag varieties. For the most part, we follow [KK90], and this is where the reader can find more details and complete proofs. Some recent developments are presented in the last section.

2.1 Flag varieties of semisimple groups

Let G be a connected, simply connected, semisimple linear complex algebraic group. Fix a Borel subgroup $B \subset G$. Let $P \subset G$ be a parabolic subgroup containing B. The coset spaces

$$Y = G/B \quad , \quad Y_P = G/P$$

are smooth projective algebraic varieties called flag varieties of G. Y is called the *complete flag variety* and Y_P is called a *partial flag variety.*

Let G_0 be a maximal compact subgroup of G. Then $T_0 = G_0 \cap B$ is a compact torus, which is a Cartan subgroup of G_0. The action of G_0 on Y is transitive and we have $Y = G_0/T_0$.

Let $T = T_0^{\mathbb{C}}$ be the complexification of T_0. Then T is a Cartan subgroup of G contained in B. Let $W = W(G,T)$ denote the Weyl group acting naturally on the Lie algebra $\mathfrak{t}$ and on its dual $\mathfrak{t}^*$. Let $R = R(G,T)$ be the root system, $R^+ = R(B,T)$ be the set of positive roots, and $\Delta = \{\alpha_1, ..., \alpha_l\} \subset R^+$ be the set of simple roots. Here $l = \dim T$. Let $s_1, ..., s_l$ be the simple reflections generating W.

2.2 K-theory of complete flag varieties

Let us start with $K^0_G(Y)$. The only irreducible G-equivariant vector bundles on X are line bundles, because the isotropy group B is solvable. A line bundle on Y is given by a character of B. Any character of B gives, via restriction, a character of T. In fact, this restriction gives an isomorphism of character groups, because $T \cong B/[B,B]$. Thus,

$$K^0_G(Y) \cong R(T) \ .$$

We now concentrate on the T-equivariant K-groups of Y. We know from theorem 1.1 that $K^0_T(Y) \cong K^T_0(Y)$. We shall show that

$$K^0_T(Y) \cong K^T_0(Y) \cong K^{\text{top}}_{T_0}(Y) \ .$$

2.2.1 A Weyl group action on $K_{T_0}^{\text{top}}(Y)$

There is a Weyl group action on Y defined as follows:

$$\begin{array}{ccc} W \times G_0/T_0 & \longrightarrow & G_0/T_0 \\ (w, gT_0) & \longmapsto & gw^{-1}T_0 . \end{array}$$

Notice that we used the diffeomorphism of Y with G_0/T_0 to define the action, rather than the representation as G/B. In fact, W does not act algebraically on the coset space G/B. There is only an action by diffeomorphisms, which induces an action of W on $K_{T_0}^{\text{top}}(Y)$.

2.2.2 Demazure operators

Let $i \in \{1, ..., l\}$. To the simple reflection s_i we shall associate an operator

$$D_{s_i} : K_{T_0}^{\text{top}}(Y) \longrightarrow K_{T_0}^{\text{top}}(Y) ,$$

called a Demazure operator. It is defined below after some preparation.

Let $P_i \subset G$ be the minimal parabolic subgroup associated to s_i, i.e., the subgroup whose Lie algebra is

$$\mathfrak{p}_i = \mathfrak{b} \oplus \mathfrak{g}^{-\alpha_i} ,$$

where $\mathfrak{g}^{-\alpha_i}$ is the root space in $\mathfrak{g}$ corresponding to the root $-\alpha_i$. Let $Y_i = G/P_i$ be the corresponding flag variety. There is a G-equivariant projection

$$\pi_i : G/B \longrightarrow G/P_i \ , \ \ gB \longmapsto gP_i \ .$$

The fibre is isomorphic to $\mathbb{P}^1 \cong P_i/B$.

Lemma 2.1. *There exists a vector bundle $E_i \to Y_i$ of rank 2, such that $G/B \cong \mathbb{P}(E_i)$, where $\mathbb{P}(E_i)$ denotes the projectivized bundle.*

Proof. Let V_i be the irreducible representation of P_i corresponding to the i-th fundamental weight χ_i of G. (The weight χ_i is characterized by $\chi_i(\alpha_j^\vee) = \delta_{i,j}$.) Then V_i is 2-dimensional. Put $E_i = G \overset{P_i}{\times} V_i$. We have a G-equivariant isomorphism:

$$\begin{array}{ccc} G \overset{P_i}{\times} \mathbb{P}(V_i) & \xrightarrow{\ \sim\ } & G/B \\ & \searrow \quad \swarrow_{\pi_i} & \\ & G/P_i & \end{array}$$

□

Theorem 2.2. (Segal, [S68], Proposition 3.9)

Let $\pi : E \to X$ be a T_0-equivariant vector bundle of rank 2 over a compact topological space X. Let $[\pi] : \mathbb{P}(E) \to X$ be the associated $\mathbb{P}^1$-bundle. Then, $K_{T_0}^{\text{top}}(\mathbb{P}(E))$ is a free module over $K_{T_0}^{\text{top}}(X)$ with basis $1, [H]$, where H is the Hopf bundle on $\mathbb{P}(E)$ (i.e., the dual of the canonical line bundle on $\mathbb{P}(E)$).

Remark 2.1. *The same holds for K_T^0 instead of $K_{T_0}^{\text{top}}$ (cf. [CG97], Theorem 5.2.31).*

Remark 2.2. *The module structure referred to in the above theorem is given by the map*

$$[\pi]^* : K_{T_0}^{\text{top}}(X) \longrightarrow K_{T_0}^{\text{top}}(\mathbb{P}(E)) .$$

The Hopf bundle is the T_0-equivariant line bundle $H \to \mathbb{P}(E)$, whose restriction to any fibre in $\mathbb{P}(E)$ is $\mathcal{O}(1)$ on that fibre.

From lemma 2.1 and the above theorem we get the following corollary about the K-theory of the flag variety Y.

Corollary 2.3. *There is an isomorphism of $R(T)$-modules*

$$K_{T_0}^{\text{top}}(Y) \cong K_{T_0}^{\text{top}}(Y_i) \oplus H_i K_{T_0}^{\text{top}}(Y_i) ,$$

where H_i is the Holf bundle on $Y = \mathbb{P}(E_i)$ with E_i as in lemma 2.1.

Definition 2.1. *The Demazure operator $D_{s_i} : K_{T_0}^{\text{top}}(Y) \to K_{T_0}^{\text{top}}(Y)$ is defined as the projection to the first factor in the direct sum decomposition given in corollary 2.3, i.e., $D_{s_i}(E + H_i F) = E$, for $E, F \in \pi_i^*(K_{T_0}^{\text{top}}(Y_i)) \subset K_{T_0}^{\text{top}}(Y)$.*

Lemma 2.4. *The Demazure operators $D_{s_1}, ..., D_{s_l}$ satisfy the following properties:*
(1) Idempotence: $D_{s_i}^2 = D_{s_i}$, for $i = 1, ..., l$.
(2) Braid property: if $w = s_{i_1}...s_{i_k} = s_{j_1}...s_{j_k}$ are two reduced expressions for w, then

$$D_{s_{i_1}} \circ \cdots \circ D_{s_{i_k}} = D_{s_{j_1}} \circ \cdots \circ D_{s_{j_k}} .$$

The above lemma allows us to associate a Demazure operator to any Weyl group element.

Definition 2.2. *For $w \in W$, define $D_w = D_{s_{i_1}} \circ \cdots \circ D_{s_{i_k}}$, where $w = s_{i_1}...s_{i_k}$ is any reduced expression.*

Remark 2.3. *The braid property given in part (2) of lemma 2.4 is satisfied if and only if the following relations are satisfied for $i \neq j$:*

$$\begin{array}{ll} D_{s_i} \circ D_{s_j} = D_{s_j} \circ D_{s_i} & \text{, if } s_i s_j = s_j s_i \text{ ,} \\ D_{s_i} \circ D_{s_j} \circ D_{s_i} = D_{s_j} \circ D_{s_i} \circ D_{s_j} & \text{, if the order of } s_i s_j \text{ is 3,} \\ (D_{s_i} \circ D_{s_j})^2 = (D_{s_j} \circ D_{s_i})^2 & \text{, if the order of } s_i s_j \text{ is 4,} \\ (D_{s_i} \circ D_{s_j})^3 = (D_{s_j} \circ D_{s_i})^3 & \text{, if the order of } s_i s_j \text{ is 6.} \end{array}$$

2.3 Algebraic model for $K_{T_0}^{\text{top}}(Y)$

Let $Q(T)$ denote the quotient field of the representation ring $R(T)$. The following theorem is an important result in equivariant K-theory.

Theorem 2.5. (Atiyah-Segal localization theorem, cf. [S68], Proposition 4.1)
Let X be a compact T_0-space. Then there is an isomorphism

$$Q(T) \otimes_{R(T)} K_{T_0}^{\text{top}}(X) \cong Q(T) \otimes_{R(T)} K_{T_0}^{\text{top}}(X^{T_0}) ,$$

where X^{T_0} denotes the set of points in X fixed by T_0.

This theorem and the specific features of flag manifolds can be applied to obtain information about, and a purely algebraic incarnation of, the K-theory of flag manifolds.

For $Y = G/B$ we have $Y^{T_0} = Y^T = \{wB : w \in W\}$. We identify W with Y^T under $w \mapsto w^{-1}B$. Thus, we have

$$K_{T_0}^{\mathrm{top}}(Y^T) \cong \mathrm{Maps}(W, R(T)) \ .$$

Let

$$Q_W = \mathrm{Maps}(W, Q(T)) \ .$$

This set has a structure of an $R(T)$-algebra under pointwise addition, multiplication and multiplication by scalars, i.e., for $f, g \in Q_W$, $a \in R(T)$ and $w \in W$,

$$\begin{aligned}(f+g)(w) &= f(w) + g(w) \ , \\ (fg)(w) &= f(w)g(w) \ , \\ (af)(w) &= a(f(w)) \ .\end{aligned}$$

There is an action of W on Q_W given by:

$$(vf)(w) = f(v^{-1}w) \ .$$

For each simple reflection s_i, we define an operator

$$A_{s_i} : Q_W \longrightarrow Q_W \quad , \quad (A_{s_i}f)(w) = \frac{f(w) - f(s_i w)e^{-w^{-1}\alpha_i}}{1 - e^{-w^{-1}\alpha_i}} \ ,$$

where $e^{-w^{-1}\alpha_i} \in R(T)$ denotes the element corresponding to the character $-w^{-1}\alpha_i$. The operators A_{s_i} satisfy the following relations analogous to those between the Demazure operators D_{s_i}.

Lemma 2.6. *The operators $A_{s_1}, ..., A_{s_l}$ satisfy the following properties:*

(1) Idempotence: $A_{s_i}^2 = A_{s_i}$ for $i = 1, ..., l$.

(2) Braid property: if $w = s_{i_1}...s_{i_k} = s_{j_1}...s_{j_k}$ are two reduced expressions for w, then

$$A_{s_{i_1}} \circ \cdots \circ A_{s_{i_k}} = A_{s_{j_1}} \circ \cdots \circ A_{s_{j_k}} \ .$$

Definition 2.3. *For $w \in W$, we define $A_w = A_{s_{i_1}} \circ \cdots \circ A_{s_{i_k}}$, where $w = s_{i_1}...s_{i_k}$ is any reduced expression.*

In the algebra Q_W, we distinguish the following subalgebra:

$$\Omega_W = \{f \in Q_W : (A_w f)(1) \in R(T), \forall w \in W\} \ .$$

Theorem 2.7. (Kostant-Kumar, [KK90], Theorem 3.13)

The localization map

$$\gamma : K_{T_0}^{\mathrm{top}}(Y) \longrightarrow K_{T_0}^{\mathrm{top}}(Y^{T_0})$$

is a monomorphism of $R(T)$-algebras, whose image is exactly Ω_W. Moreover, γ commutes with the W-action and the Demazure operators D_{s_i} (acting on the left) and A_{s_i} (acting on the right).

Lemma 2.8. *The $R(T)$-algebra $K_{T_0}^{\text{top}}(Y)$ is a free $R(T)$-module of rank $|W|$.*

Proof. Recall the Schubert cell decomposition of the flag manifold:

$$Y = \bigsqcup_{w \in W} BwB/B\ .$$

Using the length function on W we define a filtration

$$\{pt\} = Y_0 \subset Y_1 \subset \cdots \subset Y_N = Y \quad , \quad \text{where} \quad Y_n = \bigcup_{l(w) \le n} BwB/B\ .$$

Now one can use induction to prove that $K_{T_0}^{\text{top}}(Y_n)$ is a free $R(T)$-module. The argument uses relative K-groups and the fact that the topological K-group of an affine space is the same as that of a point (see the proof of Lemma 3.15 in [KK90]). □

Corollary 2.9. *The $R(T)$-algebra Ω_W is a free $R(T)$-module of rank $|W|$.*

Lemma 2.10. *There exists a canonical $R(T)$-basis $\{\psi^w\}_{w \in W}$ of Ω_W satisfying the following properties:*

$$A_{s_i}\psi^w = \begin{cases} \psi^w + \psi^{s_i w} & , \text{ if } s_i w < w \\ 0 & , \text{ otherwise} \end{cases} ,$$

$$\psi^{w_0}(w) = \begin{cases} \prod_{\alpha \in R^+} (1 - e^\alpha) & , \text{ if } w = w_0 \\ 0 & , \text{ if } w \neq w_0 \end{cases} ,$$

where $w_0 \in W$ is the longest element.

The following result is due to McLeod, [M79]. (See also Theorem 4.4 in [KK90], Theorem 6.1.22 in [CG97], and [KL87].)

Theorem 2.11. *The map*

$$\begin{array}{rccc} \varphi : & R(T) \otimes_{R(T)^W} R(T) & \longrightarrow & K_{T_0}^{\text{top}}(Y) \\ & p \otimes q & \longmapsto & p \cdot \beta(q) \end{array}$$

is an isomorphism of $R(T)$-modules. Here the map β is given by

$$\beta : R(T) \longrightarrow K_T^0(Y) \to K_{T_0}^{\text{top}}(Y) \quad , \quad \beta(e^\lambda) = [G \overset{B}{\times} \mathbb{C}_\lambda]\ ,$$

and $p\cdot$ is the action of $R(T)$ on $K_{T_0}^{\text{top}}(Y)$.

Idea of proof: We have a composition

$$R(T) \otimes_{R(T)^W} R(T) \longrightarrow K_{T_0}^{\text{top}}(Y) \longrightarrow \Omega_W\ .$$

First, one shows the surjectivity of the composed map. Thus, we have a surjective map between free $R(T)$-modules of equal rank. Such a map can only be an isomorphism, as any surjective R-linear map between R-modules of the same finite rank is an isomorphism for any commutative ring R. (It is known, from a result of Steinberg [S75], that $R(T)$ is a free $R(T)^W$-modules of rank $|W|$.)

Corollary 2.12. *There are isomorphisms of $R(T)$-modules*

$$K_{T_0}^{\mathrm{top}}(Y) \cong K_T^0(Y) \cong K_0^T(Y) \; .$$

Proof. The second isomorphism follows from theorem 1.1, since Y is smooth. We consider the first one. We have

$$\begin{array}{ccc} K_T^0(Y) & \xrightarrow{\quad\zeta\quad} & K_{T_0}^{\mathrm{top}}(Y) \\ \nwarrow & & \nearrow \varphi \\ & R(T) \otimes_{R(T)^W} R(T) & \end{array}$$

where the map ζ on the top is obtained by simply forgetting the algebraic structure of a given vector bundle and considering it as a topological vector bundle. By construction, we have Image(φ) $\subset$ Image(ζ). Since φ is an isomorphism, ζ is surjective. The two $R(T)$-modules on the top row are free and both have rank $|W|$ (cf. Lemma 2.8 for $K_{T_0}^{\mathrm{top}}(Y)$, and Section 5.2.14 and Theorem 5.4.17 in [CG97] for $K_0^T(Y) \cong K_T^0(Y)$). Thus the surjective morphism ζ must be an isomorphism. □

2.4 K-theory of partial flag varieties

Let $P \subset G$ be a parabolic subgroup containing B and $Y_P = G/P$ be the associated flag variety. Let $W_P \subset W$ denote the Weyl group of the Levi component of P associated with the chosen Cartan subgroup $T \subset B$.

Proposition 2.13. *There is an isomorphism of $R(T)$-algebras*

$$K_{T_0}^{\mathrm{top}}(Y_P) \cong [K_{T_0}^{\mathrm{top}}(Y)]^{W_P} \; .$$

Proof. We have the projection $\pi : Y = G/B \to G/P = Y_P$. By functoriality, we get a map

$$\pi^* : K_{T_0}^{\mathrm{top}}(Y_P) \to K_{T_0}^{\mathrm{top}}(Y) \cong \Omega_W \; .$$

For any $w \in W_P$, we have a commutative triangle

$$\begin{array}{ccc} G_0/T_0 & \xrightarrow{\quad \cdot w \quad} & G_0/T_0 \\ \searrow & & \swarrow \\ & G_0/(G_0 \cap P). & \end{array}$$

Thus, the image of π^* is contained in the set of W_P-invariant elements. In fact, as proved in Theorem 3.23 in [KK90], we have

$$K_{T_0}^{\mathrm{top}}(Y_P) \cong \Omega_W^{W_P} \; .$$

The $R(T)$-module $\Omega_W^{W_P}$ has basis $\{e^\rho \psi^w\}_{W^P}$, where ψ^w are the elements given in lemma 2.10, $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ and W^P is the set of shortest representatives of the cosets in $W_P \backslash W$. □

2.5 A basis of $K_{T_0}^{\text{top}}(Y)$ and its geometric identification

We denote by $\{\tau^w\}_{w\in W}$ the $R(T)$-basis of $K_{T_0}^{\text{top}}(Y)$ such that $\gamma(\tau^w) = \psi^w$, where ψ^w is given by lemma 2.10. The product in $K_{T_0}^{\text{top}}(Y)$ is then determined by the structure constants associated with this basis via

$$\tau^u \cdot \tau^v = \sum_{w\in W} p_{u,v}^w \tau^w \quad , \quad p_{u,v}^w \in R(T) \; .$$

The constants $p_{u,v}^w$ can be studied using the combinatorics of W and Ω_W. More details on that topic can be found in [KK90].

The following theorem provides a geometric interpretation of the basis $\{\tau^w\}$. Let $Y_w = \overline{BwB/B} \subset Y$ denote the Schubert variety associated to $w \in W$. Let B^- be the opposite Borel subgroup to B, so that $R^- = R(B^-) = -R^+$. Denote $Y^w = \overline{B^-wB/B} \subset Y$.

For any closed subvariety $Z \subset Y$ let $\mathcal{O}_Z$ denote the coherent sheaf on Y defined as the structure sheaf of Z extended to Y by 0 on $Y \setminus Z$. Then, if Z is T-stable, $[\mathcal{O}_Z]$ is an element of $K_0^T(Y)$ and, via theorem 2.12, we can view it as an element of $K_{T_0}^{\text{top}}(Y)$.

Theorem 2.14. (Graham-Kumar, [GK08], Proposition 2.2)
For $w \in W$ we have

$$\tau^{w^{-1}} = *(\mathcal{O}_{Y^w}[-\partial Y^w]) = *([\mathcal{O}_{Y^w}] - [\mathcal{O}_{\partial Y^w}]) \; ,$$

where $\partial Y^w = Y^w \setminus (B^-wB/B)$ and $* : K_{T_0}^{\text{top}}(Y) \to K_{T_0}^{\text{top}}(Y)$, $*[E] = [E^*]$, is the dualization operator.

2.6 Positivity of the basis $\{\tau^w\}$

Conjecture 2.15. (Graham-Kumar, [GK08])
The structure constants $\{p_{u,v}^w\}_{u,v,w\in W}$ of the $R(T)$-algebra $K_{T_0}^{\text{top}}(G/B)$ with respect to the basis $\{\tau^w\}_{w\in W}$ satisfy $(-1)^{l(u)+l(v)+l(w)} p_{u,v}^w \in \mathbb{Z}_+[x_1, ..., x_l]$, where $x_i = e^{\alpha_i} - 1$.

Theorem 2.16. (Anderson-Miller-Griffeth, [AGM11])
The above conjecture holds.

Theorem 2.17. (Kumar, [K12])
The above conjecture holds in the more general setting of symmetrizable Kac-Moody groups.

References

[AGM11] D. Anderson, S. Griffeth and E. Miller, *Positivity and Kleiman transversality in equivariant K-theory of homogeneous spaces*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 1, 57-84.

[CG97] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston, 1997.

[GK08] W. Graham and S. Kumar, *On positivity in T-equivariant K-theory of flag varieties*, Int. Math. Res. Not. 2008, Art. ID rnn 093, 43 pp.

[KK90] B. Kostant and S. Kumar, *T-equivaraint K-theory of generalized flag varieties*, J. Differential Geometry **32** (1990), 549-603.

[K12] S. Kumar, *Positivity in T-equivarant K-theory of flag varieties*, Preprint, 2012.

[KL87] D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), 153-215.

[M79] J. McLeod, *The Künneth formula in equivariant K-theory*, Algebraic Topology (P. Hoffman and V. Snaith, eds.), Lecture Notes in Math., Vol. 741, Springer, Berlin, 1979, 316-333.

[S68] G. Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 129-151.

[S75] R. Steinberg, *On a theorem of Pittie*, Topology **14** (1975), 173-177.