Eigenvalue problem for Hermitian matrices and its generalization to arbitrary reductive groups

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1 Hermitian eigenvalue problem

For any $n \times n$ Hermitian matrix A, let $\lambda_A = (\lambda_1 \geq \cdots \geq \lambda_n)$ be its set of eigenvalues written in descending order. (Recall that all the eigenvalues of a Hermitian matrix are real.) We recall the following classical problem.

Problem 1. *(The Hermitian eigenvalue problem) Given two n-tuples of nonincreasing real numbers: $\boldsymbol{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1 \geq \cdots \geq \mu_n)$, determine all possible $\boldsymbol{\nu} = (\nu_1 \geq \cdots \geq \nu_n)$ such that there exist Hermitian matrices A, B, C with $\lambda_A = \boldsymbol{\lambda}, \lambda_B = \boldsymbol{\mu}, \lambda_C = \boldsymbol{\nu}$ and $C = A + B$.*

Said imprecisely, the problem asks the possible eigenvalues of the sum of two Hermitian matrices with fixed eigenvalues.

A conjectural solution of the above problem was given by Horn in 1962.

For any positive integer $r < n$, inductively define the set S_r^n as the set of triples (I, J, K) of subsets of $[n] := \{1, \dots, n\}$ of cardinality r such that

$$\sum_{i \in I} i + \sum_{j \in J} j = r(r+1)/2 + \sum_{k \in K} k \tag{1}$$

and for all $0 < p < r$ and $(F, G, H) \in S_p^r$ the following inequality holds:

$$\sum_{f \in F} i_f + \sum_{g \in G} j_g \leq p(p+1)/2 + \sum_{h \in H} k_h. \tag{2}$$

Now, Horn conjectured the following.

Conjecture 2. *(Horn) A triple $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}$ occurs as eigenvalues of Hermitian $n \times n$ matrices A, B, C respectively such that $C = A + B$ if and only if*

$$\sum_{i=1}^n \nu_i = \sum_{i=1}^n \lambda_i + \sum_{i=1}^n \mu_i,$$

and for all $1 \le r < n$ *and all triples* $(I, J, K) \in S_r^n$*, we have*

$$\sum_{k \in K} \nu_k \le \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j.$$

Of course, the first identity is nothing but the trace identity.

Remark. Even though this problem goes back to the nineteenth century, the first significant result was given by H. Weyl in 1912:

$$\nu_{i+j-1} \le \lambda_i + \mu_j \text{ whenever } i + j - 1 \le n.$$

K. Fan found some other inequalities in 1949 followed by Lidskii (1950). The full set of inequalities given above are due to Horn. Horn's above conjecture was settled in the affirmative by combining the work of Klyachko (1998) with the work of Knutson-Tao (1999) on the 'saturation' problem. The above system of inequalities is overdetermined. Belkale came up with a subset of the above set of inequalities which forms an irredundant system of inequalities as proved by Knutson-Tao-Woodward.

2 Generalization of the eigenvalue problem

Now we will discuss a genaralization of the above Hermitian eigenvalue problem to an arbitrary complex semisimple group. (A further generalization to any reductive group follows fairly easily from the semisimple case.)

So, let G be a connected, simply-connected, semisimple complex algebraic group. We fix a Borel subgroup B, a maximal torus H, and a maximal compact subgroup K. We denote their Lie algebras by the corresponding Gothic characters: $\mathfrak{g}$, $\mathfrak{b}, \mathfrak{h}, \mathfrak{k}$ respectively. Let R^+ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$) and let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset R^+$ be the set of simple roots. There is a natural homeomorphism $C : \mathfrak{k}/K \to \mathfrak{h}_+$, where K acts on $\mathfrak{k}$ by the adjoint representation and $\mathfrak{h}_+ := \{h \in \mathfrak{h} : \alpha_j(h) \ge 0\}$ (for all the simple roots α_j) is the positive Weyl chamber in $\mathfrak{h}$. The inverse map C^{-1} takes any $h \in \mathfrak{h}_+$ to the K-conjugacy class of ih.

For any positive integer s, define the set $\Gamma(s) :=$

$$\{(h_1, \ldots, h_s) \in \mathfrak{h}_+^s \mid \exists (k_1, \ldots, k_s) \in \mathfrak{k}^s:$$

$$\sum_{j=1}^{s} k_j = 0 \text{ and } C(k_j) = h_j \forall j = 1, \ldots, s\}.$$

Following is the generalization of the Hermitian eigenvalue problem to an arbitrary G. (The case $G = GL_n$ and $s = 3$ specializes to the problem discussed in the beginning if we replace C by $-C$.)

Problem 3. *Describe the set* $\Gamma(s)$.

By virtue of the convexity result in symplectic geometry, the subset $\Gamma(s) \subset \mathfrak{h}_+^s$ is a convex polyhedral cone (defined by certain inequalities). The aim is to find these inequalities describing $\Gamma(s)$ explicitly.

Before we can give a solution of the problem, we need some more notation.

Let $P \supset B$ be a standard parabolic subgroup with Lie algebra $\mathfrak{p}$ and let $\mathfrak{l}$ be its unique Levi component containing the Cartan subalgebra $\mathfrak{h}$. Let $\Delta(P) \subset \Delta$ be the set of simple roots contained in the set of roots of $\mathfrak{l}$. For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}$ by

$$\alpha_i(x_j) = \delta_{i,j}, \ \forall\ 1 \leq i \leq \ell. \tag{3}$$

Let W_P be the Weyl group of P (which is, by definition, the Weyl Group of the Levi component L), then in each coset of W/W_P we have a unique member w of minimal length. Let W^P be the set of the minimal length representatives in the cosets of W/W_P.

For any $w \in W^P$, define the (shifted) Schubert cell:

$$\Lambda_w^P := w^{-1}BwP \subset G/P.$$

Then, it is a locally closed subvariety of G/P isomorphic with the affine space $\mathbb{A}^{\ell(w)}, \ell(w)$ being the length of w . Its closure is denoted by $\bar{\Lambda}_w^P$, which is an irreducible (projective) subvariety of G/P of dimension $\ell(w)$. Let $\mu(\bar{\Lambda}_w^P)$ denote the fundamental class of $\bar{\Lambda}_w^P$ considered as an element of the singular homology with integral coefficients $H_{2\ell(w)}(G/P, \mathbb{Z})$ of G/P. Then, from the Bruhat decomposition, the elements $\{\mu(\bar{\Lambda}_w^P)\}_{w\in W^P}$ form a $\mathbb{Z}$-basis of $H_*(G/P, \mathbb{Z})$. Let $\{\epsilon_w^P\}_{w\in W^P}$ be the dual basis of the singular cohomology with integral coefficients $H^*(G/P, \mathbb{Z})$, i.e., for any $v, w \in W^P$ we have

$$\epsilon_v^P(\mu(\bar{\Lambda}_w^P)) = \delta_{v,w}.$$

Given a standard maximal parabolic subgroup P, let ω_P denote the corresponding fundamental weight, i.e., $\omega_P(\alpha_i^\vee) = 1$, if $\alpha_i \in \Delta \setminus \Delta(P)$ and 0 otherwise, where $\alpha_i^\vee$ is the fundamental coroot corresponding to the simple root α_i.

3 Deformation of Cup Product in $H^*(G/P)$

Let P be any standard parabolic subgroup of G. Write the standard cup product in $H^*(G/P, \mathbb{Z})$ in the $\{\epsilon_w^P\}$ basis as follows:

$$[\epsilon_u^P] \cdot [\epsilon_v^P] = \sum_{w\in W^P} d_{u,v}^w [\epsilon_w^P]. \tag{4}$$

Introduce the indeterminates τ_i for each $\alpha_i \in \Delta \setminus \Delta(P)$ and define a deformed cup product $\odot$ as follows:

$$\epsilon_u^P \odot \epsilon_v^P = \sum_{w \in W^P} \Big(\prod_{\alpha_i \in \Delta \setminus \Delta(P)} \tau_i^{(u^{-1}\rho + v^{-1}\rho - w^{-1}\rho - \rho)(x_i)} \Big) d_{u,v}^w \epsilon_w^P,$$

where ρ is the (usual) half sum of positive roots of $\mathfrak{g}$.

By using the Geometric Invariant Theory one proves that whenever $d_{u,v}^w$ is nonzero, the exponent of τ_i in the above is a nonnegative integer. Moreover, the product $\odot$ is associative (and clearly commutative).

The cohomology algebra of G/P obtained by setting each $\tau_i = 0$ in $(H^*(G/P, \mathbb{Z}) \otimes \mathbb{Z}[\tau_i], \odot)$ is denoted by $(H^*(G/P, \mathbb{Z}), \odot_0)$. Thus, as a $\mathbb{Z}$-module, this is the same as the singular cohomology $H^*(G/P, \mathbb{Z})$ and under the product $\odot_0$ it is associative (and commutative). Moreover, it continues to satisfy the Poincaré duality. Further, it can be proved that for a minuscule maximal paparbolic P, the product $\odot_0$ coincides with the standard cup product.

Now we are ready to state the main result on solution of the eigenvalue problem for any G stated above.

Theorem 4. *(due to Belkale-Kumar) Let $(h_1, \ldots, h_s) \in \mathfrak{h}_+^s$. Then, the following are equivalent:*

(a) $(h_1, \ldots, h_s) \in \Gamma(s)$.

(b) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \ldots, w_s) \in (W^P)^s$ such that

$$\epsilon_{w_1}^P \odot_0 \cdots \odot_0 \epsilon_{w_s}^P = \epsilon_o^P \in \big(H^*(G/P, \mathbb{Z}), \odot_0\big),$$

the following inequality holds:

$$\omega_P\Big(\sum_{j=1}^{s} w_j^{-1} h_j\Big) \geq 0,$$

where ϵ_o^P is the (top) fundamental class (which is the oriented integral generator of $H^{\mathrm{top}}(G/P, \mathbb{Z})$).

Remark. The above theorem specializes to a solution of the Hermitian eigenvalue problem if we take $G = GL_n$. In this case, every maximal parabolic subgroup P is minuscule and hence, as mentioned earlier, the deformed product $\odot_0$ in $H^*(G/P)$ coincides with the standard cup product. In this case, the above theorem was obtained by Klyachko with a refinement by Belkale. (The set of inequalities (b) for $G = GL_n$ in general is much smaller than the set of Horn inequalities discussed earlier. Further, as shown by Knutson-Tao-Woodward, the set of inequalities (b) is an irredundant system for $G = GL_n$.) If we replace the product $\odot_0$ in (b) by the standard cup product, then the equivalence of (a) and (b) for general G was proved by Kapovich-Leeb-Millson following an analogous slightly weaker result proved by Berenstein-Sjamaar. It may be mentioned that replacing the product $\odot_0$ in (b) by the standard cup product, we get far more inequalities for groups other than GL_n (or SL_n). For example, for G of type B_3, the standard cup product gives rise to 135 inequalities, whereas the new product gives only 102 inequalities. I should also mention that by some results of Kumar-Leeb-Millson and Kapovich-Kumar-Millson, the set of inequalities (b) is an irredundant system for groups of type B_3, C_3 and D_4. It might be expected that the set of inequalities (b) is an irredundant system for any G.

My interest in the eigenvalue problem stems from the problem of tensor product decomposition. Specifically, for any dominant integral weight $\lambda \in \mathfrak{h}^*$ (i.e., $\lambda(\alpha_i^\vee) \in \mathbb{Z}_+$ for each simple coroot $\alpha_i^\vee$), let $V(\lambda)$ be the finite dimensional irreducible G-module with highest weight λ. *Given dominant integral weights $\lambda_1, \dots, \lambda_s \in \mathfrak{h}^*$, a classical and a very central problem is to determine which irreducible representations $V(\nu)$ occur in the tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_s)$?* By taking the tensor product of $V(\lambda_1) \otimes \cdots \otimes V(\lambda_s)$ with the dual representation $V(\nu)^*$, (and replacing s by $s+1$) we can reformulate the above question as follows.

Problem 5. *Determine the set of s-tuples $(\lambda_1, \dots, \lambda_s)$ of dominant integral weights such that the tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_s)$ has a nonzero G-invariant subspace.*

This problem in general seems quite hard. So, let us pose the following weaker *saturated tensor product* problem.

Problem 6. *Determine the set $\hat{\Gamma}(s)$ of s-tuples $(\lambda_1, \dots, \lambda_s)$ of dominant rational weights such that the tensor product $V(N\lambda_1) \otimes \cdots \otimes V(N\lambda_s)$ has a nonzero G-invariant subspace for some positive integer N, where we call a weight a dominant rational weight if its some positive integral multiple is a dominant integral weight.*

The above saturated tensor product problem is parallel to the eigenvalue problem because of the following result. Let $D := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{R}_+ \forall i\}$ be the set of dominant real weights. Then, under the Killing form, we have an identification $\mathfrak{h}_+ \to D$. Under this identification, x_i corresponds with $2\omega_i / \langle \alpha_i, \alpha_i \rangle$, where ω_i is the i-th fundamental weight.

Proposition 7. *Under the identification of $\mathfrak{h}_+$ with D (and hence of $\mathfrak{h}_+^s$ with D^s), $\Gamma(s)$ corresponds to the closure of $\hat{\Gamma}(s)$. In fact, $\hat{\Gamma}(s)$ consists of the rational points of the image of $\Gamma(s)$.*

The following theorem is the main result on the saturated tensor product decomposition.

Theorem 8. *(due to Belkale-Kumar) Let $(\lambda_1, \dots, \lambda_s)$ be a s-tuple of dominant integral weights. Then, the following are equivalent:*

(i) For some integer $N > 0$, the tensor product $V(N\lambda_1) \otimes \cdots \otimes V(N\lambda_s)$ has a nonzero G-invariant subspace.

(ii) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \ldots, w_s) \in (W^P)^s$ such that

$$\epsilon_{w_1}^P \odot_0 \cdots \odot_0 \epsilon_{w_s}^P = \epsilon_o^P \in \big(H^*(G/P, \mathbb{Z}), \odot_0\big),$$

the following inequality holds:

$$\sum_{j=1}^{s} \lambda_j(w_j x_{i_P}) \geq 0, \tag{5}$$

where α_{i_P} is the simple root in $\Delta \setminus \Delta(P)$.

I have said nothing so far about the proofs, nor can I say much for lack of time. But let me mention that Theorem 4 on the eigenvalue problem for an arbitrary G follows from Theorem 8 and Proposition 7. The proof of Theorem 8 makes essential use of the Geometric Invariant Theory, specifically the Hilbert-Mumford criterion for semistability and Kempf's *maximally destabilizing one parameter subgroups* and Kempf's *parabolic subgroups* associated to unstable points. In addition, the notion of Levi-movability (defined below) plays a fundamental role in the proofs. Also, the new product $\odot_o$ in the cohomology of the flag variety G/P is intimately connected with the Lie algebra cohomology of the nil-radical of the parabolic subalgebra $\mathfrak{p}$.

Here is the definition of Levi-movability: Let P be any standard parabolic subgroup of G with Levi component L. Let $w_1, \ldots, w_s \in W^P$ be such that

$$\sum_{j=1}^{s} \operatorname{codim} \Lambda_{w_j}^P = \dim G/P. \tag{6}$$

This of course is equivalent to the condition:

$$\sum_{j=1}^{s} \ell(w_j) = (s-1) \dim G/P. \tag{7}$$

Then, the s-tuple $(w_1, \ldots, w_s)$ is called *Levi-movable* for short *L-movable* if, for generic $(l_1, \ldots, l_s) \in L^s$, the intersection $l_1\Lambda_{w_1} \cap \cdots \cap l_s\Lambda_{w_s}$ is transverse at e.

Final Remark. I have no time to discuss the Geometric Horn Problem or the Saturation theorems which have close connections with what I talked about. Nor do I have time to talk about the multiplicative analog of the eigenvalue problem.